3 Complex Differentiation

Recall that a real function $f : (a,b) \to \mathbb{R}$ is differentiable at $x \in (a,b)$ if

$$(*) \qquad f(x+h) = f(x) + Lh + o(h)$$

where here we are using "little-oh notation":
The above equation means that

$$f(x+h) = f(x) + Lh + E(h)$$

for some function $E(h)$ (the "error term") satisfying

$$E(0) = 0 \quad \text{and} \quad \lim_{h \to 0} \frac{E(h)}{|h|} = 0.$$

Equation $(*)$ is just another way of saying that

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = L.$$

It is better, however, to think of differentiability in terms of the existence of a linear approximation $(*)$ than to think of it in terms of the difference quotient.

* Complex Differentiation

There are two kinds of differentiability for complex functions $f(z)$ corresponding to two ways in which a map $L : \mathbb{C} \to \mathbb{C}$ can be said to be linear ($\mathbb{R}$-linear or $\mathbb{C}$-linear).

Definition

Let $V \subseteq \mathbb{C}$ be an open set and $f: V \to \mathbb{C}$.

We say that f is real-differentiable ($\mathbb{R}$-diff$^{\text{ble}}$) at $z \in V$ if

$$f(z+h) = f(z) + Lh + o(|h|)$$

where $L: \mathbb{C} \to \mathbb{C}$ is an $\mathbb{R}$-linear map.

> Thinking of f as a map from $V \subseteq \mathbb{R}^2$ to $\mathbb{R}^2$, this is just the usual notion of differentiability from multivariable calculus and $L: \mathbb{R}^2 \to \mathbb{R}^2$ is just the linear map given by the matrix of partial derivatives.

The map L is called the Fréchet derivative (or real derivative) of f at z and denoted $Df(z)$ (or just Df).
If $f = u + iv$, then considered as a map from $\mathbb{R}^2$ to $\mathbb{R}^2$

$$Df = \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix}.$$

Replacing $\mathbb{R}$-linearity by the stronger condition of $\mathbb{C}$-linearity we have:

Definition

Let $V \subseteq \mathbb{C}$ be open and $f: V \to \mathbb{C}$. We say that f is complex-differentiable ($\mathbb{C}$-diff$^{\text{ble}}$, or just diff$^{\text{ble}}$) at z if

$$f(z+h) = f(z) + Lh + o(|h|)$$

where $L: \mathbb{C} \to \mathbb{C}$ is $\mathbb{C}$-linear (i.e. L is a complex number acting by multiplication).
In this case we write $f'(z) = L$.

Equivalently, f is complex differentiable at z if and only if the complex derivative

$$f'(z) = \lim_{h \to 0} \frac{f(z+h) - f(z)}{h}$$

exists.

In this case we simply refer to $f'(z)$ as the derivative of f at z. (Alternative notation: $f' = \frac{df}{dz}$.)

Note: f $\mathbb{C}$-diff^{ble} at z $\Rightarrow$ f $\mathbb{R}$-diff^{ble} at z
(but the converse does not hold!)

Proposition 0.0
Suppose $V \subseteq \mathbb{C}$ is open and $f: V \to \mathbb{C}$. Then f is $\mathbb{C}$-diff^{ble} at $z \in V$ if and only if f is $\mathbb{R}$-diff^{ble} at z and $Df(z)$ is $\mathbb{C}$-linear.

* The Cauchy-Riemann Equations:

If $V \subseteq \mathbb{C}$ and $f: V \to \mathbb{C}$ is $\mathbb{C}$-diff^{ble} at $z \in V$ then (writing $f = u + iv$)

$$Df = \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix}$$

must correspond to multiplication by a complex number (namely $f'(z) = a + ib$) and hence must be of the form

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix}.$$

Hence u and v must solve the Cauchy-Riemann equations:

$$u_x = v_y \quad \text{and} \quad u_y = -v_x$$

at z.

