

RESOLUTIONS OF PINCHED POWER IDEALS

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ABSTRACT. In this paper, we construct resolutions of ideals obtained by removing a small number of generators from the generators of $(x_1, \dots, x_n)^d$.

1. INTRODUCTION

Let $S = k[x_1, \dots, x_n]$, $I = (x_1, \dots, x_n)^d$, and $\hat{I} = I \setminus \{x_1^{d_1} \cdots x_n^{d_n}\}$ the ideal obtained by removing $m = x_1^{d_1} \cdots x_n^{d_n}$ from the minimal generators of I . In this paper, we describe a minimal polytopal resolution of I , based on the complex-of-boxes resolution (see [NR]), which has cyclic symmetry about m , and then use this complex to construct a minimal polytopal resolution for \hat{I} .

Understanding the minimal resolutions of monomial ideals has been a major driver of research in commutative algebra for more than fifty years. The free resolution contains all homological information about the ideal, including its regularity and Hilbert function. Unfortunately, the problem of describing a minimal free resolution, even for monomial ideals, without requiring prohibitive amounts of computation, seems to be intractable in general.

The most substantial case with a computationally easy solution is that of Borel-fixed ideals (which include powers of the maximal ideal), where both the Eliahou-Kervaire resolution ([EK, PS]) and the complex-of-boxes construction ([NR]) allow one to read off, from the list of monomial generators, both a basis for the minimal resolution and the formulas for the differential maps. The two constructions are different, meaning that they produce different choices of basis for the same resolution. These bases are fundamentally non-canonical; in the case of a power of the maximal ideal they also rely on an ordering of the variables and so fail to maintain the symmetries of the ideal.

It is, in principle, possible to construct canonical explicit symmetry-respecting resolutions by refusing to choose a basis. In the case of a power of the maximal ideal, this goes back at least as far as [BE], where I is realized as a special case of a generic determinantal ideal; the resulting construction relies on considerable functorial wizardry and has full GL_n symmetry. More recently, sylvan resolutions ([EMO]) and dynamical systems resolutions ([Tc]) provide explicit descriptions of the minimal resolutions of arbitrary monomial ideals; Tchernev notes that dynamical systems resolutions respect all symmetries of the underlying ideal, and we suspect that the same is true of sylvan resolutions. Both these constructions require a homology computation at every node of the LCM lattice, which is prohibitive in general.

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In recent decades, an important trend in the study of resolutions has been to identify the resolution with some topological or combinatorial object which supports its structure, such as a simplicial complex or a poset. (See [BS, BPS, GPW, No, Me2, Cl, Ma, CM, CT, TV].) While Velasco shows that even the class of CW complexes is not large enough to describe all monomial resolutions, these techniques have been very useful where they work, allowing us to read off the Betti numbers, as well as bases and the corresponding differential matrices, from the combinatorics of the underlying object. Our results are very much in this tradition; the resolutions of I and \hat{I} we construct are supported on a polytopal complex which has cyclic symmetry about the monomial m .

The paper is organized as follows. In section 2, we introduce some notation and describe the complex-of boxes resolution. In section 3, we establish some technical results that are necessary for the rest of the paper. In section 4, we construct symmetric minimal resolutions of $I = (x_1, \dots, x_n)^n$ and $\hat{I} = (x_1, \dots, x_n) \setminus \{x_1 \dots x_n\}$. These constructions are special cases of those found in the next two sections, so we focus on the geometric intuition, suppressing the proofs completely. In section 5, we construct a polytopal minimal resolution of a power of the maximal ideal which is cyclically symmetric around a monomial m . In section 6, we construct a minimal resolution of $\hat{I} = I \setminus \{x_{i_1}^{d_{i_1}} \dots x_{i_s}^{d_{i_s}}\}$. In section 7, we use the mapping cone exact sequence to derive the graded Betti numbers of \hat{I} from those of I , and provide explicit formulas. We observe that this is a highly unusual use of the mapping cone, as the deduction almost always goes in the other direction. Finally, in section 8, we discuss the question of when our techniques can work with more than one deleted generator, and provide some partial answers in terms of staircase diagrams.

2. BACKGROUND AND NOTATION

Throughout the paper, $S = k[x_1, \dots, x_n]$ is a polynomial ring over an arbitrary field k .

2.1. Free resolutions.

Definition 2.1. Let I be an ideal of S . A *free resolution* of $\frac{S}{I}$ is an exact sequence

$$\mathbf{F}: \dots \longrightarrow F_i \xrightarrow{d_i} F_{i-1} \xrightarrow{d_{i-1}} \dots \longrightarrow F_0 \xrightarrow{d_0} \frac{S}{I} \longrightarrow 0,$$

where each F_i is a free S -module.

If I is a homogeneous ideal of S , then the free resolution is said to be *minimal* if $d_{i+1}(F_{i+1}) \subset (x_1, \dots, x_n)F_i$ for all i .

Up to isomorphism, there exists a unique minimal free resolution of $\frac{S}{I}$. If \mathbf{F} is the minimal free resolution of $\frac{S}{I}$, the i^{th} *Betti number* (or *total Betti number*) of $\frac{S}{I}$ over S , denoted by β_i^S , is

$$\beta_i^S \left(\frac{S}{I} \right) = \text{rank}(F_i),$$

and if $F_i = \bigoplus_{p \in \mathbb{Z}} S(-p)^{\beta_{i,p}}$ for each i , the exponents $\beta_{i,p}$ are called *graded Betti numbers* of $\frac{S}{I}$.

Notation 2.2. Following [Me2], we write multigradings multiplicatively. That is, for each monomial m of S , set S_m equal to the k -vector space spanned by m . Then $S = \bigoplus S_m$, and $S_m \cdot S_n = S_{mn}$, so this decomposition is a grading. We say that the monomial m has multidegree m . We allow multidegrees to have negative exponents, so, for example, the twisted module $S(m^{-1})$ is a free module with generator in multidegree m , and $S(m^{-1})_n \cong S_{m^{-1}n}$ as a vector space; this is one-dimensional if no exponent of $m^{-1}n$ is negative, and trivial otherwise. Note that $S = S(1)$.

Notation 2.3. All ideals considered in this paper are monomial ideals. Recall that every monomial ideal J has a unique minimal monomial generating set, which we refer to as $G(J)$.

Definition 2.4. We say that a monomial ideal is *equigenerated (in degree d)* if all its minimal generators have the same degree d . All ideals considered in this paper will be equigenerated.

