7. Contour Integrals

Motivation (the Cauchy Integral Formula):

If f is holomorphic on and inside the simple closed (positively oriented) contour γ then

$$f(z_0) = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z - z_0}\,dz$$

for any z_0 inside γ (i.e. enclosed by γ).

Comments:

In fact the Cauchy Integral Formula can be understood in a way that makes it valid even when γ is merely continuous. (See Fulton)

- We'll assume our curves are "piecewise continuously differentiable" meaning that they have a piecewise C^1 parametrization. This will be fine for our purposes, though we could just assume the existence of a Lipshitz continuous parametrization (and really it is the image + orientation of the curve that matters, not the parametrization).
- What does "inside" mean?

A nontrivial answer: Jordan Curve Theorem

> A continuous simple closed plane curve divides the plane into two components (i.e. its complement has two connected components), one bounded and the other unbounded. The bounded component defines the inside of the curve.

Unfortunately, we don't have time to go into this.

For a proof see: Fulton, Algebraic Topology: A First Course, pages 68-72. (Springer GTM)

(Fulton also discusses the Cauchy Integral Formula, etc., in Ch 9.)

Since this is an analysis course (and for the sake of time) we are avioding using too much topology. We do lose something by doing this, however, and you are strongly recommended to read the first 72 pages of Fulton's book as a complement to this course (the presentation is elementary and you will learn a lot of beautiful mathematics on the way to the proof of the Jordan Curve Theorem).

We will be content, for a start, just to prove that the Cauchy Integral Formula holds when γ is a circle (this will already allow us to prove a lot of things). Here we make some formal definitions:

Definition

A (parametrized) curve in a metric space (or topological space) X is a continuous map $\gamma : [a,b] \to X$.

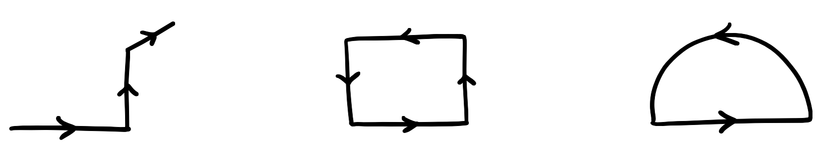
The curve is closed if $\gamma(a) = \gamma(b)$.
A non-closed curve is simple if $\gamma : [a,b] \to X$ is injective.
A closed curve is simple if $\gamma|_{[a,b)}$ is injective.

The image (or trace) of γ will be denoted $\gamma^* = \gamma([a,b])$.

We often get lazy and talk about a point lying on γ when we mean that it lies on γ^*.

Definition

A piecewise continuously differentiable (or piecewise $\mathcal{C}^1$) curve in $\mathbb{C}$ is a curve $\gamma : [a,b] \to \mathbb{C}$ which is continuously differentiable except at finitely many exceptional points, at each of which the derivative has one-sided limits.



Definition

A <u>contour</u> is a piecewise continuously differentiable curve.

> Ullrich uses the term "smooth curve" rather than "contour." The problem with the word "smooth" is that it is commonly used to mean C^∞ rather than piecewise C^1; the term is highly ambiguous and generally means "having sufficiently many derivatives."

Definition

Let $\gamma : [a,b] \to \mathbb{C}$ be a contour and $f : \gamma^* \to \mathbb{C}$ a continuous function. The <u>contour integral</u> of f over γ is

$$\int_\gamma f(z)\,dz = \int_a^b f(\gamma(t)) \cdot \gamma'(t)\,dt.$$

- The contour integral is also referred to as a "line integral" or a "path integral."

- Note that we are taking a shortcut with our definition (since for us the contour integral is just a tool). In particular, note that we could have defined $\int_\gamma f(z)dz$ by taking the limit of a sequence of "Riemann-Stieltjes sums"

$$\sum_j f(z_j)\,\Delta z_j = \sum_j f(\gamma(t_j)) \cdot (\gamma(t_j) - \gamma(t_{j-1})).$$

  Since we will only be needing piecewise C^1 curves, taking a shortcut is no problem.