The condition that "$Df(z)$ is $\mathbb{C}$-linear" in Proposition 0.0 can be replaced by "$u_x = v_y$ and $u_y = -v_x$ at z" giving:

Proposition 0.1

Suppose $V \subseteq \mathbb{C}$ is open and $f : V \to \mathbb{C}$. Then f is $\mathbb{C}$-diff^{ble} at $z \in V$ if and only if f is $\mathbb{R}$-diff^{ble} and $u = \operatorname{Re} f$ and $v = \operatorname{Im} f$ satisfy

$$u_x = v_y, \quad u_y = -v_x \quad \text{at } z.$$

Remark

$$f \ \mathbb{C}\text{-diff}^{ble} \text{ at } z \implies u_x = v_y \ \& \ u_y = -v_x \text{ at } z$$

but

$$u_x = v_y \ \& \ u_y = -v_x \text{ at } z \not\Rightarrow f \ \mathbb{C}\text{-diff}^{ble} \text{ at } z.$$

Consider, e.g., the function

$$f(z) = \begin{cases} 0 & \text{if } x=0 \text{ or } y=0 \\ 1 & \text{otherwise} \end{cases}$$

at $z=0$. The Cauchy-Riemann equations hold, but the function is not even continuous.

On the other hand, if the first partial derivatives of $u = \operatorname{Re} f$ and $v = \operatorname{Im} f$ exist and are continuous at z, then f is $\mathbb{R}$-diff^{ble} at z (from "advanced calculus"). So:

Proposition (Corollary 0.3):

Suppose $V \subseteq \mathbb{C}$ is open and $f : V \to \mathbb{C}$. If the first order partial derivatives exist and are continuous at z and satisfy the Cauchy-Riemann equations there then f is $\mathbb{C}$-diff^{ble} at z.

Examples

① A polynomial $p(z) = a_n z^n + a_{n-1} z^{n-1} + \dots + a_1 z + a_0$ $(a_0, a_1, \dots, a_n \in \mathbb{C})$ is $\mathbb{C}$-diff^ble at any point $z_0 \in \mathbb{C}$.

(The proof is essentially the same as showing that a polynomial in one real variable $p(x)$ is differentiable at any point $x_0 \in \mathbb{R}$.)

② $f(z) = \bar{z}$ is $\mathbb{R}$-diff^ble at any point $z_0 \in \mathbb{C}$, but there is no point where it is $\mathbb{C}$-diff^ble.

↳ Df is complex antilinear rather than complex linear.
(A complex antilinear map $L : \mathbb{C} \to \mathbb{C}$ is an $\mathbb{R}$-linear map such that $L(ih) = -iL(h)$.)

↳ Later we will say that $f(z) = \bar{z}$ is <u>antiholomorphic</u>.

③ $f(z) = z\bar{z} = |z|^2 = x^2 + y^2 \quad (z = x + iy)$

$\rightsquigarrow$ $f = u + iv$ with $u = x^2 + y^2$, $v = 0$

$\rightsquigarrow$ $u_x = 2x$, $u_y = 2y$, $v_x = v_y = 0$.

The Cauchy-Riemann equations hold <u>only when</u>

$x = 0$ and $y = 0$.

$\rightsquigarrow$ f is $\mathbb{R}$-diff^ble everywhere, but is only $\mathbb{C}$-diff^ble at $z = 0$.

<u>Exercise</u>: Show that $f(x+iy) = x^2 + iy^2$ is $\mathbb{C}$-diff^ble only when $x = y$.

⋆ Differentiation Rules

Proposition

Let $V \subseteq \mathbb{C}$ be open, $f: V \to \mathbb{C}$ and $g: V \to \mathbb{C}$.
If f and g are $\mathbb{C}$-diff$^{\text{ble}}$ at $z \in V$, then

(i) $f+g$ is $\mathbb{C}$-diff$^{\text{ble}}$ at z and

Sum Rule
$$(f+g)'(z) = f'(z) + g'(z);$$

(ii) fg is $\mathbb{C}$-diff$^{\text{ble}}$ at z and

Product Rule
$$(fg)'(z) = f'(z)g(z) + f(z)g'(z);$$

(iii) f/g is $\mathbb{C}$-diff$^{\text{ble}}$ at z, provided $g(z) \neq 0$, and

Quotient Rule
$$\left(\frac{f}{g}\right)'(z) = \frac{f'(z)g(z) - g'(z)f(z)}{g(z)^2}.$$

The proof is essentially no different from the real variable case.

