31. Linear Fractional Transformations

Linear Fractional Transformations (LFTs):

$$w = \frac{az+b}{cz+d}, \qquad ad-bc \neq 0.$$

LFTs are also called Möbius transformations.

Remarks:

- a, b, c, d are complex numbers.
- $\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad-bc \neq 0.$
- $ad-bc \neq 0$ implies c and d cannot both be zero.

Alternative form:

$$\left.\begin{aligned} c\,zw &- az + dw - b = 0 \\ A\,zw &+ Bz + Cw + D = 0 \end{aligned}\right\} \text{write as}$$

where $\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} c & -a \\ d & -b \end{pmatrix}$, $\underbrace{\det \begin{pmatrix} A & B \\ C & D \end{pmatrix} \neq 0}_{AD-BC = -bc+ad \neq 0}.$

$\leadsto$ LFTs are also called bilinear transformations.

If $c=0$ then $ad \neq 0$ (so $a \neq 0, d \neq 0$)

$\leadsto$ $w = \frac{a}{d} z + \frac{b}{d}$ linear transformation.

Also, non-constant ($\frac{a}{d} \neq 0$).

If $c \neq 0$

$$w = \frac{\frac{a}{c}z + \frac{b}{c}}{z + \frac{d}{c}} = \frac{\frac{a}{c}(z+\frac{d}{c}) - \frac{ad}{c^2} + \frac{b}{c}}{z + \frac{d}{c}}$$

$$\rightsquigarrow \quad w = \frac{a}{c} + \frac{bc-ad}{c} \cdot \frac{1}{cz+d}. \qquad (*)$$

Since $bc-ad \neq 0$, the map is not constant.

From $(*)$ we see that an LFT is a composition of a linear transformation, an inversion $(Z \mapsto \frac{1}{Z})$ and then another linear transformation.

Explicitly:

$$z \mapsto Z = cz+d$$

$$Z \longmapsto W = \frac{1}{Z}$$

$$W \longmapsto \frac{a}{c} + \frac{bc-ad}{c}W.$$

It follows that LFTs map circles & lines in $\mathbb{C}$ to circles & lines in $\mathbb{C}$.

Extending to $\hat{\mathbb{C}}$:

If $T(z) = \dfrac{az+b}{cz+d}$ we can extend T continuously to $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, so

$$T : \hat{\mathbb{C}} \to \hat{\mathbb{C}} \qquad \text{one-to-one \& onto.}$$

If $c = 0$, just set $T(\infty) = \infty$.

If $c \neq 0$, $T(z) = \dfrac{a + \frac{b}{z}}{c + \frac{d}{z}} \to \dfrac{a}{c}$ as $z \to \infty$, so we let $T(\infty) = \frac{a}{c}$.

Also (with $c \neq 0$) $T(z) = \frac{az+b}{cz+d} \to \infty$ as $z \to -\frac{d}{c}$,

since $cz+d=0$ at $z = -\frac{d}{c}$, so we set $T(-\frac{d}{c}) = \infty$.

Inverting an LFT:

If $w = T(z) = \frac{az+b}{cz+d}$ $(ad-bc \neq 0)$

then $wcz + wd = az + b$, so

$$z = \frac{-dw+b}{cw-a}, \qquad (-d)(-a) - bc = ad - bc \neq 0.$$

$\rightsquigarrow$ $z = T^{-1}(w)$ is also a linear fractional transformation.

Composing LFTs:

If S and T are LFTs, with

$$S(z) = \frac{az+b}{cz+d}, \qquad T(z) = \frac{a'z+b'}{c'z+d'},$$

then $(S \circ T)(z) = S(T(z))$ is an LFT of the form $\frac{a''z+b''}{c''z+d''}$ where

$$\begin{pmatrix} a'' & b'' \\ c'' & d'' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}.$$

Computing the composition:

$$S(T(z)) = \frac{a \cdot \frac{a'z+b'}{c'z+d'} + b}{c \cdot \frac{a'z+b'}{c'z+d'} + d}$$

$$= \frac{a(a'z+b') + b(c'z+d')}{c(a'z+b') + d(c'z+d')}$$

$$= \frac{(aa'+bc')z + ab'+bd'}{(ca'+dc')z + cb'+dd'}.$$

Compare with:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} = \begin{pmatrix} aa'+bc' & ab'+bd' \\ ca'+dc' & cb'+dd' \end{pmatrix}.$$

Note that

$$\det\left[\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}\right] = \det\begin{pmatrix} a & b \\ c & d \end{pmatrix} \times \det\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}$$

$$\neq 0.$$

Remarks:

(i) The collection of all LFTs form a group (under composition) called the Möbius group.