- The contour integral does not really depend on the parametrization of the curve γ, but only on the image γ^* and its orientation. <u>Exercise:</u> Prove that $\int_\gamma f(z)dz$ does not change when we reparametrize $\gamma(t)$ by an orientation preserving C^1 change of variables ($\gamma \mapsto \gamma \circ \phi$, $\phi : [c,d] \to [a,b]$ C^1, $\phi' > 0$).
  $\phi(c) = a$, $\phi(d) = b$

Example:

Let $\gamma_{a,b} : [a,b] \to \mathbb{C}$ be given by $\gamma_{a,b}(t) = t$.

Then $\gamma_{a,b}'(t) = 1$ for all $t \in [a,b]$ so

$$\int_{\gamma_{a,b}} z^2 dz = \int_a^b t^2 dt = \left[\frac{t^3}{3} \right]_a^b = \frac{b^3 - a^3}{3}.$$

$\rightsquigarrow$ In general, if $f(z)$ is real for z real, then $\int_{\gamma_{a,b}} f(z)dz$ is the ordinary real integral $\int_a^b f(x)dx$ from calculus.

Contour integrals arose as a natural generalization of real integrals.

Examples:

Let $\gamma : [0,2\pi] \to \mathbb{C}$ be given by $\gamma(t) = re^{it}$ (the circle of radius $r > 0$ oriented counterclockwise). For $n \in \mathbb{Z}$ we have:

$$\int_\gamma z^n dz = \begin{cases} 2\pi i, & \text{if } n = -1 \\ 0, & \text{otherwise.} \end{cases}$$

Note: This is where the $2\pi i$ in the Cauchy Integral Formula comes from.

Calculation:

$$\int_\gamma z^n dz = \int_0^{2\pi} (re^{it})^n \cdot ire^{it} dt = ir^{n+1} \int_0^{2\pi} e^{i(n+1)t} dt.$$

When $n = -1$, $r^{n+1} = r^0 = 1$ and $e^{i(n+1)t} = e^0 = 1$, so we get $\underline{\underline{2\pi i}}$.

When $n \neq -1$, we have $\displaystyle\int_0^{2\pi} e^{i(n+1)t} dt = \int_0^{2\pi} \cos((n+1)t)\, dt + i \int_0^{2\pi} \sin((n+1)t) dt$

$$= 0 + i0 = \underline{\underline{0}} \quad \text{since } n+1 \neq 0.$$

Exercise: Compute $\int_\gamma \bar{z}^n dz$ for the same path γ.

★ Reparametrization and Changes of Coordinates

While conceptually important, the following technical results on reparametrization and changes of coordinates will not be essential for us. They are included for completeness.

Propositon (Reparametrization):

Suppose $\gamma_1 : [a,b] \to \mathbb{C}$ and $\gamma_2 : [c,d] \to \mathbb{C}$ are (parametrized) contours such that $\gamma_1^* = \gamma_2^*$ (i.e. $\gamma_1([a,b]) = \gamma_2([c,d])$).

Suppose either

(i) γ_1 and γ_2 are injective, or

(ii) $\gamma_1|_{[a,b)}$ and $\gamma_2|_{[c,d)}$ are injective and $\gamma_1(a) = \gamma_2(c) = \gamma_1(b) = \gamma_2(d)$.

Then there exists a strictly monotone continuous $\phi : [c,d] \to [a,b]$ such that $\gamma_1 \circ \phi = \gamma_2$. Moreover,

(a) if ϕ is increasing, then for every f continuous on $\gamma_1^* = \gamma_2^*$,

$$\int_{\gamma_1} f(z)\,dz = \int_{\gamma_2} f(z)\,dz \, ;$$

(b) if ϕ is decreasing, then for every f continuous on $\gamma_1^* = \gamma_2^*$,

$$\int_{\gamma_1} f(z)\,dz = -\int_{\gamma_2} f(z)\,dz .$$

Proof: Exercise. In case (i) start by proving that γ_1^{-1} is continuous on $\gamma_1([a,b])$ by using that closed subsets of $[a,b]$ are compact. For the last part ((a) & (b)) think in terms of Riemann sums. □

Usually we work with curves γ with $\gamma' \neq 0$ though.