Proposition (Chain Rule):

Let $\mathcal{U}, V \subseteq \mathbb{C}$ be open sets, $f: V \to \mathbb{C}$ and $g: \mathcal{U} \to \mathbb{C}$.
Suppose $z \in V$ and $w = f(z) \in \mathcal{U}$. If f is $\mathbb{C}$-diff$^{\text{ble}}$ at z and g is $\mathbb{C}$-diff$^{\text{ble}}$ at w, then $g \circ f : V \to \mathbb{C}$ is $\mathbb{C}$-diff$^{\text{ble}}$ at z and

$$(g \circ f)'(z) = g'(f(z)) \cdot f'(z).$$

Again, this can be proved in the same way as in the real variable case. Alternatively, it can be seen as a special case of the Chain Rule for real-differentiable maps from (open subsets of) $\mathbb{R}^2$ to $\mathbb{R}^2$.

* Holomorphic Functions

Definition

Let $V \subseteq \mathbb{C}$ be an open set. A function $f: V \to \mathbb{C}$ is **holomorphic** if f is $\mathbb{C}$-diff'ble at every point.

Examples

① $f(z) = z^n$ is holomorphic on $\mathbb{C}$ $(n = 0, 1, 2, 3, \ldots)$.

$$f(z+h) = (z+h)^n = z^n + \underline{nz^{n-1}}h + \underbrace{\cdots + nzh^{n-1} + h^n}_{o(|h|)}$$

$$\rightsquigarrow \quad f'(z) = nz^{n-1}.$$

② a polynomial $p(z)$ is holomorphic on $\mathbb{C}$.

③ $e^z = e^x \cos y + i e^x \sin y$ $(z = x+iy)$ is holomorphic on $\mathbb{C}$.

$$u = e^x \cos y, \quad v = e^x \sin y$$

$$\rightsquigarrow \quad u_x = e^x \cos y = v_y, \qquad u_y = -e^x \sin y = -v_x$$

Exercise: Show that $\frac{d}{dz}(e^z) = e^z$.

↳ suggestion: use the relationship between Df and f' to show that if $f = u + iv$ is $\mathbb{C}$-diff'ble at z then $f'(z) = u_x + iv_x$ at z. Then apply this to $f(z) = e^z$.

④ $\sin z$ and $\cos z$ are holomorphic on $\mathbb{C}$.

⑤ $\frac{1}{z^n}$ is holomorphic on $\mathbb{C} \setminus \{0\}$ for $n = 1, 2, 3, \ldots$.

From the Quotient Rule: $$\left(\frac{1}{z^n}\right)' = \frac{-n}{z^{n+1}}.$$

★ <u>Wirtinger Derivatives</u>

Recall that if $V \subseteq \mathbb{R}^2$ is open and $u: V \to \mathbb{R}$ is a differentiable function, then the <u>differential</u> of u at $(x,y) \in V$ is

$$du = u_x\,dx + u_y\,dy.$$

If $f: V \to \mathbb{C}$ is an $\mathbb{R}$-diff^{ble} function, then we may similarly define the <u>differential</u> of $f = u + iv$ at $(x,y) \in V$:

$$df = f_x\,dx + f_y\,dy = (u_x + iv_x)dx + (u_y + iv_y)dy$$

i.e.

$$df = du + i\,dv.$$

Identifying $\mathbb{R}^2$ with $\mathbb{C}$ it makes sense to write the differential df in terms of $dz = dx + i\,dy$ and $d\bar{z} = dx - i\,dy$ instead of dx and dy. (This amounts to a change of basis in the complex vector space of differentials at a given point in $\mathbb{C}$.)

Rewriting df in terms of $dz = dx + i\,dy$ and $d\bar{z} = dx - i\,dy$ we have:

$$df = \tfrac{1}{2}(f_x - i f_y)\,dz + \tfrac{1}{2}(f_x + i f_y)\,d\bar{z}.$$

This motivates the following definition:

<u>Def^n</u>: We write

$$\frac{\partial}{\partial z} \stackrel{\text{def}}{=} \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right), \qquad \frac{\partial}{\partial \bar{z}} \stackrel{\text{def}}{=} \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right).$$

This allows us to write (for $\mathbb{R}$-diff^{ble} functions f):

$$df = \frac{\partial f}{\partial z}\,dz + \frac{\partial f}{\partial \bar{z}}\,d\bar{z}.$$

! Note: $\frac{\partial}{\partial z}$ and $\frac{\partial}{\partial \bar{z}}$ are not partial derivatives w.r.t. z and $\bar{z}$!

