6. Analytic Functions

Functions that may be locally represented by a convergent power series are known as <u>analytic</u>. While this notion is common for functions of a real variable (and one may talk about "real-analytic" functions) it is clearly more natural to think about analytic functions in terms of <u>complex variables</u>.

Definition

Let $V \subseteq \mathbb{C}$ be open. A function $f: V \to \mathbb{C}$ is <u>analytic</u> if for every $z_0 \in V$ there exists an $r > 0$ with $\Delta_r(z_0) \subseteq V$ and a power series $\sum_{n=0}^{\infty} c_n (z-z_0)^n$ converging to f on $\Delta_r(z_0)$.

From last time (Proposition 1.1) we have:

Proposition

An analytic function $f: V \to \mathbb{C}$ is holomorphic; in fact it is infinitely complex differentiable and each derivative of f is analytic.

> After we prove the Cauchy Integral Formula we will see that holomorphic functions are analytic, and hence also infinitely complex differentiable.

Examples:

1. Any polynomial $p(z)$ is analytic.

② $1/z$ is analytic on $\mathbb{C} \setminus \{0\}$.

↳ **Exercise:** Prove this by explicitly writing down a power series representation at any $z_0 \in \mathbb{C} \setminus \{0\}$ using the geometric series.

③ e^z, $\sin z$, $\cos z$ are analytic on $\mathbb{C}$.

↑ It suffices to prove this for e^z.

Proposition

The power series $\sum_{n=0}^{\infty} \frac{z^n}{n!}$ has infinite radius of convergence and

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

for all $z \in \mathbb{C}$.

Proof:

That the power series $f(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}$ has infinite radius of convergence is easy to see, e.g., by the Ratio Test. In order to show that $f(z) = e^z$ for all $z \in \mathbb{C}$ note that

$f(z)$ is the unique power series in z solving $f(0) = 1$ & $f' = f$

$$f(0) = 1 \quad \text{and} \quad f'(z) = \sum_{n=1}^{\infty} \frac{z^{n-1}}{(n-1)!} = \sum_{n=0}^{\infty} \frac{z^n}{n!} = f(z)$$

for all $z \in \mathbb{C}$. Thus (noting that e^z is never 0) we have

$$\frac{d}{dz}\left[\frac{f(z)}{e^z}\right] = \frac{f'(z)e^z - e^z f(z)}{(e^z)^2} = \frac{f(z)e^z - f(z)e^z}{(e^z)^2} = 0$$

for all $z \in \mathbb{C}$ and hence $f(z)/e^z$ is constant on $\mathbb{C}$. Since $f(0) = e^0 = 1$ we must have $f(z) = e^z$ for all $z \in \mathbb{C}$.

$\square$

Note: To see that e^z may be represented by a power series centered at z_0 for any $z_0 \in \mathbb{C}$ one may simply write $e^z = e^{z_0 + (z-z_0)} = e^{z_0} e^{(z-z_0)}$ and expand e^{z-z_0} as $\sum_{n=0}^{\infty} \frac{(z-z_0)^n}{n!}$.

☆ Zeros of Analytic Functions

Definition

Let $V \subseteq \mathbb{C}$ be an open set and $f: V \to \mathbb{C}$ an analytic function. We say that f has a zero of order $N \in \{1,2,3,\ldots\}$ at $z \in U$ if

$$f^{(n)}(z) = 0 \quad \text{for } 0 \leq n < N \quad \text{but } f^{(N)}(z) \neq 0.$$

A zero of order 1 is also known as a simple zero.

Examples:

1. z^N has a zero of order N at $z=0$.
2. $\sin z$ has a simple zero at $z = 0, \pm\pi, \pm 2\pi, \ldots$.

Exercise: show that these are the only zeros of $\sin z$.