> **Remark:** Any piecewise C^1 curve $\gamma : [a,b] \to \mathbb{C}$ can be reparametrized to a C^1 curve $\gamma \circ \phi$ (by making the derivative at the "corners"/"exceptional points" zero). The proof is an exercise.

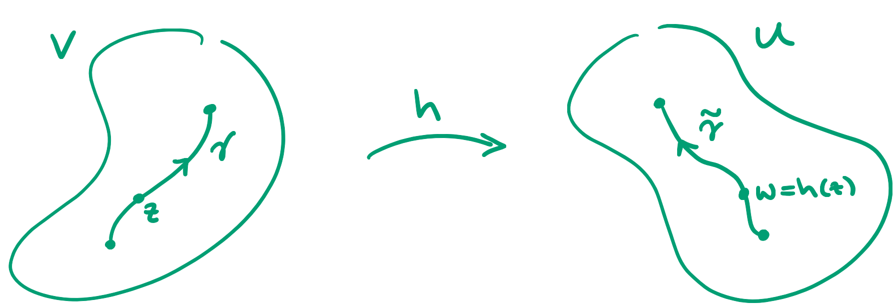
Notation: Due to the invariance of the contour integral under reparametrizations we sometimes write down integrals over an oriented curve in the plane (usually circular arcs or lines) without specifying a parametrization. In particular, we apply this to the boundaries of simple regions in the plane (such as discs), where the boundary orientation is determined by the following rule: when walking along the boundary the region should always be to your left.

> For example, $\int_{\partial \mathbb{D}} f(z)\,dz$ can be understood to mean $\int_\gamma f(z)\,dz$ where $\gamma : [0, 2\pi] \to \mathbb{C}$ is given by $\gamma(t) = e^{it}$.

Contour integrals are also invariant under biholomorphic changes of coordinates ($z \mapsto w = h(z)$):

Proposition (Change of Coordinates):
Let $\mathcal{U}, V \subseteq \mathbb{C}$ be open sets and $h : V \to \mathcal{U}$ a biholomorphism. If γ is a contour in V and $f : V \to \mathbb{C}$ is continuous, then $\tilde{\gamma} = h \circ \gamma$ is a contour in $\mathcal{U}$, $\tilde{f} = f \circ h^{-1} : \mathcal{U} \to \mathbb{C}$ is continuous and

$$\int_\gamma f(z)\,dz = \int_{\tilde{\gamma}} \tilde{f}(w)\,dw.$$



Exercise: Let $f : \partial \mathbb{D} \to \mathbb{C}$ be a continuous function. Prove that

$$\int_{\partial \mathbb{D}} f(z)\,dz = \int_{\partial \mathbb{D}} \frac{f\left(\frac{1}{z}\right)}{z^2}\,dz = \int_{\partial \mathbb{D}} f(\bar{z})\,\bar{z}^2\,dz.$$

* Chains

Definition

A chain in $U \subseteq \mathbb{C}$ is a formal expression

$$\Gamma = a_1\gamma_1 + a_2\gamma_2 + \cdots + a_n\gamma_n$$

where $a_1, \dots, a_n \in \mathbb{Z}$ and $\gamma_1, \dots, \gamma_n$ are piecewise C^1 curves (contours) in U. The trace of Γ is $\Gamma^* = \bigcup_{j=1}^n \gamma_n^*$. If $f : \Gamma^* \to \mathbb{C}$ is continuous, then we define the integral

$$\int_\Gamma f(z)\,dz = a_1 \int_{\gamma_1} f(z)\,dz + \cdots + a_n \int_{\gamma_n} f(z)\,dz.$$

Two chains Γ_1 and Γ_2 are said to be equivalent (as chains) if

$$\int_{\Gamma_1} f(z)\,dz = \int_{\Gamma_2} f(z)\,dz$$

for all continuous functions $f : U \to \mathbb{C}$; in this case we write $\Gamma_1 = \Gamma_2$ (as chains). We write $\Gamma = 0$ (the zero chain) if

$$\int_\Gamma f(z)\,dz = 0$$

for all continuous functions $f : U \to \mathbb{C}$.

