Complex Analysis I

1. The Complex Plane

Math 5283, Spring 2021

Oklahoma State University

Roadmap

Complex Numbers

The *complex number field* $\mathbb{C}$ is defined to $\mathbb{R}^2$ together with the addition and multiplication given by

$$(a,b)+(c,d) \stackrel{def}{=} (a+b, c+d)$$
$$(a,b)(c,d) \stackrel{def}{=} (ac-bd, bc+ad).$$

We write

$$x=(x,0), \quad x\in\mathbb{R}$$

and

$$i \stackrel{def}{=} (0,1).$$

Defintions/Notation

The *complex conjugate* of $z = x + iy$ is

$$\bar{z} = x - iy.$$

Also, if $z = x + iy$ then

$$\operatorname{Re} z = x = \frac{z + \bar{z}}{2} \qquad \operatorname{Im} z = y = \frac{z - \bar{z}}{2i}.$$

Useful Application ("$f(x, y) = f(z, \bar{z})$"):

$$x^2 - ixy = \left(\frac{z + \bar{z}}{2}\right)^2 - i\left(\frac{z + \bar{z}}{2}\right)\left(\frac{z - \bar{z}}{2i}\right)$$

Modulus and Distance

The *modulus* of $z = x + iy$ ia

$$|z| = \sqrt{z\overline{z}} = \sqrt{x^2 + y^2}.$$

Proposition

(i) $|\mathrm{Re}\, z\overline{w}| \leq |z||w|$ (Cauchy-Schwarz)

(ii) $|z + w| \leq |z| + |w|$ (Triangle inequalty)

Distance: $\mathbb{C}$ is a complete metric space, with

$$d(z, w) = |z - w|.$$

Limit Laws

Proposition

Complex addition, multiplication, division and conjugation are continuous: If $\{a_n\}$ and $\{b_n\}$ are convergent sequences of complex numbers, then

(i) $$\lim_{n\to\infty}(a_n+b_n)=\left(\lim_{n\to\infty}a_n\right)+\left(\lim_{n\to\infty}b_n\right),$$

(ii) $$\lim_{n\to\infty}a_nb_n=\left(\lim_{n\to\infty}a_n\right)\left(\lim_{n\to\infty}b_n\right),$$

(iii) $$\lim_{n\to\infty}\frac{1}{a_n}=\frac{1}{\lim_{n\to\infty}a_n},\text{ provided }\lim_{n\to\infty}a_n\neq 0,$$

(iv) $$\lim_{n\to\infty}\overline{a_n}=\overline{\lim_{n\to\infty}a_n}.$$

Common Sets

The disc of radius $r > 0$ centered at $p \in \mathbb{C}$ is denoted

$$\Delta_r(p) = \{\, z \in \mathbb{C} \,:\, |z - p| < r \,\}.$$

Frequently we will be concerned with the *unit disc*

$$\mathbb{D} = \Delta_1(0) = \{\, z \in \mathbb{C} \,:\, |z| < 1 \,\}.$$

The unit disc will turn out to be "equivalent" to the *upper half plane*

$$\mathbb{H} = \{\, z \in \mathbb{C} \,:\, \operatorname{Im} z > 0 \,\}.$$

Functions $f : [a, b] \to \mathbb{C}$

We set $u = \operatorname{Re} f$ and $v = \operatorname{Im} f$ and write

$$f = u + iv$$

If u and v are differentiable, then we write

$$f' = u' + iv'.$$

If u and v are Riemann integrable, then we say that f is *Riemann integrable* and write

$$\int_a^b f(t)\, dt = \int_a^b u(t)\, dt + i \int_a^b v(t)\, dt.$$

Functions $f : [a, b] \to \mathbb{C}$

From the triangle inequality applied to approximating Riemann sums we have:

Proposition

If $f : [a, b] \to \mathbb{C}$ is Riemann integrable, then $|f|$ is Riemann integrable and

$$\left| \int_a^b f(t)\, dt \right| \leq \int_a^b |f(t)|\, dt.$$

Functions $f : X \to \mathbb{C}$ ($X \subseteq \mathbb{C}$ Open)

We write

$$f(z) = u(x, y) + iv(x, y).$$

If $u, v : \mathbb{R}^2 \to \mathbb{R}$ have partial derivatives then we write

$$\frac{\partial f}{\partial x} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \quad \text{and} \quad \frac{\partial f}{\partial y} = \frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}$$

or simply $f_x = u_x + iv_x$ and $f_y = u_y + iv_y$.

Complex Numbers as Maps $\mathbb{R}^2 \to \mathbb{R}^2$

If $w = a + ib \in \mathbb{C}$, then the map $z \mapsto wz$ $(z \in \mathbb{C})$ can be thought of as a linear map $\mathbb{R}^2 \to \mathbb{R}^2$ and is therefore represented by a real 2×2 matrix M_w.

Easy Exercise

$$M_w = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}.$$

Morever, $M_{w_1} + M_{w_2} = M_{w_1 + w_2}$ and $M_{w_1} M_{w_2} = M_{w_1 w_2}$.

Note:

$$M_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad M_i = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

The Exponential

Euler's Formula

Note that $i^2 = -1$, $i^3 = -i$, $i^4 = 1$, $i^5 = i$, $i^6 = -1$ ...

