4 Holomorphic Functions as Maps

* Biholomorphisms

- When are two complex domains "the same" (from the point of view of complex analysis)?
- What kind of changes of variables/coordinates should be allowed when working with holomorphic functions?

Definition

Let $U, V \subseteq \mathbb{C}$ be open sets. A holomorphic function $f : U \to V$ that is bijective and such that the inverse f^{-1} is also holomorphic is called a biholomorphism.

> A biholomorphism is also called a conformal mapping, and the sets U and V are said to be conformally equivalent.

Remark

Later we will see that

$$f \text{ holomorphic} + \text{bijective} \implies f^{-1} \text{ holomorphic.}$$

(Note that in the real variables case

$$f \text{ (real) diff}^{\text{ble}} + \text{bijective} \not\Rightarrow f^{-1} \text{ (real) diff}^{\text{ble}},$$

consider, e.g., $f(x) = x^3$ as a map from $\mathbb{R}$ to $\mathbb{R}$.)

Example

The Cayley map $C(z) = \frac{z-i}{z+i}$ restricts to a biholomorphism from the upper half plane $\mathbb{H} = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$ to the unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$.

Definition

Let $V \subseteq \mathbb{C}$ be an open set. A biholomorphism $f: V \to V$ is called an automorphism (or biholomorphic/conformal automorphism) of V.

The set of automorphisms of V is denoted $\mathrm{Aut}(V)$. It is easy to check that $\mathrm{Aut}(V)$ forms a group under composition.

Examples

1. For any $a, b \in \mathbb{C}$ with $a \neq 0$ the map $z \mapsto az+b$ is an automorphism of $\mathbb{C}$.

2. Exercise: If $a, b, c, d \in \mathbb{R}$ with $ad-bc > 0$, then the LFT $\frac{az+b}{cz+d}$ is an automorphism of the upper half plane $\mathbb{H}$.

Since a, b, c, d can be scaled by $\lambda > 0$ so that $ad-bc=1$ (without changing the LFT) we see that $\mathrm{Aut}(\mathbb{H})$ contains a group of LFTs isomorphic to

$$PSL(2,\mathbb{R}) \stackrel{\text{def}}{=} SL(2,\mathbb{R})/\{\pm I\}.$$

multiplying a,b,c,d by -1 does not change $\frac{az+b}{cz+d}$

Later we will see that $\mathrm{Aut}(\mathbb{H}) \cong PSL(2,\mathbb{R})$, i.e. these LFTs give all of the automorphisms.

* Inverse Function Theorem

A biholomorphism f has the property that $f'(z) \neq 0$ for any z.

Why? Differentiate $f^{-1}(f(z)) = z$

$$\rightsquigarrow \quad (f^{-1})'(f(z)) \cdot f'(z) = 1$$

This tells us that $f'(z) \neq 0$ and $(f^{-1})'(f(z)) = \frac{1}{f'(z)}$.

Locally, we have a converse:

Theorem (Inverse Function Theorem for Holomorphic Maps):
Suppose $U \subseteq \mathbb{C}$ is open, $f: U \to \mathbb{C}$ is holomorphic, $p \in U$, and $f'(p) \neq 0$. Suppose further that f is continuously diff^{ble}. Then there exist open sets $V, W \subseteq \mathbb{C}$ such that $p \in V \subseteq U$, $W = f(V)$, and $f|_V : V \to W$ is a biholomorphism. The map $g = f|_V^{-1} : W \to V$ satisfies

$$g'(w) = \frac{1}{f'(g(w))} \quad \text{for all } w \in W.$$

Remark: Later we will see that the assumption that f be continuously diff^{ble} can be dropped (it follows from the assumption that f is holomorphic).