2.2. Cellular resolutions and polyhedral complexes. The idea of parametrizing a resolution by a simplicial complex was first introduced in [BPS] by Bayer, Peeva, and Sturmfels, and then extended to more general cellular complexes in [BS] by Bayer and Sturmfels. For details, see the original papers or the textbooks [MS, Chapter 4] or [Pe, Chapters 57–60]. In this paper, we are specifically interested in resolutions supported on a polytopal complex; the definitions below loosely follow the treatment in [MS]

Definition 2.5. A *polyhedral cell complex* X is a finite collection of convex polytopes (in \mathbb{R}^n), called *faces* of X , satisfying two properties:

- If $\mathcal{P} \in X$ and \mathcal{F} is a face of \mathcal{P} , then $\mathcal{F} \in X$.
- If $\mathcal{P}, \mathcal{Q} \in X$, then $\mathcal{P} \cap \mathcal{Q}$ (computed as a subset of \mathbb{R}^n) is a face of both \mathcal{P} and \mathcal{Q} .

The embedding of X in \mathbb{R}^n equips it with an *orientation function*,

$$\text{sign} : X \times X \rightarrow \{\pm 1, 0\}$$

with the property that $\text{sign}(G, F) = \pm 1$ if and only if G is a facet of F . For all $F, G \in X$ we have $\sum_{H \in X} \text{sign}(G, H) \text{sign}(H, F) = 0$.

Definition 2.6. The cell complex X is called a *labelled* complex if it is equipped with a labeling function

$$\ell : X \rightarrow \{\text{monomials of } S\}$$

that assigns a monomial $\ell(v)$ to each vertex v of X and satisfies $\ell(F) = \text{lcm}\{\ell(v) : v \in F\}$ for every face F of X . We refer to $\ell(F)$ as the *label* of F .

Definition 2.7. Let X be a labelled polytopal complex. We construct a complex of free modules \mathcal{F}_X with basis given by the faces of X and differential arising from the labels and the topological boundary maps. More precisely, for an i -dimensional face $F \in X$, we create a basis element (also called F) with homological degree i and internal degree $\ell(F)$. Then the differential ∂ of \mathcal{F}_X is

$$\partial(F) = \sum_{\text{facets } G \text{ of } F} \text{sign}(G, F) \frac{\ell(F)}{\ell(G)} G$$

If \mathcal{F}_X is acyclic, we say that it is a *polytopal resolution*.

For any multidegree m , denote

$$X_{\leq m} = \{F \in X : \ell(F) \text{ divides } m\}.$$

Theorem 2.8. *Let X be a labelled polytopal complex. The free complex \mathcal{F}_X supported on X is a (polytopal) resolution if and only if $X_{\leq m}$ is acyclic for all m . In this case, it is a free resolution of the quotient module $\frac{S}{I}$, where $I = (\ell(v) : v \in X \text{ is a vertex})$ is generated by the monomial labels of vertices.*

2.3. Borel ideals.

Definition 2.9. We say that a monomial ideal B is *Borel* if it satisfies the condition

$$\text{If } mx_j \in B \text{ and } i < j, \text{ then } mx_i \in B.$$

Definition 2.10. We refer to the operation sending mx_j to mx_i as a *Borel move*, and the operation sending mx_{i+1} to mx_i as an *elementary Borel move*. Observe that a monomial ideal is Borel if and only if it is closed under (elementary) Borel moves.

Definition 2.11. Let m be a monomial. Define $\text{Borel}(m)$ to be the smallest Borel ideal containing m . We say that $\text{Borel}(m)$ is the *principal Borel ideal* generated by m and m is the Borel generator of $\text{Borel}(m)$.

Remark 2.12. Let $\mathbf{m} = (x_1, \dots, x_n)$ be the homogeneous ideal. Then $\mathbf{m}^d = \text{Borel}(x_n^d)$ is a (principal) Borel ideal.

The notion of Q -Borel ideals was introduced in [FMS2] to study ideals satisfying a weaker exchange property than Borel ideals. We need it here to simultaneously study ideals that are Borel with different orders on the variables.

Definition 2.13. Let Q be a poset on $\{x_1, \dots, x_n\}$ and m be a monomial in S . A Q -Borel move on m is an operation that sends m to a monomial $m \cdot \frac{x_{i_1}}{x_{j_1}} \dots \frac{x_{i_s}}{x_{j_s}}$, where $i_t <_Q j_t$ for all t , and all x_{j_t} divide m .

Definition 2.14. A monomial ideal B is a Q -Borel ideal if B is closed under Q -Borel moves. That is, if $m \in B$, then any monomial obtained from a Q -Borel move on m is also in B .

For a monomial m , define the *principal Q -Borel ideal generated by m* , written $\text{Borel}_Q(m)$, to be the smallest Q -Borel ideal containing m .

There are several explicit ways to resolve a Borel ideal. Two of the most important are the Eliahou-Kervaire resolution [EK, PS] and the complex-of-boxes resolution [NR].

The Eliahou-Kervaire resolution allows us to write out a basis for each module explicitly, and thus compute the (graded) Betti numbers of a Borel ideal by counting the basis elements associated to each monomial generator. (See [PS, Corollary 3.1] or [FMS1, Section 6] for formulas.)

The Eliahou-Kervaire resolution is supported on a regular cellular complex [Me1], but not a polytopal complex.

2.4. The complex-of-boxes resolution. In this section, we recall the complex-of-boxes resolutions for equigenerated Borel ideals. The complex of boxes, introduced by Nagel and Reiner [NR], is always polytopal.

Definition 2.15. An *elementary box* or *interval* is a simplex whose vertices are a subset of variables. Anticipating Notation 2.25, we set $\Delta_{[i,j]}$ equal to the simplex on vertices $\{x_i, \dots, x_j\}$.

Definition 2.16. A *box* is a finite product of intervals. The simplicial structure of the intervals induces a polytopal structure on the box.

Example 2.17. In Figure 1, $\Delta_{[a,c]} \times \Delta_{[c,d]}$ is a box, as is $\Delta_{[a,b]} \times \Delta_{[b,c]} \times \Delta_{[c,d]}$.

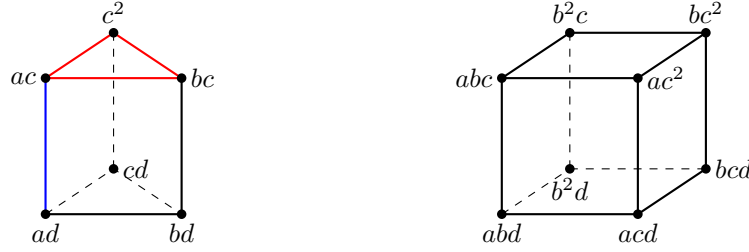


FIGURE 1. The polytopal structures of $\Delta_{[a,c]} \times \Delta_{[c,d]}$ and $\Delta_{[a,b]} \times \Delta_{[b,c]} \times \Delta_{[c,d]}$. The vertices are labelled with monomials rather than ordered tuples, using the labelling of Definition 2.18.

