5 Power Series

One of our first goals in the course is to prove that functions that are complex differentiable on an open set (holomorphic functions) are locally represented by convergent power series. (A remarkable fact!)

Here we record some basics on power series:

Definition: A (formal) power series about $z_0 \in \mathbb{C}$ is a series

$$\sum_{n=0}^{\infty} c_n (z-z_0)^n$$

where the c_n are complex constants $(n = 0, 1, 2, \ldots)$ and $z \in \mathbb{C}$. Where the series converges it defines a function of z.

Note: By convention $(z-z_0)^0 = 1$ for any z.

Recall: $\Delta_r(z_0) \overset{\text{def}}{=} \{z \in \mathbb{C} : |z-z_0| < r\}$.

Lemma 1.0: Suppose $(c_n)_{n=0}^{\infty}$ is a sequence of complex numbers, and define R by

R could be ∞

$$R = \sup\{r \geq 0 : \text{the sequence } (c_n r^n) \text{ is bounded}\}.$$

Then the power series $\sum_{n=0}^{\infty} c_n (z-z_0)^n$ converges absolutely and uniformly* on every compact subset of the disc $\Delta_R(z_0)$ and diverges at every point z with $|z-z_0| > R$.

* In fact we will see that $\sum_{n=0}^{\infty} |c_n||z-z_0|^n$ converges uniformly on compact subsets of $\Delta_R(z_0)$, i.e. the absolute convergence <u>is</u> uniform.

Proof:

Note that the second part of the conclusion follows from the definition of R, since if $|z-z_0|>R$ then the terms in the series $\sum_{n=0}^{\infty} c_n(z-z_0)^n$ are not bounded so the series cannot converge (by the "n^{th} term test").

For the first part of the conclusion it suffices to show that the series converges uniformly and absolutely on $\overline{\Delta_r(z_0)}$ for any r with $0 \le r < R$. Fix such an r and choose a number ρ with $r<\rho<R$. The definition of R tells us that $(c_n\rho^n)$ is bounded; take $A>0$ s.t. $|c_n|\rho^n < A$ for all n. Then for $z \in \overline{\Delta_r(z_0)}$,

$$(*) \qquad |c_n(z-z_0)^n| \le |c_n| r^n = |c_n|\rho^n \cdot \left(\frac{r}{\rho}\right)^n \le A \cdot \left(\frac{r}{\rho}\right)^n .$$

Since $\frac{r}{\rho}<1$, the geometric series $\sum_{n=0}^{\infty} \left(\frac{r}{\rho}\right)^n$ converges, and hence $(*)$ shows that $\sum_{n=0}^{\infty} c_n(z-z_0)^n$ converges absolutely and uniformly on $\overline{\Delta_r(z_0)}$. $\square$

The R in Lemma 1.0 is known as the radius of convergence. A more usual way to define R is via the Root Test; we have

$$R = \left(\limsup_{n\to\infty} \sqrt[n]{|c_n|}\right)^{-1}$$

where we take ∞^{-1} to be 0 and 0^{-1} to be ∞. It is a good exercise to prove that this is equivalent to the definition of R in Lemma 1.0.

Q: What happens on the boundary ($|z-z_0|=R$)?

→ A power series may or may not converge at a point z on the boundary of the disc of convergence.

Examples:

The series $\sum_{n=1}^{\infty} z^n$, $\sum_{n=1}^{\infty} \frac{z^n}{n}$, and $\sum_{n=1}^{\infty} \frac{z^n}{n^2}$ all have radius of convergence $R=1$. However,

1. $\sum_{n=1}^{\infty} z^n$ diverges for all z with $|z|=1$
2. $\sum_{n=1}^{\infty} \frac{z^n}{n}$ converges for all z with $|z|=1$ except $z=1$.
3. $\sum_{n=1}^{\infty} \frac{z^n}{n^2}$ converges for all z with $|z|=1$.

Exercise: Prove that $\sum_{n=1}^{\infty} \frac{z^n}{n}$ converges for all $z \in \overline{\mathbb{D}} \setminus \{1\}$.