Proof

The existence of V, W and the local inverse map $g: W \to V$ follows from the inverse function theorem for continuously diff^{ble} maps from (open subsets of) $\mathbb{R}^2$ to $\mathbb{R}^2$ since $f'(p) \neq 0$ implies the real derivative $Df(p)$ is invertible (why?).
From the real inverse function theorem one then also has that $Dg(w) = Df(g(w))^{-1}$. Since f is holomorphic, $Df(z)$ corresponds to multiplication by a complex number (which must be nonzero when $z \in V$). Hence, for $w \in W$, $Dg(w) = Df(g(w))^{-1}$ also corresponds to multiplication by a complex number (namely $f'(g(w))^{-1}$), i.e. g is holomorphic on W and

$$g'(w) = f'(g(w))^{-1} \quad \text{for } w \in W. \qquad \square$$

Example Application

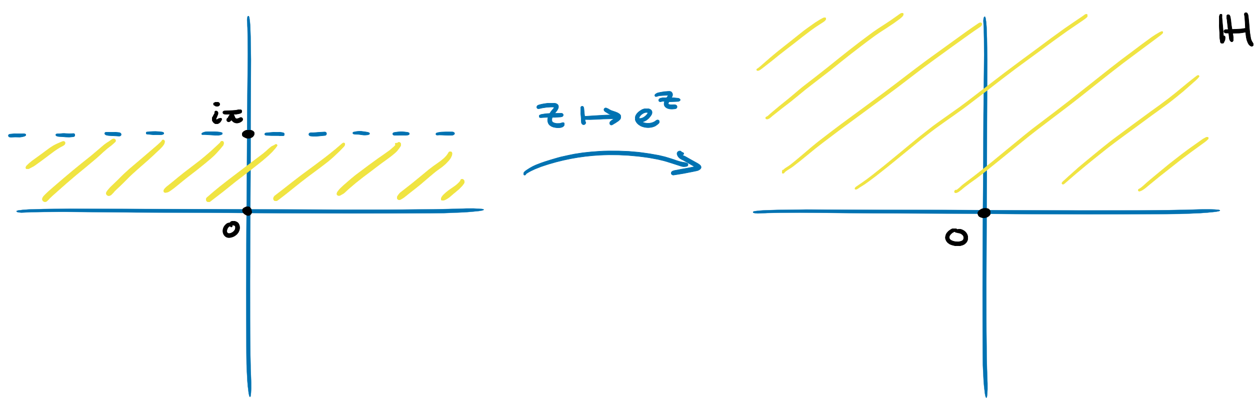
The map $f(z) = e^z$ restricts to a biholomorphism

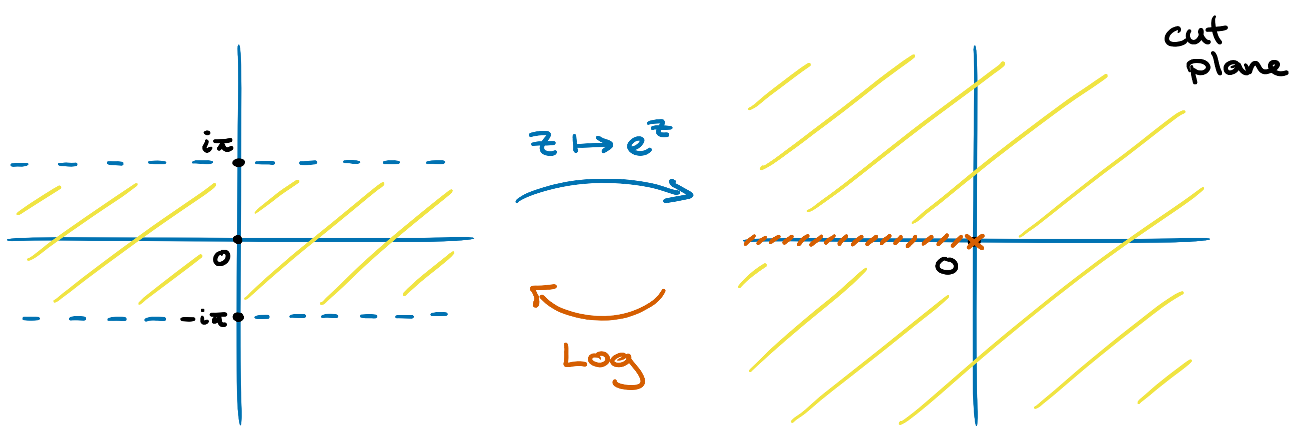
from $\{z \in \mathbb{C} : 0 < \operatorname{Im} z < \pi\}$ to $\mathbb{H}$

and from $\{z \in \mathbb{C} : -\pi < \operatorname{Im} z < \pi\}$ to $\mathbb{C} \setminus (-\infty, 0]$.