Definition 2.18. Let $\Gamma = \prod_{i=1}^n \Delta_i$ be a box. Every vertex of Γ is an ordered n -tuple of variables, $v = (\dots, x_{j_i} \in \text{supp } \Delta_i, \dots)$. We label the vertices with monomials by forgetting the ordering, $\ell(v) = \prod_{i=1}^n x_{j_i}$.

We say that Γ is an *admissible box* if all vertices have distinct monomial labels.

Example 2.19. As shown in Figure 2, $\Delta_{[a,c]} \times \Delta_{[c]} \times \Delta_{[c,d]}$ is an admissible box, whereas $\Delta_{[a,b]} \times \Delta_{[a,b]}$ is not admissible.

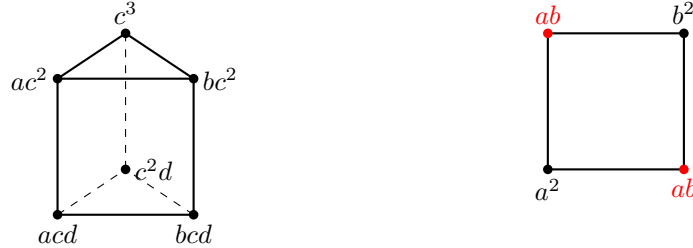


FIGURE 2. The polytopal structures of $\Delta_{[a,c]} \times \Delta_{[c]} \times \Delta_{[c,d]}$ and $\Delta_{[a,b]} \times \Delta_{[a,b]}$.

Remark 2.20. Every monomial is itself an admissible box. More precisely, if $m = \prod_{i=1}^d x_i$ (with indices possibly repeated), then the box $\Gamma = \prod_{i=1}^d \Delta_{\{i\}}$ is a zero-dimensional box consisting of the single vertex m .

Construction 2.21. For a monomial $m = x_{i_1} \dots x_{i_d}$, set $\Gamma(m)$ equal to the $(i_d - 1)$ -dimensional box

$$\Gamma(m) = \Delta_{[1, i_1]} \times \Delta_{[i_1, i_2]} \times \Delta_{[i_2, i_3]} \times \dots \times \Delta_{[i_{d-1}, i_d]}.$$

Observe that $\Gamma(m)$ is admissible.

If I is an equigenerated Borel ideal with minimal generating set $I = (m_1, \dots, m_s)$, then the *complex of boxes* associated to I is the polytopal complex generated by the faces $\{\Gamma(m_1), \dots, \Gamma(m_s)\}$. (This generating set is not minimal, since some faces, e.g. $\Gamma(x_1^d)$, are not facets.)

Remark 2.22. If m is a monomial and \sqrt{m} is its squarefree part, then $\Gamma(m) \cong \Gamma(\sqrt{m})$, since, after rearranging the factors, we have $\Gamma(m) = \Gamma(\sqrt{m}) \times (\frac{m}{\sqrt{m}})$.

Example 2.23. Like the Eliahou-Kervaire resolution, the complex of boxes associated to $(x_1, \dots, x_n)^d$ is a subdivision of the topological n -simplex with vertices labeled by powers of the variables. In three variables, the two complexes coincide (modulo a permutation) as shown in Figure 3. But with more variables, differences begin to emerge. Most notably, the complex of boxes is polytopal, whereas the Eliahou-Kervaire resolution is not, as seen in Figure 4. For example, the complex-of-boxes resolution of $(a, b, c, d)^2$ can be decomposed into four polytopes as in Figure 5.

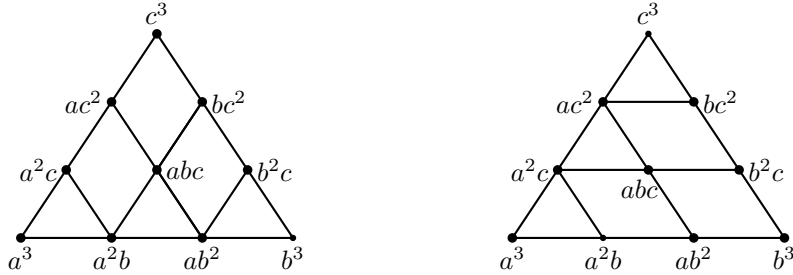


FIGURE 3. The Eliahou-Kervaire resolution (left) and the complex-of-boxes resolution (right) of $(a, b, c)^3$.

Theorem 2.24. [NR, Theorem 3.12] *Let I be an equigenerated Borel ideal. Then the complex of boxes associated to I is a polytopal complex, and supports a minimal resolution of I .*

2.5. Notation for cyclic intervals. Many of the constructions and proofs throughout the paper will involve working with a cyclically symmetric collection of ideals. To avoid notational nightmares, we introduce some notation for dealing with cyclic intervals. Let $\sigma = (1 \ 2 \ 3 \ \dots \ n)$ be the standard generator for the cyclic group on $\{1, 2, \dots, n\}$.

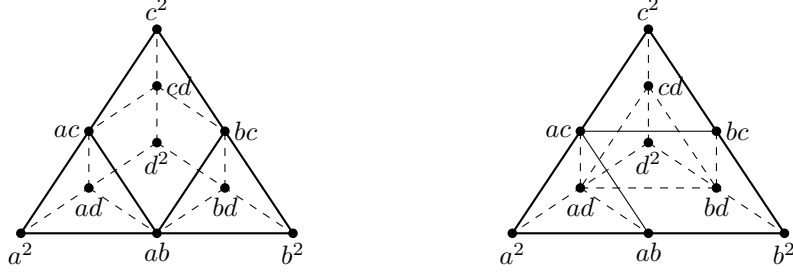


FIGURE 4. The complexes supporting the Eliahou-Kervaire resolution and the complex-of-boxes resolution of $(a, b, c, d)^2$. The Eliahou-Kervaire complex is not polytopal, since the cell containing d^2 contains ac and bc but none of the segment connecting them. The box complex consists of two tetrahedra and two triangular prisms.