↳ Suggestion: Multiply the series by $(1-z)$ to get $z - \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1}\right) z^{n+1}$ and then prove absolute convergence (for $|z| \leq 1$, $z \neq 1$) by comparison with the telescoping series $\sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1}\right)$.

The exercise can also be done using either Abel's test or Dirichlet's test.

Q: Is $f(z) = \sum_{n=0}^{\infty} c_n (z-z_0)^n$ complex differentiable in its disc of convergence $\Delta_R(z_0)$?

Note that for each N, $f_N(z) = \sum_{n=0}^{N} c_n (z-z_0)^n$ is $\mathbb{C}$-diff^{ble} (being a polynomial) and

$$f_N \xrightarrow{\text{uniformly}} f$$

on any compact subset of $\Delta_R(z_0)$.

$\rightsquigarrow$ Does this imply f is $\mathbb{C}$-diff^{ble}? (Dumb question?)

Surprisingly, the answer to this is yes: If a sequence h_n of holomorphic functions converges uniformly on V to some function h, then the limit function is also holomorphic! (We'll prove this later.)

$\rightsquigarrow$ Why is this surprising?

Consider the real variables case:

$$\left.\begin{array}{l} f_n \to f \text{ uniformly} \\ \& \\ f_n' \text{ (real) diff}^{ble}\ \forall n \end{array}\right. \;\not\Rightarrow\; f \text{ (real) diff}^{ble}$$

For example, $\sqrt{x^2 + \frac{1}{n^2}} \to |x|$ uniformly on $\mathbb{R}$.

If, however, we assume $f_n \to f$ (pointwise) and $f_n' \to g$ uniformly, then f must be real diff^{ble} and $f' = g$. We could adapt this argument to prove our power series is $\mathbb{C}$-diff^{ble} in $\Delta_R(z_0)$ if we wished.

$\rightsquigarrow$ For now we will set aside this discussion of uniform convergence and simply prove the differentiability directly.

Proposition 1.1

Suppose that the power series $\sum_{n=0}^{\infty} c_n(z-z_0)^n$ has radius of convergence $R>0$. Then the function

$$f(z) = \sum_{n=0}^{\infty} c_n(z-z_0)^n$$

is differentiable on the disc $\Delta_R(z_0)$, with derivative

$$f'(z) = \sum_{n=1}^{\infty} nc_n(z-z_0)^{n-1} = \sum_{n=0}^{\infty} (n+1)c_{n+1}(z-z_0)^n.$$

Proof:

We may assume without loss of generality that $z_0=0$.

warm up

To differentiate z^n at a point w we consider the difference quotient

(*) $$\frac{z^n-w^n}{z-w} = \sum_{k=0}^{n-1} z^k w^{n-1-k}$$

which goes to nw^{n-1} as $z\to w$.

Applying (*) to f term-wise we get, for $z,w\in\Delta_R(0)$

(**) $$\frac{f(z)-f(w)}{z-w} = \sum_{n=1}^{\infty} c_n \frac{z^n-w^n}{z-w} = \sum_{n=1}^{\infty} c_n \sum_{k=0}^{n-1} z^k w^{n-1-k}$$

provided $z\neq w$. We wish to take the limit as $z\to w$ in the right hand side expression to get $\sum_{n=1}^{\infty} c_n\cdot nw^{n-1}$, but this requires passing the limit inside the infinite sum.

We need to justify this exchanging the order of the limits.

Trick: Let $F : \Delta_R(0) \times \Delta_R(0) \to \mathbb{C}$ be the function of two complex variables given by

$$F(z,w) = \sum_{n=1}^{\infty} c_n \sum_{k=0}^{n-1} z^k w^{n-1-k}.$$

It is easy to see that the series defining F converges uniformly on compact subsets of $\Delta_R(0) \times \Delta_R(0)$:
Fix $r \in (0,R)$, choose $\rho \in (r,R)$, choose A such that $|c_n|\rho^n \le A$ for all n, and note that for $(z,w) \in \overline{\Delta_r(0)} \times \overline{\Delta_r(0)}$ we have

$$\left| c_n \sum_{k=0}^{n-1} z^k w^{n-k-1} \right| \le n|c_n| r^{n-1} = n|c_n|\rho^n \frac{r^{n-1}}{\rho^n} \le n\frac{A}{r}\left(\frac{r}{\rho}\right)^n;$$

e.g., by the Ratio Test

since $\sum_{n=1}^{\infty} n\left(\frac{r}{\rho}\right)^n$ converges, the series for $F(z,w)$ converges uniformly on $\overline{\Delta_r(0)} \times \overline{\Delta_r(0)}$.