Example

$$d(z^2\bar{z}^5) = 2z\bar{z}^5\,dz + 5z^2\bar{z}^4\,d\bar{z}$$

[Why? $(z+h)^2(\overline{z+h})^5 = z^2\bar{z}^5 + 2z\bar{z}^5\cdot h + 5z^2\bar{z}^4\cdot\bar{h} + o(|h|)$]

So

$$\frac{\partial}{\partial z}(z^2\bar{z}^5) = 2z\bar{z}^5, \qquad \text{and}$$

$$\frac{\partial}{\partial \bar{z}}(z^2\bar{z}^5) = 5z^2\bar{z}^4.$$

Exercise: For nonegative integers n, m, compute

$$\frac{\partial}{\partial z}(z^n\bar{z}^m) \quad \text{and} \quad \frac{\partial}{\partial \bar{z}}(z^n\bar{z}^m).$$

Proposition

Let $V \subseteq \mathbb{C}$ be open and $f : V \to \mathbb{C}$. Then f is $\mathbb{C}$-diff$^{\text{ble}}$ at $z \in V$ if and only if f is $\mathbb{R}$-diff$^{\text{ble}}$ and

$$\frac{\partial f}{\partial \bar{z}} = 0 \quad \text{at } z.$$

Proof: Exercise (check that $\frac{\partial f}{\partial \bar{z}} = 0 \iff u_x = v_y,\ u_y = -v_x$). $\square$

It is also easy to check that if f is $\mathbb{C}$-diff$^{\text{ble}}$ at z then

$$f'(z) = \frac{\partial f}{\partial z}.$$

Appendix: When is a Function that Satisfies the Cauchy-Riemann Equations Holomorphic?

A function $f(z)$ that satisfies the Cauchy-Riemann equations at one point certainly need not be $\mathbb{C}$-diff$^{\text{ble}}$ at that point (as it need not be $\mathbb{R}$-diff$^{\text{ble}}$ or even continuous!), but what about a function that satisfies the Cauchy-Riemann equations in an open set?

Question: Let $V \subseteq \mathbb{C}$ be open, $f: V \to \mathbb{C}$. If f satisfies the Cauchy-Rieman equations on V then is f holomorphic on V?

If f is also continuous on V then the answer is "yes":

Theorem (Looman-Menchoff):
Let $V \subseteq \mathbb{C}$ open, $f: V \to \mathbb{C}$. If $f = u + iv$ is continuous on V and the first partial derivatives of u and v exist and satisfy $u_x = v_y$ and $u_y = -v_x$ on V, then f is holomorphic on V.

Proof: See J.D. Gray and S.A. Morris, When is a Function that Satisfies the Cauchy-Riemann Equations Analytic? *The American Mathematical Monthly*, Vol. 85, No. 4 (Apr. 1978), pp. 246-256. $\square$

Comments:

- This theorem allows us to replace the condition of real-differentiability by the condition of continuity (and the existence of first partial derivatives so that the Cauchy-Riemann equations make sense).
- The result is a prime example of a general phenomenon known as "elliptic regularity." The Cauchy-Riemann

equations form what we call an "elliptic" system of partial differential equations and these equations force their solutions to be differentiable (even C^∞, as we shall see later for the Cauchy-Riemann equations).

In fact, even the assumption of continuity and that u_x, u_y, v_x and v_y exist can be dropped if you are willing to work in the distributional sense:

Theorem

Let $V \subseteq \mathbb{C}$ be an open set and $f : V \to \mathbb{C}$. If f is locally integrable and, as a distribution, satisfies the Cauchy-Riemann equations, then f agrees almost everywhere with a function that is holomorphic in V.

Proof

The proof is sketched in the article of Gray and Morris referenced above. A full proof can be found in:

L. Zalcman, Real proofs of complex theorems (and vice versa). *The American Mathematical Monthly*, Vol. 81, No. 2 (Feb. 1974), pp. 115-137.

$\square$