(ii) Note that $S(z) = T(z) \ \forall z \in \hat{\mathbb{C}}$ if and only if $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \lambda \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}$ for some $\lambda \in \mathbb{C} - \{0\}$.

(iii) These observations show that the Möbius group is isomorphic to $PGL(2, \mathbb{C})$, i.e. $GL(2,\mathbb{C}) / \{\lambda I : \lambda \in \mathbb{C} - \{0\}\}$.

(iv) The projective general linear group $PGL(2,\mathbb{C})$ is isomorphic to the projective special linear group $PSL(2,\mathbb{C}) = SL(2,\mathbb{C})/\{\pm I\}$ and one usually writes $PSL(2,\mathbb{C})$ for the Möbius group.

$\rightsquigarrow$ Normalize $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ up to sign by requiring $\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = 1$.

<u>Theorem</u> (T3): Given 3 distinct points z_1, z_2, z_3 in $\hat{\mathbb{C}}$ and 3 distinct points w_1, w_2, w_3 in $\hat{\mathbb{C}}$, <u>there is a unique</u> LFT $T: \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ such that $T(z_1) = w_1$, $T(z_2) = w_2$, $T(z_3) = w_3$.

<u>Examples</u>:

① $$\left.\begin{array}{lll} z_1 = -1, & z_2 = 0, & z_3 = 1 \\ w_1 = -i, & w_2 = 1, & w_3 = i \end{array}\right\} \quad \text{Find } T.$$

$$w = T(z) = \frac{az+b}{cz+d}, \qquad ad-bc \neq 0.$$

<u>$T(z_2)=w_2$</u>: $T(0) = 1 \rightsquigarrow \frac{b}{d} = 1 \rightsquigarrow b = d.$

Note that this gives $b(a-c) \neq 0$.

<u>$T(z_1)=w_1$</u>: $T(-1) = -i \rightsquigarrow \frac{-a+b}{-c+b} = -i \rightsquigarrow ic-ib = -a+b$

<u>$T(z_3)=w_3$</u>: $T(1) = i \rightsquigarrow \frac{a+b}{c+b} = i \rightsquigarrow ic+ib = a+b$

$\rightsquigarrow \quad 2ic = 2b \quad \& \quad 2ib = 2a.$

$\quad (c = -ib) \qquad (a = ib)$

$$\rightsquigarrow \quad w = \frac{ibz+b}{-ibz+b} = \frac{b}{b} \cdot \frac{iz+1}{-iz+1},$$

but $b(a-c) \neq 0 \Rightarrow b \neq 0$, so

$$T(z) = \frac{iz+1}{-iz+1}.$$

(2) $\quad z_1 = 2, \quad z_2 = i, \quad z_3 = 2i$

$\quad w_1 = 0, \quad w_2 = \infty, \quad w_3 = 2+2i$

$w = \dfrac{az+b}{cz+d}, \qquad z = i: \quad ci + d = 0 \rightsquigarrow d = -ci.$

$\qquad z = 2: \quad 2a + b = 0 \rightsquigarrow b = -2a.$

$\rightsquigarrow \quad w = \dfrac{a(z-2)}{c(z-i)}, \qquad z = 2i: \quad \dfrac{a}{c} \cdot \underbrace{\dfrac{2i-2}{i}}_{= 2+2i} = 2+2i$

$\rightsquigarrow \quad \dfrac{a}{c} = 1.$

$$\text{So} \quad T(z) = \frac{z-2}{z-i}.$$

<u>Def^n</u>: By a fixed point of a transformation $T: \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ we mean a point z_0 such that $T(z_0) = z_0$.

