2. The Riemann Sphere and Linear-Fractional Transformations

Math 5283, Spring 2021

Oklahoma State University

Roadmap

1 The Complex Plane

2 The Riemann Sphere & LFTs

---

3 Complex Differentiation

4 Conformal Mapping

5 Power Series

6 Analytic Functions

---

7 Contour Integrals

. . .

The Riemann Sphere

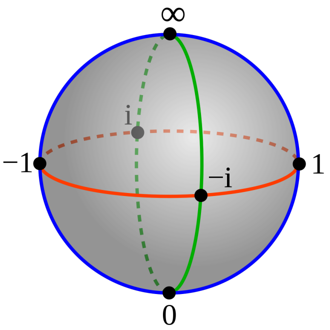
As a set the *Riemann sphere* is

$$\mathbb{C}_\infty = \mathbb{C} \cup \{\infty\},$$

but we want to think of $\{\infty\}$ as being on the same footing as the other points.

Define $g : \mathbb{C}_\infty \to \mathbb{C}_\infty$ by

$$g(z) = \begin{cases} \frac{1}{z} & \text{if } z \neq 0 \text{ and } z \neq \infty \\ \infty & \text{if } z = 0, \\ 0 & \text{if } z = \infty. \end{cases}$$



wikipedia.org/wiki/Riemann_sphere

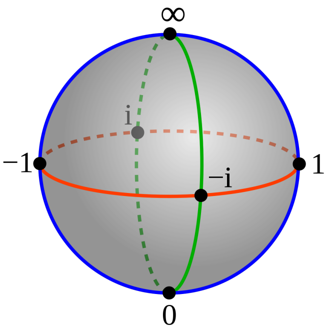
The Riemann Sphere (Topology)

Using the bijetion $g : \mathbb{C}_\infty \to \mathbb{C}_\infty$ we can define a *neighborhood* of ∞ to be the image of a neighborhood of 0.

But we really want to think of the Riemann sphere has having local complex coordinates:

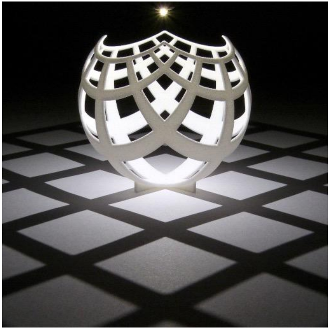
z works away from ∞;

$w = g(z)$ works away from 0.



wikipedia.org/wiki/Riemann_sphere

Sterographic Projection



https://twitter.com/henryseg

This gives $\mathbb{C}_\infty$ a metric space structure.

Limits Involving ∞

Definition

If z_0 is a limit point of $U \subseteq \mathbb{C}_\infty$ and $f : U \to \mathbb{C}_\infty$, then we define

$$\lim_{z \to z_0} f(z)$$

using the topology/metric space structure of $\mathbb{C}_\infty$.

In Practical Terms

$$\lim_{z \to z_0} f(z) = \infty \quad \Leftrightarrow \quad \lim_{z \to z_0} \frac{1}{f(z)} = 0;$$

$$\lim_{z \to \infty} f(z) = L \quad \Leftrightarrow \quad \lim_{z \to 0} f\left(\frac{1}{z}\right) = L;$$

$$\lim_{z \to \infty} f(z) = \infty \quad \Leftrightarrow \quad \lim_{z \to 0} \frac{1}{f\left(\frac{1}{z}\right)} = 0.$$

Limits Involving ∞ (Examples)

$\mathbb{C}_\infty$ as a Riemann Surface

The Riemann Sphere as $\mathbb{CP}^1$

Linear-Fractional Transformations

Linear-Fractional Transformations

Definition

A *Linear-Fractional Transformation* is a map $f : \mathbb{C}_\infty \to \mathbb{C}_\infty$ given by

$$f(z) = \frac{az + b}{cz + d}$$

when $z \in \mathbb{C}$, where $a, b, c, d \in \mathbb{C}$ are constants with $ad - bc \neq 0$.

See Complex Variables Notes:

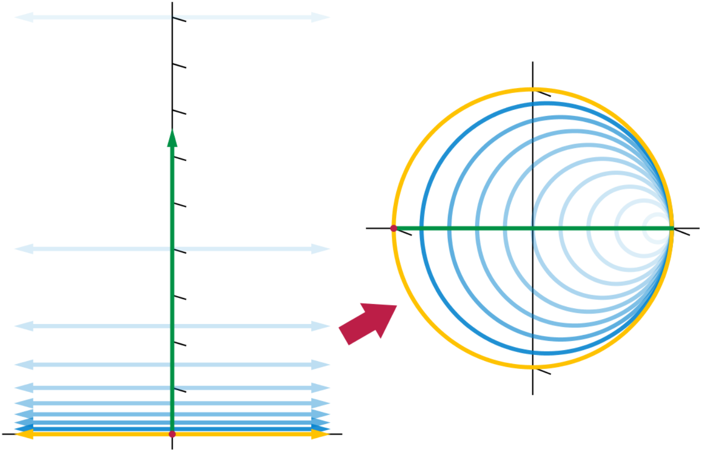
30 Linear Transformations and Inversions (Link)
31 Linear-Fractional Transformations (Link)

The Cayley Map

The LFT $f(z) = \dfrac{z - i}{z + i}$ maps the upper half plane $\mathbb{H}$ to the unit disc $\mathbb{D}$ and is known as the *Cayley map*.

Why?

The Cayley Map...



wikipedia.org/wiki/Cayley_transform