Why?

- Step 1 : prove f is bijective so that f^{-1} exists.
- Step 2 : check that f' is never zero and use the inverse function theorem to show that f^{-1} is holomorphic.





The inverse of the map $\{z \in \mathbb{C} : -\pi < \operatorname{Im} z < \pi\} \to \mathbb{C} \setminus (-\infty, 0]$ which sends z to e^z is denoted Log. $\operatorname{Log}(z)$ is holomorphic on the cut plane $\mathbb{C} \setminus (-\infty, 0]$ and is known as the principal branch of the log function.

Remark: Note that $z \mapsto e^z$ is many-to-one on the complex plane (since, e.g., $e^{2\pi i} = e^0 = 1$) and only becomes injective after we restrict to an appropriate subset of $\mathbb{C}$. Thus there are many possible choices for a "complex logarithm." This is an important point and we will come back to this later!

Exercise: Show that for $z \in \mathbb{C} \setminus (-\infty, 0]$ and Log as defined above we have

$$\mathrm{Log}(z) = \ln(|z|) + i\,\mathrm{Arg}(z)$$

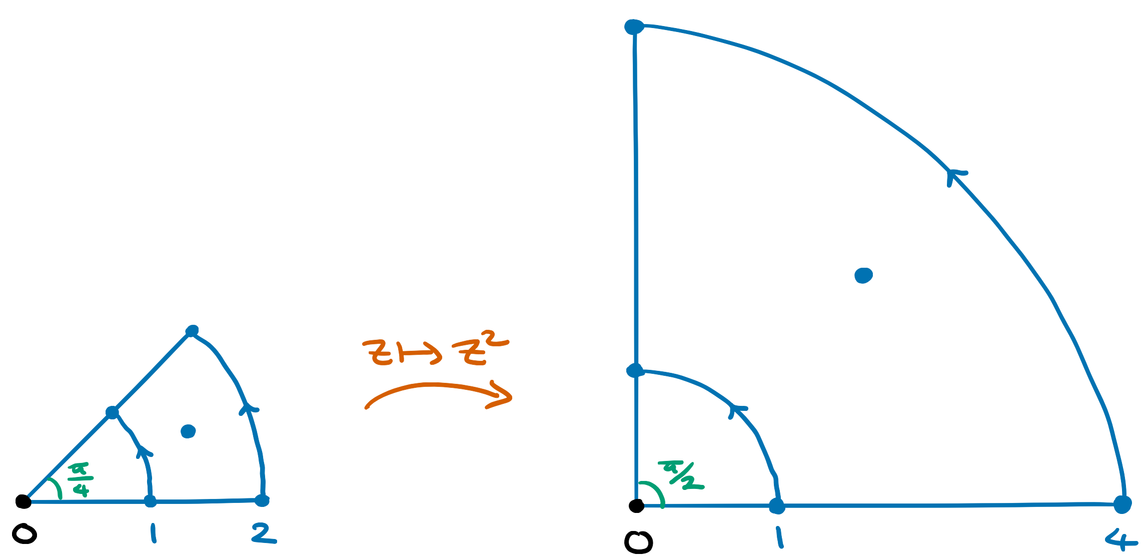
where ln is the ordinary (real) natural logarithm and Arg is the principal argument function (so $-\pi < \mathrm{Arg}(z) \leq \pi$).