<u>Lemma</u>: A linear fractional transformation has <u>at most 2 fixed points</u>, unless it is the identity.

Proof of Lemma: $T(z) = \dfrac{az+b}{cz+d}$, $ad-bc \neq 0$.

$T : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$.

Case 1 $(c=0)$: $T(z) = \frac{a}{d} z + \frac{b}{d}$, $ad \neq 0$,

$T(\infty) = \infty \leftarrow$ fixed point.

If $z \in \mathbb{C}$ is a fixed point of T, $az+b = dz$,

i.e. $(a-d)z = -b$.

This has at most one solution, unless $a=d$ & $b=0$, i.e. $T(z) = z$, the identity.

Case 2 $(c \neq 0)$: $T(-\frac{d}{c}) = \infty$, so ∞ is not a fixed point.

For $z \in \mathbb{C}$,

$$T(z) = \frac{az+b}{cz+d} = z \iff cz^2 + (d-a)z - b = 0$$

which has at most 2 solutions. $\square$

Proof of Theorem (T3):

Uniqueness: Let S & T be LFTs and suppose $S(z_j) = T(z_j) = w_j$, $j = 1,2,3$.

Then $S^{-1} \circ T$ is a LFT and

$$(S^{-1} \circ T)(z_j) = S^{-1}(w_j) = z_j, \quad j=1,2,3.$$

By the above Lemma $(\underbrace{S^{-1} \circ T}_{=\text{ the identity}})(z) = z \quad \forall z \in \hat{\mathbb{C}}$

and hence $S = T$.

<u>Existence:</u> Given z_j and w_j $(j=1,2,3)$ as in Theorem (T3) we may define a LFT implicitly.

<u>Case 1</u> $(z_j \in \mathbb{C},\ w_j \in \mathbb{C},\ j=1,2,3)$:

Observe that by clearing denominators, the equation

$$(*) \qquad \frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

can be written in the form

$$(**) \qquad Azw + Bz + Cw + D = 0$$

where A, B, C, D depend on z_j & w_j $(j=1,2,3)$. It is easy to check that $(*)$ has unique solution $w = w_j$ when we plug in $z = z_j$ $(j=1,2,3)$. It follows that $(**)$ does not define a constant function (hence $AD - BC \neq 0$) and hence $(*)$ or $(**)$ define a LFT $T(z)$ such that $T(z_j) = w_j$, $j = 1,2,3$.

<u>Case 2</u> $(z_j \in \hat{\mathbb{C}},\ w_j \in \hat{\mathbb{C}},\ j=1,2,3)$:

$\rightsquigarrow$ What happens if one of the z_j's, or one of the w_j's (or one of both) is ∞?

We simply proceed as above, but if say $z_2 = \infty$ we treat z_2 as if it were finite (for a start) but then let $z_2 \to \infty$.

Sending $z_2 \to \infty$ in the RHS of $(*)$ above we get

check!

$$\lim_{z_2 \to \infty} \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)} = \frac{z-z_1}{z-z_3}$$

so we replace $(*)$ by

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{z-z_1}{z-z_3}.$$

We can then argue just as we did in case 1 to get the LFT.

In general, if one of the z_j's or one of the w_j's is ∞ we just delete the factors involving that z_j or w_j from $(*)$. For example, if $z_1 = \infty$ and $w_2 = \infty$ then we replace $(*)$ by

$$\frac{w-w_1}{w-w_3} = \frac{z_2-z_3}{z-z_3}. \qquad \square$$

<u>Example:</u> $z_1 = 1, \quad z_2 = 0, \quad z_3 = -1$

$w_1 = i, \quad w_2 = \infty, \quad w_3 = 1.$

$$\rightsquigarrow \quad \frac{w-w_1}{w-w_3} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

$$\rightsquigarrow \quad \frac{w-i}{w-1} = \frac{(z-1)(0+1)}{(z+1)(0-1)} = \frac{1-z}{z+1}$$

$$\rightsquigarrow \quad (w-i)(z+1) = (1-z)(w-1)$$

and solving for w we get

$$w = T(z) = \frac{(i+1)z + (i-1)}{2z}.$$

<u>Mapping the line $y=0$ onto the unit circle $|w|=1$ by a LFT</u>:

Note that the line $y=0$ is really a "circle" going through ∞ in $\hat{\mathbb{C}}$. If a LFT maps $y=0$ to the circle $|w|=1$, then it will also map ∞ to the circle $|w|=1$. This can be seen by continuity, or just by the fact that LFTs map "circles" in $\hat{\mathbb{C}}$ to "circles" in $\hat{\mathbb{C}}$.

We look for the general LFT sending $\{y=0\}\cup\{\infty\}$ onto $\{|w|=1\}$.

$$w = \frac{az+b}{cz+d}, \qquad z=0: \quad \left|\frac{b}{d}\right| = 1, \quad |b| = |d|.$$

$$z=\infty: \quad \left|\frac{a}{c}\right| = 1, \quad |a| = |c|.$$

$$\rightsquigarrow \quad w = \frac{a}{c}\cdot\frac{z+\frac{b}{a}}{z+\frac{d}{c}}, \qquad \frac{a}{c} = e^{i\alpha}, \quad \alpha\in\mathbb{R}$$

$$\& \quad \left|\frac{b}{a}\right| = \left|\frac{d}{c}\right|.$$

So $\quad w = e^{i\alpha}\,\dfrac{z-z_0}{z-z_1}$ where $|z_1| = |z_0|$.

$z=1: \quad \left|\dfrac{1-z_0}{1-z_1}\right| = 1, \quad |1-z_0| = |1-z_1|.$

$$\rightsquigarrow \quad |1-z_0|^2 = |1-z_1|^2$$

$$(1-z_0)(1-\overline{z_0}) = (1-z_1)(1-\overline{z_1})$$

$$1 + \underbrace{z_0\overline{z_0}}_{=|z_0|^2} - (z_0+\overline{z_0}) = 1 + \underbrace{z_1\overline{z_1}}_{=|z_1|^2} - (z_1+\overline{z_1})$$

Since $|z_0| = |z_1|$ we get $z_0 + \overline{z_0} = z_1 + \overline{z_1}$,

i.e. $\operatorname{Re} z_0 = \operatorname{Re} z_1$.

Writing $z_0 = x_0 + iy_0$, $z_1 = x_0 + iy_1$, from

$|z_0|^2 = x_0^2 + y_0^2 = |z_1|^2 = x_0^2 + y_1^2$ we get

$y_0^2 = y_1^2$, i.e. $y_1 = \pm y_0$.

If $y_1 = y_0$ then $z_1 = z_0$ & $w = e^{i\alpha}$,

a constant (not LFT!), so $y_1 \neq y_0$.

$\rightsquigarrow \quad y_1 = -y_0$, i.e. $z_1 = \overline{z_0}$.

Also $z_0 \neq \overline{z_0}$ (else w would be constant).

So T must be of the form

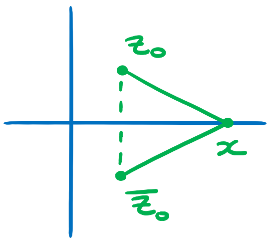
$$(*) \qquad w = T(z) = e^{i\alpha}\,\frac{z-z_0}{z-\overline{z_0}}$$

with $\alpha \in \mathbb{R}$ and $\operatorname{Im} z_0 \neq 0$.

Conversely, if $T(z)$ is of the form $(*)$

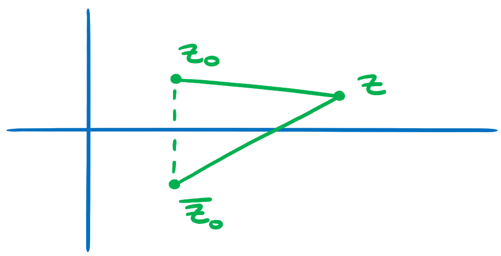
then $|w| = \left|\dfrac{z-z_0}{z-\overline{z_0}}\right|$ and one indeed has

$|w| = 1$ for $z = x$ real $(y = 0)$.