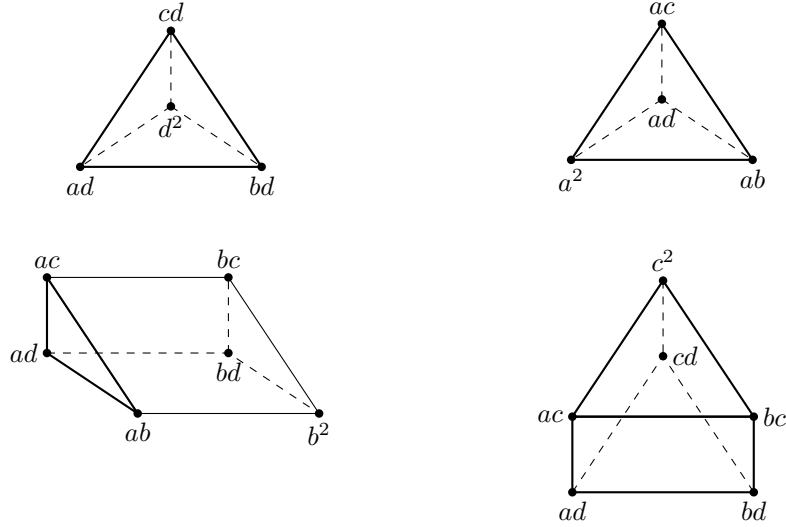


FIGURE 5. The four boxes comprising the complex-of-boxes resolution of $(a, b, c, d)^2$.

Notation 2.25. For any $t \in \{1, \dots, n\}$, the *initial interval* $[1, t]$ refers to the set $\{1, 2, \dots, t\}$. For any $i, j \in \{1, \dots, n\}$, the *cyclic interval* refers to the interval obtained by applying σ^{i-1} to the appropriate initial interval. That is, $[i, j] = \{i, i+1, \dots, j\}$ if $i \leq j$ and $[i, j] = \{i, i+1, \dots, n, 1, \dots, j\}$ if $i > j$.

We will define many objects in terms of cyclic intervals.

- The *simplicial complex supported on* $[i, j]$ is $\Delta_{[i, j]} = \langle \{x_s : s \in [i, j]\} \rangle$.

- The *monomial supported on* $[i, j]$ is $m_{[i,j]} = \prod_{s \in [i,j]} x_s$.
- The *prime ideal supported on* $[i, j]$ is $\mathbf{m}_{[i,j]} = (x_s : s \in [i, j])$.
- The poset Q_i is the chain on $[i, i-1]$ defined by the covering relations $x_i < x_{i+1} < \cdots < x_n < x_1 < \cdots < x_{i-1}$.

3. GLUING OF COMPLEXES

The goal of this paper is to build (new) polytopal resolutions for powers and pinched powers of the maximal ideal by gluing simpler complexes together. The gluing process is unfortunately quite technical, and we rely on a number of lemmas due to Sinefakopoulos [Si]. We collect the necessary results in this section.

Lemma 3.1. [Si, Lemma 3.1] *Let I and J be two monomial ideals in S such that $G(I + J) = G(I) \cup G(J)$ set-theoretically. Suppose that*

- (i) *X and Y are regular cell complexes in \mathbb{R}^N that support a (minimal) free resolution of I and J , respectively, and*
- (ii) *$X \cap Y$ is a regular cell complex that supports a (minimal) free resolution of $I \cap J$.*

Then $X \cup Y$ supports a (minimal) free resolution of $I + J$.

Remark 3.2. [Si, Remark 3.2] For any two monomial ideals I and J , we have $G(I + J) \subseteq G(I) \cup G(J)$. Our assumption that $G(I + J) = G(I) \cup G(J)$ guarantees the right labelling of the cell complex $X \cup Y$. A case where equality holds is when all elements of $G(I) \cup G(J)$ are of the same degree.

Remark 3.3. From assumption (ii) of Lemma 3.1, $I \cap J$ is generated by the labels of the vertices of $X \cap Y$. We conclude that $G(I \cap J) = G(I) \cap G(J)$. This is an extremely strong assumption, as intersecting ideals almost always produces generators in higher degree. But it does hold in the cases of interest, where, among other things, the ideals in question all have linear quotients, and the boundaries of the complexes coincide.

Lemma 3.4. *Let M_1, M_2, \dots, M_n be equigenerated monomial ideals, all generated in the same degree. Assume that*

- (i) *W_1, \dots, W_n are regular cell complexes in \mathbb{R}^N that support (minimal) free resolutions of M_1, \dots, M_n , respectively, and*
- (ii) *For all subsets $\{i_1, \dots, i_\ell\}$, $W_{i_1} \cap W_{i_2} \cap \cdots \cap W_{i_\ell}$ is a regular cell complex that supports a (minimal) free resolution of $M_{i_1} \cap M_{i_2} \cap \cdots \cap M_{i_\ell}$.*

Then $W_1 \cup W_2 \cup \cdots \cup W_n$ supports a (minimal) free resolution of $M_1 + M_2 + \cdots + M_n$.

Proof. Since M_1, M_2, \dots, M_n are all equigenerated monomial ideals,

$$G(M_{i_1} + M_{i_2} + \cdots + M_{i_\ell}) = G(M_{i_1}) \cup G(M_{i_2}) \cup \cdots \cup G(M_{i_\ell}).$$

We will prove the statement by induction on the number of ideals, n .

The case $n = 1$ is trivial and the case $n = 2$ is given by Lemma 3.1. Assume that the statement holds for n . We shall show that it is also true for $n + 1$. Let

$$\begin{aligned} M'_1 &= M_1 + M_{n+1}, & W'_1 &= W_1 \cup W_{n+1}, \\ M'_2 &= M_2 + M_{n+1}, & W'_2 &= W_2 \cup W_{n+1}, \\ &\vdots & &\vdots \\ M'_n &= M_n + M_{n+1}, & W'_n &= W_n \cup W_{n+1}. \end{aligned}$$

It follows from Lemma 3.1 that W'_i is a (minimal) free resolution of M'_i , for $1 \leq i \leq n$, and that

$$W'_{i_1} \cap W'_{i_2} \cap \cdots \cap W'_{i_\ell} = (W_{i_1} \cap W_{i_2} \cap \cdots \cap W_{i_\ell}) \cup W_{n+1}$$

is a (minimal) free resolution of

$$(M_{i_1} \cap M_{i_2} \cap \cdots \cap M_{i_\ell}) + M_{n+1} = M'_1 \cap M'_2 \cap \cdots \cap M'_{i_\ell}.$$

By the inductive hypothesis (applied to M'_i and W'_i), we obtain that

$$W_1 \cup W_2 \cup \cdots \cup W_{n+1} = W'_1 \cup W'_2 \cup \cdots \cup W'_n$$

is a (minimal) free resolution of

$$M_1 + M_2 + \cdots + M_{n+1} = M'_1 + M'_2 + \cdots + M'_n.$$

□

Corollary 3.5. *Let M_1, M_2, \dots, M_n be equigenerated monomial ideals, all generated in the same degree. Assume that*

- (i) W_1, \dots, W_n are regular cell complexes in \mathbb{R}^N that support (minimal) free resolutions of M_1, \dots, M_n , respectively, and
- (ii) $W_{i_1} \cap W_{i_2} \cap \cdots \cap W_{i_\ell}$ is a regular cell complex that supports a (minimal) free resolution of $M_{i_1} \cap M_{i_2} \cap \cdots \cap M_{i_\ell}$.