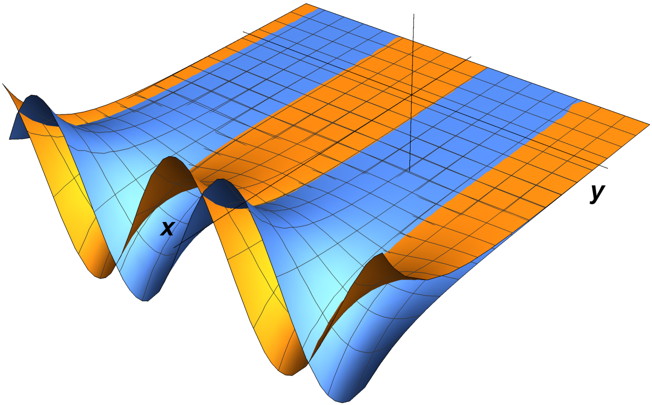
Following Euler, we expand

$$\begin{aligned} e^{ix} &= 1 + ix + \frac{(ix)^2}{2} + \frac{(ix)^3}{3!} + \frac{(ix)^4}{4!} + \frac{(ix)^5}{5!} + \cdots \\ &= \left(1 - \frac{x^2}{2} + \frac{x^4}{4!} - \cdots\right) + i\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right) \\ &= \cos x + i \sin x, \qquad x \in \mathbb{R}. \end{aligned}$$

$$\rightsquigarrow e^z = e^{x+iy} \stackrel{def}{=} e^x e^{iy} = e^x \cos y + i e^x \sin y$$

The Real and Imaginary Parts of e^z

Plot of $e^x \cos y$ (orange) and $e^x \sin y$ (blue).



Properties of e^z

Modulus: $|e^{iy}| = 1$, so $|e^z| = |e^x e^{iy}| = e^x$.

Easy Exercise: Use $e^{tz} = e^{tx} \cos ty + ie^{tx} \sin ty$ to show that

$$\frac{d}{dt}\left(e^{tz}\right) = ze^{tz}.$$

Consequence (ODE uniqueness/trig. identities):
For any $z, w \in \mathbb{C}$,

$$e^z e^w = e^{z+w}.$$

Complex Sine and Cosine

Since $e^{i\theta} = \cos\theta + i\sin\theta$ and $e^{-i\theta} = \cos\theta - i\sin\theta$,

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} \quad \text{and} \quad \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}.$$

Definition

$$\cos z \stackrel{def}{=} \frac{e^{iz} + e^{-iz}}{2}, \qquad \sin z \stackrel{def}{=} \frac{e^{iz} - e^{-iz}}{2i}.$$

Note:

$$\rightsquigarrow \cos z = \cosh iz, \qquad i\sin z = \sinh iz$$

Polar Coordinates ($z \neq 0$)

In polar coordinates $(x, y) = (r\cos\theta, r\sin\theta)$.
Hence

$$z = x + iy = r\cos\theta + ir\sin\theta = re^{i\theta},$$

where $r = |z| = \sqrt{x^2 + y^2}$, and θ (the *argument*) is the angle that $z = x + iy$ makes with the positive real axis.

Multiplication:

$$z_1 z_2 = \left(r_1 e^{i\theta_1}\right)\left(r_2 e^{i\theta_2}\right) = r_1 r_2 e^{i(\theta_1+\theta_2)}.$$

The Argument

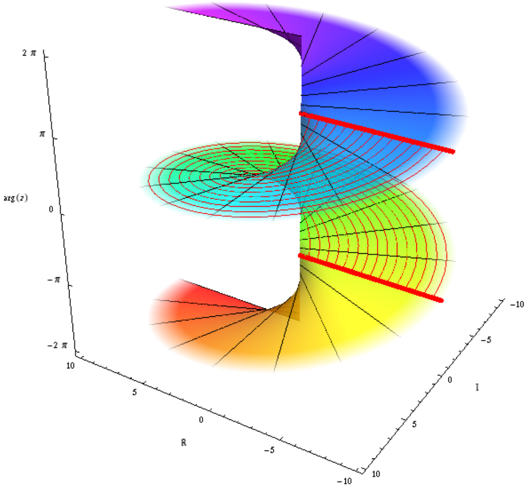
Rotating around the origin by an angle of 2π gets us back where we started. So for $z \neq 0$, $z = re^{i\theta}$, we define the multivalued function $\arg z$ by

$$\arg z \stackrel{def}{=} \ldots, \theta - 4\pi, \theta - 2\pi, \theta, \theta + 2\pi, \theta + 4\pi, \ldots$$

Note that $\arg z$ is not a function in the ordinary sense. One way to remedy this is to define the *principal branch* of $\arg$ to be

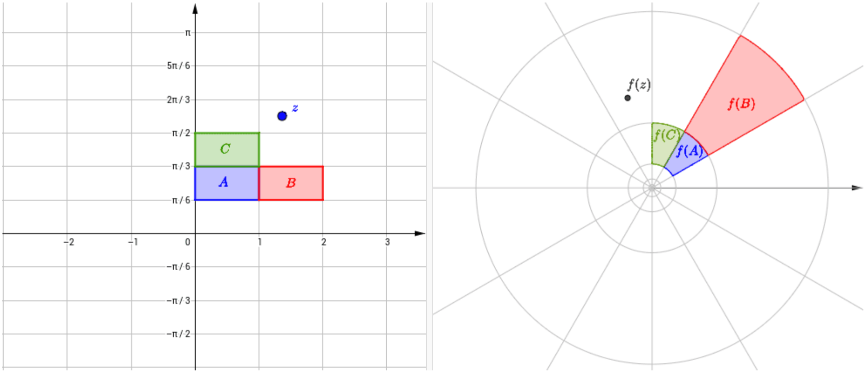
$$\operatorname{Arg} z \stackrel{def}{=} \theta, \text{ where } z = re^{i\theta} \text{ with } -\pi < \theta \leq \pi.$$

arg vs Arg



https://commons.wikimedia.org/wiki/File:Principle_branch_of_arg_on_Riemann.png

The Exponential as a Map $\mathbb{C} \to \mathbb{C}$



https://www.geogebra.org/m/Ac43DNNP