Power Functions (z^n, $n = 1, 2, 3, \ldots$):

For $z \neq 0$ we can write $z = re^{i\theta}$, and hence

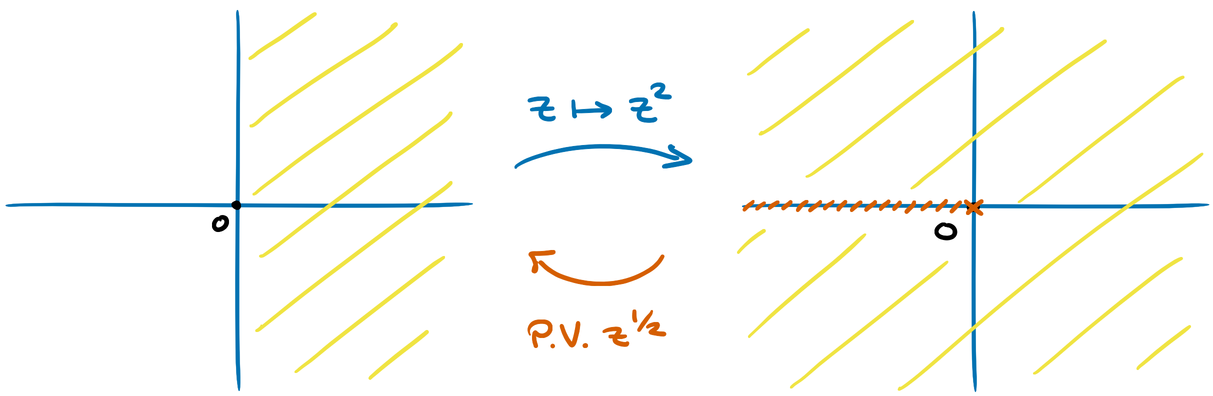
$$z^n = r^n e^{in\theta}.$$

$\rightsquigarrow$ The map $z \mapsto z^n$ sends 0 to 0 and takes the circle of radius r and wraps it around the circle of radius r^n n times.



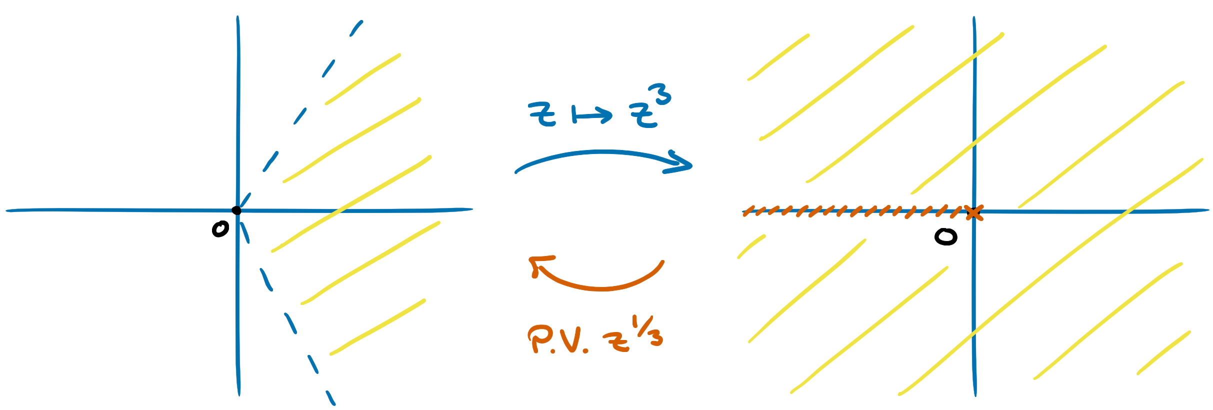
Exercise: Show that z^2 maps the right half plane $\{z \in \mathbb{C} : \operatorname{Re} z > 0\}$ biholomorphically onto the cut plane $\mathbb{C} \setminus (-\infty, 0]$.

(The inverse of this biholomorphism is denoted $z \mapsto \text{P.V. } z^{1/2}$, where "P.V." stands for "principal value"; $f(z) = \text{P.V. } z^{1/2}$ is called the principal branch of the square root function.)