Then $(W_1 \cup W_2 \cup \cdots \cup W_{n-1}) \cap W_n$ supports a (minimal) free resolution of

$$(M_1 + M_2 + \cdots + M_{n-1}) \cap M_n.$$

Proof. Apply Lemma 3.4 to $M_1 \cap M_n, M_2 \cap M_n, \dots, M_{n-1} \cap M_n$. □

Lemma 3.6. [Si, Lemma 3.3] *Let $I \subset k[x_1, \dots, x_s]$ and $J \subset k[x_{s+1}, \dots, x_n]$ be two monomial ideals. Suppose that X and Y are regular cell complexes in \mathbb{R}^N of dimension $s - 1$ and $n - s - 1$, respectively, that support (minimal) free resolutions of I and J , respectively. Then the regular cell complex $X \times Y$ supports a (minimal) free resolution for IJ .*

Lemma 3.7. [Si, Lemma 3.5] *Let $I \subset k[x_1, \dots, x_s]$ and $J \subset k[x_s, \dots, x_n]$ be two monomial ideals such that $|G(IJ)| = |G(I)| \cdot |G(J)|$. Suppose that X and Y are regular cell complexes in some \mathbb{R}^N of dimension $s - 1$ and $n - s$, respectively, that support (minimal) free resolutions of I and J , respectively. Then the regular cell complex $X \times Y$ supports a (minimal) free resolution for IJ .*

4. SYMMETRIC POLYTOPAL RESOLUTIONS OF $I = (x_1, \dots, x_n)^n$ AND $I \setminus x_1 \cdots x_n$

In this section we build the intuition for the rest of the paper. Unfortunately, when we work in the full generality of $(x_1, \dots, x_n)^d$, the notation gets messy enough to obscure the ideas. This section covers the special case where the degree is equal to the number of variables and we have the full symmetry arising from working around a “central” monomial (in this case, the product of the variables). Everything here is a special case of §5 and §6, so we further emphasize the intuition by suppressing the proofs. Readers who don’t need our geometric intuition can safely skip this section and jump directly to §5.

The goal of this section is to build a resolution of $I = \mathfrak{m}^n = (x_1, \dots, x_n)^n$ which is a polytopal subdivision of the $(n-1)$ -dimensional simplex, and is also symmetric about its center $m = x_1 \cdots x_n$. (In particular, no variable is preferred over the others, so all corners of the simplex will look the same.) Our strategy is to let the cyclic group act on the principal Borel ideal $I_1 = \text{Borel}(m)$ and its complex of boxes. This produces n polytopal complexes which intersect in positive codimension; gluing them together yields the desired resolution (Theorem 4.5).

Using this symmetric resolution, we then construct a resolution for the “pinched power” ideal \tilde{I} , generated by all degree n monomials except the central monomial $m = x_1 \cdots x_n$. This is accomplished in Construction 4.10 by merging all boxes containing m into a single larger polyhedron.

Notation 4.1. For each i , let $I_i = \text{Borel}_{Q_i}(m)$ be the principal Q_i -Borel ideal generated by m . Denote by X_1 the complex of boxes resolving I_1 , and by X_i the appropriate cyclic permutation of X_1 . Observe that X_i minimally resolves I_i ; we abuse notation by referring to X_i as the *complex of boxes resolving I_i* .

Example 4.2. When $n = 3$, the ideals I_i are $I_1 = (abc, a^2c, ab^2, a^2b, a^3)$, $I_2 = (abc, ab^2, bc^2, b^2c, b^3)$ and $I_3 = (abc, bc^2, a^2c, ac^2, c^3)$. In Figure 6, the complexes of boxes resolving these ideals are shown in red, yellow, and blue, respectively.

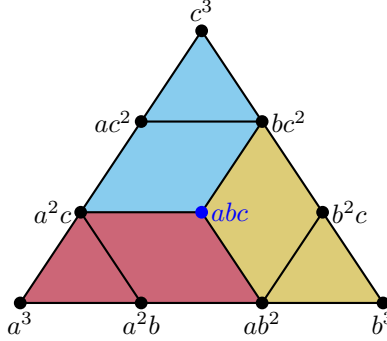


FIGURE 6. The ideals I_1 , I_2 , and I_3 inside $(a, b, c)^3$.

When $n = 4$, we have $I_1 = (abcd, ab^2d, a^2cd, a^2bd, a^3d, abc^2, a^2c^2, ab^2c, a^2bc, a^3c, ab^3, a^2b^2, a^3b, a^4)$. I_2 , I_3 , and I_4 are cyclic permutations of I_1 . The complexes of boxes resolving these ideals are shown in Figure 7.

Proposition 4.3. *The I_i , and the box complexes X_i resolving them, satisfy the hypotheses of Lemma 3.4.*

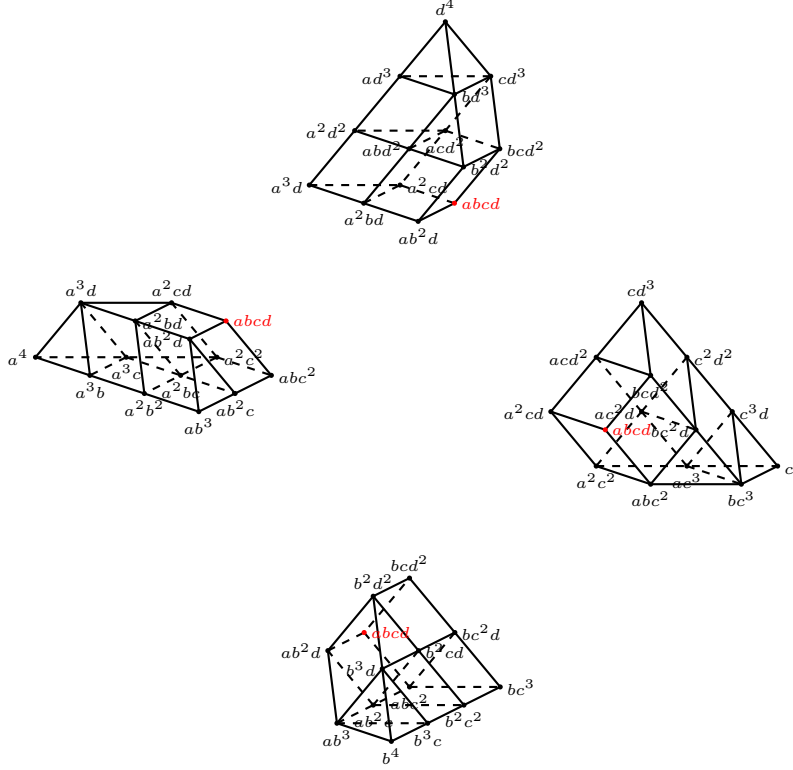


FIGURE 7. The complexes of boxes resolving I_1 (left), I_2 (bottom), I_3 (right), and I_4 (top) inside $(a, b, c, d)^4$.