Comments:

- The notion of a "chain" comes from algebraic topology. What we are calling chains are really "1-chains" (though the curves are usually taken to be merely continuous, rather than piecewise C^1). "0-chains" in U are formal sums of points in U. If $\gamma : [a,b] \to U$ is a curve then its oriented boundary is the 0-chain $\partial\gamma = \gamma(b) - \gamma(a)$. This definition extends naturally to 1-chains: the boundary of $\Gamma = a_1\gamma_1 + \cdots + a_n\gamma_n$ is $\partial\Gamma = a_1\partial\gamma_1 + \cdots + a_n\partial\gamma_n$. A 1-chain Γ is closed if $\partial\Gamma = 0$. A closed 1-chain is called a 1-cycle.

"<u>2-chains</u>" in $\mathcal{U}$ are the formal sums of continuous maps from two dimensional simplices (filled triangles $\blacktriangleright$) into $\mathcal{U}$. These also have a well defined notion of <u>oriented boundary</u> ($\partial \blacktriangleright = \triangleright$). The boundary of a 2-chain is necessarily a 1-cycle (a <u>closed</u> 1-chain). A basic question concerning the topology of $\mathcal{U}$ is this:

<u>When is a 1-cycle in $\mathcal{U}$ the boundary of a 2-chain in $\mathcal{U}$?</u>

This will not always be the case if the set $\mathcal{U} \subseteq \mathbb{C}$ has "holes." We measure the presence of such "holes" using the <u>first homology group</u>, defined to be the quotient (of abelian groups under addition):

$$H_1(\mathcal{U}) = \left\{ \begin{matrix} \text{1-cycles} \\ \text{in } \mathcal{U} \end{matrix} \right\} \Big/ \left\{ \begin{matrix} \text{1-cycles in } \mathcal{U} \text{ that are} \\ \underline{\text{boundaries}} \text{ of 2-chains in } \mathcal{U} \end{matrix} \right\}.$$

For example, $\mathbb{C}$ has no holes and $H_1(\mathbb{C}) = \{0\}$. Similarly $H_1(\mathbb{D}) = \{0\}$. On the other hand, $\mathbb{C} \setminus \{0\}$ has one "hole" and $\underline{H_1(\mathbb{C} \setminus \{0\}) \cong \mathbb{Z}}$; a curve such as the unit circle taken counterclockwise is not the boundary of a 2-chain in $\mathbb{C} \setminus \{0\}$ because such a 2-chain would have to stretch over 0 which is not in $\mathcal{U} = \mathbb{C} \setminus \{0\}$; we get the integers $\mathbb{Z}$ for our H_1 since a 1-cycle can wind around 0 an integer number of times; any two 1-cycles in $\mathbb{C} \setminus \{0\}$ that wind around 0 k times in the counterclockwise direction ($k \in \mathbb{Z}$) differ from k copies of the counterclockwise unit circle by the boundary of a 2-chain.

Example: γ_1 and γ_2 differ by the boundary of a 2-chain in $\mathbb{C} \setminus \{0\}$. Hence they define the same equivalence class of 1-cycles in $H_1(\mathbb{C} \setminus \{0\})$; we say that γ_1 and γ_2 are <u>homologous</u>.

- While we will not develop the concept of homology formally in this course we will need it later (we'll understand it using winding numbers. It would pay though to read more about this (e.g. from Fulton's Algebraic Topology).

- Note that we defined two chains Γ_1 and Γ_2 in U to be equivalent if $\int_{\Gamma_1} f(z)\,dz = \int_{\Gamma_2} f(z)\,dz$ for all continuous functions $f: U \to \mathbb{C}$.