Lemma (Finite order zeros of analytic functions are isolated):

Let $V \subseteq \mathbb{C}$ be open and $f: V \to \mathbb{C}$ an analytic function. If f has a zero of order N at $z_0 \in V$ then there exists $r > 0$ such that $f(z) \neq 0$ for $z \in \Delta_r(z_0) \setminus \{z_0\}$.

Proof: Take $r_0 > 0$ s.t. $f(z) = \sum_{n=0}^{\infty} c_n (z-z_0)^n$ on $\Delta_{r_0}(z_0)$.

Since $f(z_0) = f'(z_0) = \cdots = f^{(N-1)}(z_0) = 0$ we have $c_0 = c_1 = \cdots = c_{N-1} = 0$ (as $c_n = \frac{f^{(n)}(z_0)}{n!}$). Hence

$$f(z) = \sum_{n=N}^{\infty} c_n (z-z_0)^n = (z-z_0)^N \sum_{n=N}^{\infty} c_n (z-z_0)^{n-N} \quad \text{on } \Delta_{r_0}(z_0)$$

Define $g(z)$ on $\Delta_{r_0}(z_0)$ by $g(z) = \sum_{n=N}^{\infty} c_n (z-z_0)^{n-N}$ (convergent since the series for f is convergent). Then $g(z)$ is a continuous function on $\Delta_{r_0}(z_0)$ (being $\mathbb{C}$-diff$^{\text{ble}}$) and $g(z_0) = c_N \neq 0$ (since $c_N = \frac{f^{(N)}(z_0)}{N!}$ and $N! \neq 0$). Hence there exists $r > 0$ s.t. $g(z) \neq 0$ on $\Delta_r(z_0)$, which implies

$$f(z) = (z-z_0)^N g(z)$$

is nonvanishing on $\Delta_r(z_0) \setminus \{z_0\}$.

$\square$

On the other hand, if an analytic function vanishes to <u>infinite order</u> at a point z_0 in its domain V (assumed connected) then that function is identically zero.

<u>Lemma</u>

Let $V \subseteq \mathbb{C}$ be open and connected, and $f : V \to \mathbb{C}$ an analytic function. If f vanishes to infinite order at $z_0 \in V$, i.e.

$$f^{(n)}(z_0) = 0 \quad \text{for all } n \in \{0, 1, 2, 3, \ldots\},$$

then f is identically zero.

<u>Proof:</u> Since f is analytic there exists $r > 0$ such that

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} (z-z_0)^n = 0 \qquad \text{for } z \in \Delta_r(z_0).$$

Let Z_f denote the set of points in V where f is zero. Since f is continuous, Z_f is closed. Let $Z'_f \subseteq Z_f$

denote the set of points in Z_f that are not isolated. Clearly $\Delta_r(z_0) \subseteq Z'_f$. Note that the set Z'_f must also be closed as Z_f is closed and no sequence of points in Z'_f can converge to an isolated point of Z_f.

We now show that Z'_f is open. Suppose $z_1 \in Z'_f$ is not an interior point of Z'_f, then f cannot vanish to infinite order at z_1 (else z_1 would be an interior point of Z'_f) so it must vanish to some finite order N; but this implies that z_1 is an isolated zero, contradicting that $z_1 \in Z'_f$; we conclude that Z'_f consists solely of interior points, i.e. Z'_f is open.

Combining the previous two lemmas we have:

Theorem (The Identity Theorem):

Let $V \subseteq \mathbb{C}$ be open and connected and $f: V \to \mathbb{C}$ be an analytic function. If $Z_f = \{z \in V : f(z) = 0\}$ has a limit point in V, then f is identically zero.

In other words, the zeros of f are all isolated unless $f \equiv 0$.

Proof: The result follows immediately from the preceeding two lemmas. $\square$

Corollary

Suppose $V \subseteq \mathbb{C}$ is an open and connected set and $f, g: V \to \mathbb{C}$ are analytic functions. If $\{z \in V : f(z) = g(z)\}$ has a limit point, then $f \equiv g$.