Proof. This is a special case of Lemma 5.5 and Proposition 5.8. \square

Definition 4.4. Set $X = \bigcup_{i=1}^n X_i$.

Theorem 4.5. The polytopal complex X supports the minimal resolution of I .

Example 4.6. The complex in Figure 6 supports the minimal resolution of $(a, b, c)^3$.

The complex obtained by gluing the four complexes in Figure 7 supports the minimal resolution of $(a, b, c, d)^4$.

We now modify the minimal resolution X to describe a symmetric polytopal resolution of the “pinched power” ideal \hat{I} obtained by deleting $m = x_1 \dots x_n$ from the generating set for I . The plan is to simply remove from X all faces containing m . Naively, doing so will cause the n facets adjoining m to lose their boundaries; we fix the issue by merging these cells into a single polytope.

Notation 4.7. Let \hat{I} be the ideal generated by all degree n monomials except $m = x_1 \dots x_n$. We also use hats on our previous notation to denote the removal of m , to wit:

- \hat{I}_i is the Q_i -Borel ideal generated by all monomials of I_i except m .
- \hat{X}_i is the complex of boxes resolving \hat{I}_i .

We also define, for each i , the box $Y_i = [x_i] \times [x_i, x_{i+1}] \times \cdots \times [x_{i-2}, x_{i-1}]$, the unique facet of X_i containing m , and $Y = \bigcup Y_i$.

Finally, set J_i equal to the ideal generated by the vertices of Y_i , $J = \sum J_i$, and \hat{J} the ideal generated by all monomials of J except m .

Example 4.8. In three variables, Y_1, Y_2 , and Y_3 are rhombi and Y is their union. In four variables, Y_1, \dots, Y_4 are parallelepipeds and Y is their union.

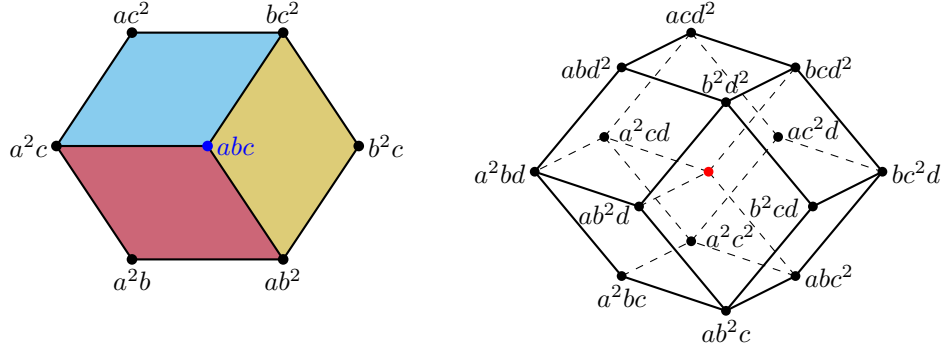


FIGURE 8. Y for the ideals $(a, b, c)^3$ (on the left) and $(a, b, c, d)^4$ (on the right).

Proposition 4.9. Y supports the minimal resolution of J .

Proof. This is a special case of Theorem 6.4. \square

Construction 4.10. Refer to the polytope obtained by removing m from Y as \hat{Y} . More precisely, consider the geometric realization of Y attained by placing each vertex $\prod x_i^{e_i}$ at the point (e_1, \dots, e_n) inside \mathbb{R}^n . Let \hat{Y} be the convex hull of all these vertices. Then \hat{Y} is an $(n-1)$ dimensional polytope (inside the hyperplane $\sum x_i = n$) with a single facet, which contains m on its interior. The lower-dimensional faces of \hat{Y} are precisely the faces of Y which don't contain m .

Example 4.11. In three variables, \hat{Y} is the hexagon on the left in Figure 4.11. In four variables, \hat{Y} is the rhombic dodecahedron on the right.

Proposition 4.12. \hat{Y} supports the minimal resolution of \hat{J} .

Proof. This is a special case of Lemma 6.6. \square

Construction 4.13. Let $\hat{X} = \hat{Y} \cup \bigcup_{i=1}^n \hat{X}_i$ be the polytopal complex obtained from X by replacing Y with \hat{Y} .

Theorem 4.14. \hat{X} supports the minimal resolution of \hat{I} .

Proof. This is a special case of Theorem 6.9. \square

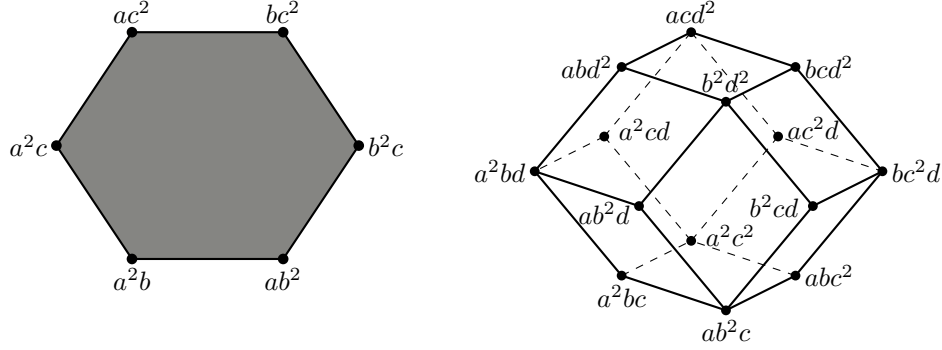


FIGURE 9. The polytope \hat{Y} , in three and four variables.

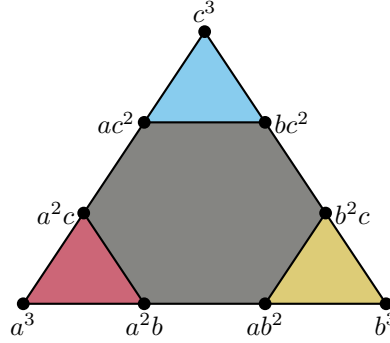


FIGURE 10. The minimal resolution of $\hat{I} = (a, b, c)^3 \setminus \{abc\}$.