The uniform limit of continuous functions is continuous, thus F is continuous on $\overline{\Delta_r(0)} \times \overline{\Delta_r(0)}$ for any $r \in (0,R)$ and hence on $\Delta_R(0) \times \Delta_R(0)$. Hence, by $(**)$

$$f'(w) \left\{ \begin{aligned} \lim_{z\to w} \frac{f(z)-f(w)}{z-w} &= \lim_{z \to w} F(z,w) = F(w,w) \\ &= \sum_{n=1}^{\infty} c_n \cdot n w^{n-1} \end{aligned} \right.$$

for any $w \in \Delta_R(0)$.

This proves the result. $\square$

Concluding Remark:

Note that if $f(z)$ and $g(z)$ are both defined by convergent power series in the disc $\Delta_r(z_0)$, then so are

$$f(z)+g(z) \quad \text{and} \quad f(z)g(z).$$

(The fact that $f(z)g(z)$ is also represented by a convergent power series is a consequence of absolute convergence; the Cauchy product of two absolutely convergent power series is absolutely convergent.)

Appendix: Complex Series

Definition: We say that a series $\sum_{n=0}^{\infty} a_n$ (where $a_n \in \mathbb{C}$, $n=0,1,2,\dots$) <u>converges</u> if the corresponding sequence of partial sums $\left(\sum_{n=0}^{N} a_n\right)_{N=0}^{\infty}$ converges in $\mathbb{C}$ (as $N \to \infty$).

↳ This is the same as saying $\sum_{n=0}^{\infty} \mathrm{Re}\, a_n$ converges and $\sum_{n=0}^{\infty} \mathrm{Im}\, a_n$ converges.

Proposition: If $\sum_{n=0}^{\infty} |a_n|$ converges, then $\sum_{n=0}^{\infty} a_n$ converges.

| i.e. Absolute convergence $\Rightarrow$ Convergence.

Proof:

Suppose $\sum_{n=0}^{\infty} |a_n|$ converges. Then $\sum_{n=0}^{\infty} a_n$ is Cauchy since

$$\left| \sum_{n=N+1}^{M} a_n \right| \leq \sum_{n=N+1}^{M} |a_n| \qquad \text{for any integers } M > N > 0$$

(and since $\sum_{n=0}^{\infty} |a_n|$ is Cauchy). Hence $\sum_{n=0}^{\infty} a_n$ converges. $\square$

* Geometric Series

By expanding and telescoping

$$(1-z)(1+z+z^2+\dots+z^m)$$

we get

$$1 + z + z^2 + \cdots + z^m = \frac{1 - z^{m+1}}{1 - z} \qquad \text{for } z \in \mathbb{C} \setminus \{1\}.$$

Proposition:

(i) For $z \in \mathbb{D}$, $\quad \dfrac{1}{1-z} = \sum_{n=0}^{\infty} z^n$;

(ii) For $z \notin \mathbb{D}$, $\quad \sum_{n=0}^{\infty} z^n$ diverges;

(iii) Given $0 < r < 1$, for all $z \in \overline{\Delta_r(0)}$ (i.e. $|z| \leq r$) we have

$$\left| \frac{1}{1-z} - \sum_{n=0}^{\infty} z^n \right| \leq \frac{r^m}{1-r}$$

and hence $\sum_{n=0}^{\infty} z^n$ converges uniformly for $z \in \overline{\Delta_r(0)}$.

Proof: Easy exercise. $\square$

Note that the uniform convergence in (iii) also follows from Lemma 1.0 above.

Note also that (by the reverse triangle inequality) if $|z| \leq r$ then

$$|1 - z| \geq \big|1 - |z|\big| = 1 - |z| \geq 1 - r$$

and hence

$$\left| \frac{1}{1-z} \right| \leq \frac{1}{1-r}.$$