$\leftarrow$ To see this, observe that for $z = x$ real, the distance from $z = x$ to z_0 is the same as the distance from $z = x$ to $\bar{z}_0$. So $|w| = \frac{|z - z_0|}{|z - \bar{z}_0|} = 1$.

Now, if $\operatorname{Im} z_0 > 0$ then $|z - z_0| < |z - \bar{z}_0|$ for $\operatorname{Im} z > 0$.



We therefore obtain the following theorem:

<u>Theorem:</u> All LFTs mapping the upper half plane $\operatorname{Im} z > 0$ onto the unit disk $|w| < 1$ are of the form

$$T(z) = e^{i\alpha} \frac{z - z_0}{z - \bar{z}_0}$$

with $\alpha \in \mathbb{R}$ and $\operatorname{Im} z_0 > 0$. The boundary $\operatorname{Im} z = 0$ is mapped onto the circle (∞ is mapped to $e^{i\alpha}$). The point z_0 is mapped to 0 and $\bar{z}_0$ is mapped to ∞.

<u>Example:</u> $T(z) = \frac{z - i}{z + i}$ sends the upper half plane onto the unit disk.

<u>Exercise:</u> Find all LFTs that send the unit disk onto itself. Show that they are of the form

$$T(z) = e^{i\beta} \frac{z - \zeta}{1 - \bar{\zeta} z}$$

"zeta"

with $\beta \in \mathbb{R}$, $\zeta \in \mathbb{C}$, $|\zeta| < 1$.

<u>Suggestion:</u> Use the theorem above, and the map $T(z)$ in the previous example. Find and use the inverse of $T(z)$.

<u>Example:</u> Show that the map

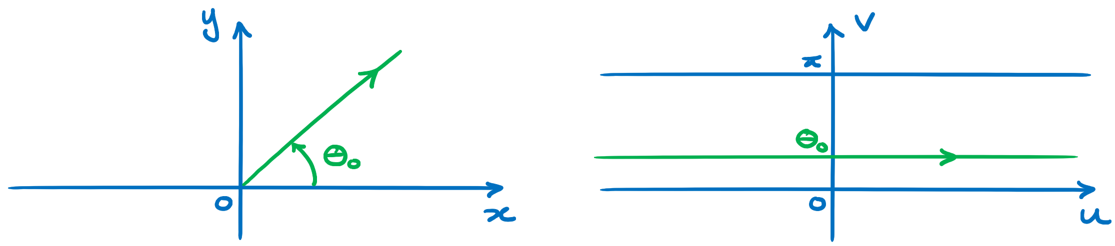
$$w = \operatorname{Log}\left(\frac{z-1}{z+1}\right)$$

is one-to-one from the half plane $y > 0$ onto the strip $0 < v < \pi$.
(Here $z = x + iy$, $w = u + iv$.)

<u>Step 1:</u> consider first the LFT $T(z) = \frac{z-1}{z+1}$.
Since $y = 0$ is mapped to a circle or line, and $T(x) = \frac{x-1}{x+1}$ is real for $x \in \mathbb{R} - \{-1\}$, the image of the "circle" $\mathbb{R} \cup \{\infty\}$ is $\mathbb{R} \cup \{\infty\}$ ($T(-1) = \infty$, $T(\infty) = 1$).
Now $v = \operatorname{Im} w = \operatorname{Im} \frac{(z-1)(\bar{z}+1)}{(z+1)(\bar{z}+1)} = \frac{2y}{|z+1|^2}$, $z \neq -1$.

Hence T maps the half plane $y > 0$ one-to-one and onto the half plane $v > 0$.

Step 2: The map $w = \operatorname{Log} z = \ln|z| + i \operatorname{Arg} z$ maps the half plane $y > 0$ onto the strip $0 < v < \pi$ and is one-to-one.



Hence the desired result is obtained by composing the maps from steps 1 & 2:

$$w = \operatorname{Log}\left(\frac{z-1}{z+1}\right) = (\operatorname{Log} \circ T)(z).$$