Example 4.15. The complex in Figure 10 supports the minimal resolution of $\hat{I} = (a, b, c)^3 \setminus \{abc\}$ in $k[a, b, c]$.

The complex \hat{X} , obtained by gluing the five complexes in Figure 11, supports a minimal resolution of $\hat{I} = (a, b, c, d)^4 \setminus \{abcd\}$ in $k[a, b, c, d]$.

5. A CYCLIC RESOLUTION OF $(x_1, \dots, x_n)^d$

Fix $m = x_1^{d_1} x_2^{d_2} \dots x_n^{d_n}$ and put $d = d_1 + \dots + d_n$. We will build a polytopal resolution of $I = \mathfrak{m}^d = (x_1, \dots, x_n)^d$ which is symmetric around m .

We shall see that the general shape of the resolution depends substantially on which variables divide m . Intuitively, the complex of boxes arising from the ordering $Q_i : x_i < x_{i+1} < \dots < x_{i-1}$ will appear if and only if x_{i-1} divides m .

Definition 5.1. Fix $m = x_1^{d_1} x_2^{d_2} \dots x_n^{d_n}$. Denote $\text{supp}(m) := \{x_i : d_i \neq 0\}$ the support of m . For convenience, we will abuse notation by setting $\text{supp}(m) := \{i : x_i \in \text{supp}(m)\}$.

Notation 5.2. For each i , denote by I_i the principal Q_i -Borel ideal generated by m , $I_i = \text{Borel}_{Q_i}(m)$. Let X_i be the complex of boxes resolving I_i .

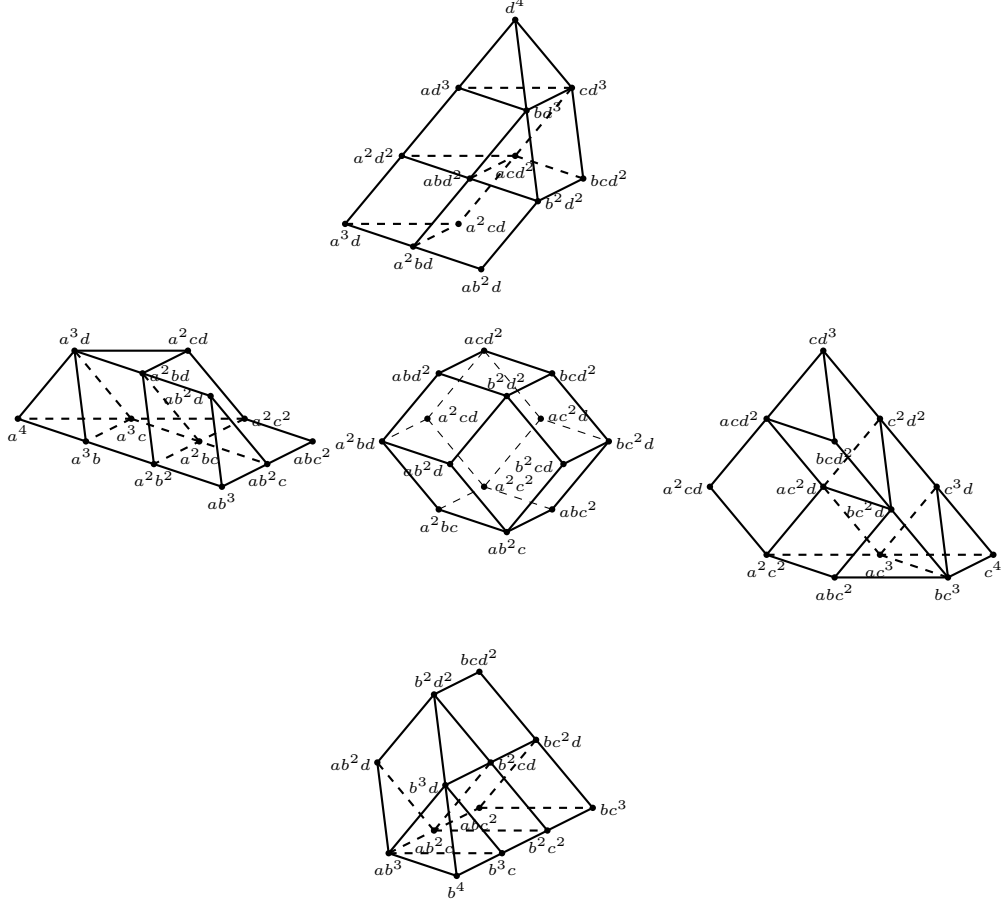


FIGURE 11. Gluing the five complexes gives us a symmetric minimal resolution of $(a, b, c, d)^4 \setminus \{abcd\}$ in $k[a, b, c, d]$.

Lemma 5.3. *For any $1 \leq i \leq n$, I_i is the cyclic product of primes*

$$I_i = \prod_{j=i}^{i-1} \mathfrak{m}_{[i,j]}^{d_j}.$$

Proof. This is immediate from [Si, Proposition 2.7], which states that $\text{Borel}(\prod x_i^{d_i}) = \prod (\text{Borel}(x_i))^{d_i}$. \square

Example 5.4. Let $m = x_2^2 x_4^3 = x_1^0 x_2^2 x_3^0 x_4^3$ and $I = (x_1, \dots, x_4)^5$ in $k[x_1, x_2, x_3, x_4]$. Then

$$\begin{aligned} I_1 &= (x_1)^0 (x_1, x_2)^2 (x_1, x_2, x_3)^0 (x_1, x_2, x_3, x_4)^3, \\ I_2 &= (x_2)^2 (x_2, x_3)^0 (x_2, x_3, x_4)^3 (x_2, x_3, x_4, x_1)^0, \\ I_3 &= (x_3)^0 (x_3, x_4)^3 (x_3, x_4, x_1)^0 (x_3, x_4, x_1, x_2)^2, \\ I_4 &= (x_4)^3 (x_4, x_1)^0 (x_4, x_1, x_2)^2 (x_4, x_1, x_2, x_3)^0. \end{aligned}$$

Lemma 5.5. We have $I = I_1 + \dots + I_n$.

Proof. It suffices to show that, for an arbitrary monomial f of degree d , we have $f \in I_i$ for some i .

Write $f = \prod x_j^{e_j}$, and set $\delta_{[r,s]}(f) = \sum_{j \in [r,s]} (e_j - d_j)$. Then $f \in I_i$ if and only if

$\delta_{[i,s]}(f) \geq 0$ for all s (including $s = i$, which is the statement that $x_i^{d_i}$ divides f).