Exercise: Show that z^3 maps the set $\{z \in \mathbb{C}\setminus\{0\} : -\frac{\pi}{3} < \operatorname{Arg} z < \frac{\pi}{3}\}$ biholomorphically onto the cut plane $\mathbb{C} \setminus (-\infty, 0]$.

(The inverse is denoted $\text{P.V. } z^{1/3}$.)



Exercise: Show that z^3 maps the set $\{z \in \mathbb{C} : \operatorname{Re} z > 0 \text{ and } \operatorname{Im} z > 0\}$ biholomorphically onto the set $\{z \in \mathbb{C} : \operatorname{Im} z > 0 \text{ or } \operatorname{Re} z < 0\}$.

* Conformality

Def^n: A bijection $f: U \to V$ between open sets $U, V \subseteq \mathbb{R}^2$ is said to be conformal if it is (real) differentiable and for any $p \in U$, $Df(p)$ preserves angles and orientation.

> Preserving angles means that the angle between $Df(p)v_1$ and $Df(p)v_2$ is the same as the angle between v_1 and v_2 for any vectors $v_1, v_2 \in \mathbb{R}^2$.

Proposition

A 2×2 real matrix M preserves angles and orientation if and only if it corresponds to multiplication by a nonzero real number, i.e. if and only if it is of the form $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$.

Proof:

The reverse implication is easy since multiplication by a nonzero complex number, say $re^{i\theta}$, amounts to a rotation (by θ) and scaling (by $r>0$) which preserves angles.

For the forward implication, suppose M preserves angles and orientation. Then M preserves orthogonality and since

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 0 \quad \text{we must have} \quad M\begin{pmatrix} 1 \\ 0 \end{pmatrix} \cdot M\begin{pmatrix} 0 \\ 1 \end{pmatrix} = 0.$$

This forces M to be of the form $\begin{pmatrix} a & -rb \\ b & ra \end{pmatrix}$ where $\begin{pmatrix} a \\ b \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. and $r > 0$ (since M preserves orientation, i.e. $\det M > 0$).

There are now many ways to see that r must be 1, e.g., since $\begin{pmatrix} 1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -1 \end{pmatrix} = 0$ we must have

$$M\begin{pmatrix} 1 \\ 1 \end{pmatrix} \cdot M\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} a-rb \\ b+ra \end{pmatrix} \cdot \begin{pmatrix} a+rb \\ b-rb \end{pmatrix} = a^2 - r^2b^2 + b^2 - r^2a^2 = (1-r^2)(a^2+b^2) = 0$$

which implies $r=1$ since $\begin{pmatrix} a \\ b \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ and $r>0$. $\square$

Note: The above proof also shows us that a 2×2 real matrix M preserves angles and orientation if and only if it preserves orthogonality and orientation.

The group of 2×2 real matrices that preserve angles and orientation is called the conformal special orthogonal group, denoted $CSO(2)$.

We have just shown that the matrix group $CSO(2)$ can be identified with $\mathbb{C}^*$, the group of nonzero complex numbers under multiplication.

$$re^{i\theta} \longleftrightarrow \begin{pmatrix} r\cos\theta & -r\sin\theta \\ r\sin\theta & r\cos\theta \end{pmatrix}$$

From the previous proposition we have:

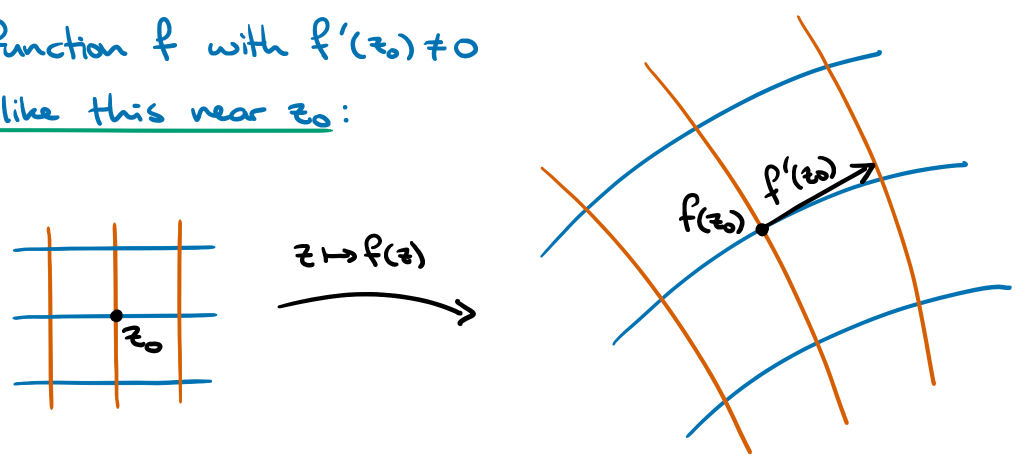
Proposition: Let $U \subseteq \mathbb{C}$ be open and $f : U \to \mathbb{C}$. The function f preserves angles and orientation if and only if f is holomorphic and f' is nowhere zero.

The image of f must be open (why?) so in other words:

f is conformal if and only if it is a biholomorphism onto its image.

* Intuition for Thinking About Holomorphic Functions as Maps

A holomorphic function f with $f'(z_0) \neq 0$ looks something like this near z_0:

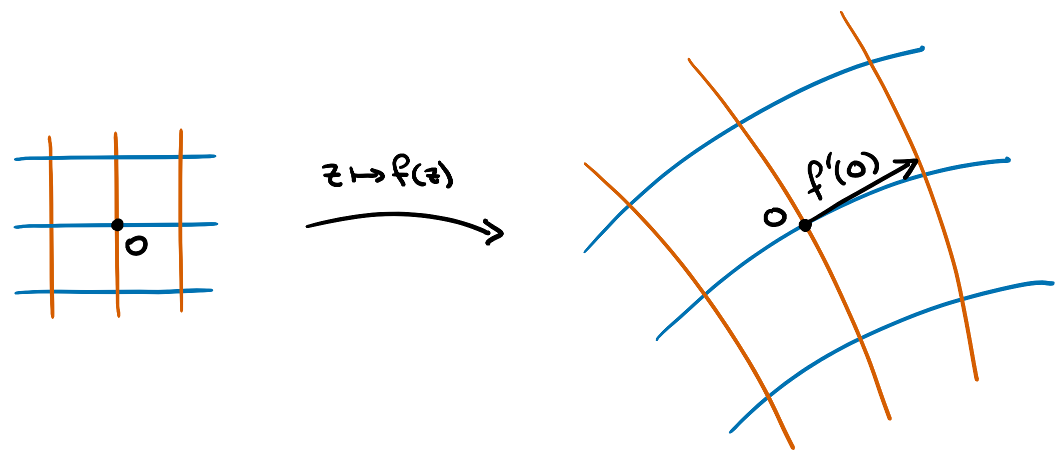


Q: What does a holomorphic map f look like near a point where $f'(z_0) = 0$?

We will answer this question later (after we have shown that a holomorphic function may be locally represented as a power series).

To give an idea of what to expect, however, let's suppose for simplicity that our point z_0 is 0 and $f(0) = 0$ (this can be accomplished by translating before and after applying f, so there is no real loss of generality).

- Note that in the case where $f'(0) \neq 0$, f is conformal near 0 and so looks like a curvy version of the map $z \mapsto \alpha z$ (where $\alpha = f'(0)$).



- In the case where $f'(0) = 0$ we will see that $f(z)$ looks like a "deformed" version of the map $z \mapsto \alpha z^k$ for some k (corresponding to the first nonzero term in the Taylor series for f at k).

Such a map is not conformal at 0, but instead multiplies angles by k. →

More precisely, the map will locally be of the form $z \mapsto g(z)^k$, where g is a holomorphic function with $g(0) = 0$ and $g'(0) \neq 0$ (so g is a local biholomorphic change of coordinates just like the map drawn above).