  If we replace continuous functions by holomorphic functions then we get a different notion of equivalence (which is easier to satisfy). It turns out that this is the notion of being homologous! A version of Cauchy's theorem states:

  > Let U be an open set and let Γ_1 and Γ_2 be chains in U. Then $\int_{\Gamma_1} f(z)\,dz = \int_{\Gamma_2} f(z)\,dz$ for all holomorphic functions $f: U \to \mathbb{C}$ if and only if Γ_1 and Γ_2 are homologous in U.

For the most part we will simply use the notion of chains to give an easy way of writing down the concatenation of several curves for the purposes of integration. Chain arithmetic also gives a nice way of reversing a curve: $-\gamma$ is equivalent to γ but with the reversed orientation and

$$\int_{-\gamma} f(z)\,dz = -\int_{\gamma} f(z)\,dz.$$

(easy) Exercise (Concatenation):

Let $\gamma_1 : [a,b] \to \mathbb{C}$ and $\gamma_2 : [b,c] \to \mathbb{C}$ be two contours, with $\gamma_1(b) = \gamma_2(b)$. Prove that the contour $\gamma : [a,c] \to \mathbb{C}$ defined by $\gamma(t) = \gamma_1(t)$ for $t \in [a,b]$ and $\gamma(t) = \gamma_2(t)$ for $t \in [b,c]$ is equivalent (as a chain) to $\gamma_1 + \gamma_2$. That is, show that for all continuous functions $f: \mathbb{C} \to \mathbb{C}$ (or $f: \gamma^* \to \mathbb{C}$) we have

$$\int_{\gamma} f(z)\,dz = \int_{\gamma_1+\gamma_2} f(z)\,dz.$$

In situations like in the above exercise, we may talk about $\gamma_1 + \gamma_2$ as if it were a contour (γ) rather than a chain (sum of contours).

Exercise (Going around n times):
Let $\partial \mathbb{D}$ denote the counterclockwise path around the unit disc. Show that for any integer n, the chain $n\partial\mathbb{D}$ is equivalent to the curve $\gamma : [0, 2\pi] \to \mathbb{C}$ given by $\gamma(t) = e^{int}$ (the path that goes n times around the unit circle counterclockwise).

★ Estimating Contour Integrals

Note that integrating $f(z)$ with respect to dz along a contour is different from the familiar arclength integral (integral with respect to ds) from calculus. But we have notation for this too:

Definition
If f is continuous on a contour $\gamma : [a,b] \to \mathbb{C}$, then the integral of f with respect to arclength along γ is

$$\int_\gamma f(z)\,|dz| = \int_a^b f(\gamma(t)) \cdot \underbrace{|\gamma'(t)|\,dt}_{\text{"ds"}}.$$

The following proposition is a consequence of the triangle inequality (applied to Riemann sums):

Proposition (Estimating Contour Integrals / "ML Inequality"):
Suppose $\gamma : [a,b] \to \mathbb{C}$ is a contour and f is a continuous function on γ^*. Then

$$\left| \int_\gamma f(z)\,dz \right| \leq \int_\gamma |f(z)|\,|dz|.$$

In particular, if $|f(z)| \leq M$ on γ^* and $L = \int_\gamma |dz|$ is the length of γ, then

$$\left| \int_\gamma f(z)\,dz \right| \leq ML.$$

Exercise (Arclength and Uniform Convergence):

(a) Show that arclength is not preserved under uniform convergence of paths: Find a sequence of contours $\gamma_n : [0,1] \to \mathbb{C}$ that uniformly converge to a constant path (a contour of length 0) but such that $\int_{\gamma_n} |dz| \geq n$ for all n.

(b) Suppose a sequence of C^1 curves $\gamma_n : [0,1] \to \mathbb{C}$ converges uniformly to a C^1 curve $\gamma : [0,1] \to \mathbb{C}$ and that γ_n' also converges uniformly to γ'. Show that $\int_{\gamma_n} |dz| \to \int_{\gamma} |dz|$.