Choose t such that $\delta_{[1,t]}(f)$ is minimal. If $\delta_{[1,t]}(f) \geq 0$, then $f \in I_1$. If not, we claim that $f \in I_{t+1}$. Indeed, setting $r = t + 1$, we have

$$\delta_{[r,s]}(f) = \begin{cases} \delta_{[1,s]}(f) - \delta_{[1,t]}(f) & \text{if } r \leq s \leq n \\ \delta_{[r,n]}(f) + \delta_{[1,s]}(f) & \text{if } 1 \leq s < r \end{cases}$$

Observe that $\delta_{[1,t]}(f) + \delta_{[r,n]}(f) = \delta_{[1,n]}(f) = \deg(f) - d = 0$. Thus $\delta_{[r,n]}(f) = -\delta_{[1,t]}(f)$ and in either case we have $\delta_{[r,s]}(f) = \delta_{(1,s)}(f) - \delta_{[1,t]}(f)$, which is non-negative by the choice of t . \square

The following notation is convenient to describe the intersection of any collection of I_i .

Notation 5.6. Let $m = x_1^{d_1} \dots x_n^{d_n}$ and $i_1 < \dots < i_\ell$ in $\{1, \dots, n\}$. For each index $j \in \{i_1, \dots, i_\ell\}$, denote by \mathbf{p}_j the product of cyclic primes

$$\mathbf{p}_j := \mathbf{m}_{[i_j, i_j]}^{d_{i_j}} \mathbf{m}_{[i_j, i_j+1]}^{d_{i_j+1}} \dots \mathbf{m}_{[i_j, i_{j+1}-1]}^{d_{i_{j+1}-1}}.$$

Observe that $\mathbf{p}_j = \text{Borel}_{Q_{i_j}}(x_{i_j}^{d_{i_j}} \dots x_{i_{j+1}-1}^{d_{i_{j+1}-1}})$ in $k[x_{i_j}, \dots, x_{i_{j+1}-1}]$. Denote by $\Gamma_{[i_j, i_{j+1}-1]}$ the box complex resolving \mathbf{p}_j .

Example 5.7. Let $m = x_2^2 x_3 x_5^3 x_8 = x_1^0 x_2^2 x_3 x_4^0 x_5^3 x_6^0 x_7^0 x_8 x_9^0$ and $I = (x_1, \dots, x_9)^7$ in $k[x_1, \dots, x_9]$ and $(i_1, i_2, i_3) = (2, 4, 8)$. Then

$$\begin{aligned} \mathbf{p}_1 &= (x_2)^2 (x_2, x_3), \\ \mathbf{p}_2 &= (x_4)^0 (x_4, x_5)^3 (x_4, x_5, x_6)^0 (x_4, x_5, x_6, x_7)^0, \\ \mathbf{p}_3 &= (x_8)(x_8, x_9)^0 (x_8, x_9, x_1)^0. \end{aligned}$$

Observe that

$$\begin{aligned} I_2 \cap I_4 \cap I_8 &= \mathbf{p}_1 \mathbf{p}_2 \mathbf{p}_3, \\ X_2 \cap X_4 \cap X_8 &= \Gamma_{[2,3]} \times \Gamma_{[4,7]} \times \Gamma_{[8,1]}. \end{aligned}$$

We will prove this property generally in the following theorem.

Theorem 5.8. Let $m = x_1^{d_1} \dots x_n^{d_n}$ and $\{i_1, \dots, i_\ell\} \subset \{1, \dots, n\}$. Then:

(1) We have

$$I_{i_1} \cap \cdots \cap I_{i_\ell} = \mathbf{p}_1 \mathbf{p}_2 \cdots \mathbf{p}_\ell.$$

(2) The intersection of the I_j is minimally resolved by the product of the box complexes resolving the \mathbf{p}_j . That is,

$$X_{i_1, \dots, i_\ell} := \Gamma_{[i_1, i_2-1]} \times \Gamma_{[i_2, i_3-1]} \times \cdots \times \Gamma_{[i_\ell, i_1-1]},$$

gives a minimal resolution of $I_{i_1} \cap \cdots \cap I_{i_\ell}$.

(3) The intersection of the I_j is minimally resolved by the intersection of the box complexes resolving the $\text{Borel}_{Q_{i_j}}(m)$,

$$X_{i_1, \dots, i_\ell} = X_{i_1} \cap \cdots \cap X_{i_\ell}.$$

Proof. To show that $I_{i_1} \cap \cdots \cap I_{i_\ell} \subseteq \prod \mathbf{p}_j$, observe that $I_{i_j} \subset \mathbf{p}_j$ for all j by Lemma 5.3. Since the \mathbf{p}_j have disjoint support, we have $\bigcap \mathbf{p}_j = \prod \mathbf{p}_j$. Thus

$$\bigcap_{j=1}^{\ell} I_{i_j} \subset \bigcap_{j=1}^{\ell} \mathbf{p}_j = \prod_{j=1}^{\ell} \mathbf{p}_j.$$

To show that $\prod \mathbf{p}_j \subseteq I_{i_1} \cap \cdots \cap I_{i_{\ell+1}}$, it suffices to show that, without loss of generality, $\prod \mathbf{p}_j \subseteq I_{i_1}$. Observe that

$$\begin{aligned} \mathbf{p}_j &= \mathbf{m}_{[i_j, i_j]}^{d_{i_j}} \mathbf{m}_{[i_j, i_j+1]}^{d_{i_j+1}} \cdots \mathbf{m}_{[i_j, i_{j+1}-1]}^{d_{i_{j+1}-1}} \\ &\subseteq \mathbf{m}_{[i_1, i_j]}^{d_{i_j}} \mathbf{m}_{[i_1, i_j+1]}^{d_{i_j+1}} \cdots \mathbf{m}_{[i_1, i_{j+1}-1]}^{d_{i_{j+1}-1}}. \end{aligned}$$

Thus,

$$\prod \mathbf{p}_j \subseteq \left[\mathbf{m}_{[i_1, i_1]}^{d_{i_1}} \cdots \mathbf{m}_{[i_1, i_2-1]}^{d_{i_2-1}} \right] \left[\mathbf{m}_{[i_1, i_2]}^{d_{i_2}} \cdots \mathbf{m}_{[i_1, i_3-1]}^{d_{i_3-1}} \right] \cdots \left[\mathbf{m}_{[i_1, i_\ell]}^{d_{i_\ell}} \cdots \mathbf{m}_{[i_1, i_1-1]}^{d_{i_1-1}} \right] = I_{i_1}$$

as desired. This proves (1).

For (2), observe that the supports of the \mathbf{p}_{i_j} are pairwise disjoint and induct on Lemma 3.6; we conclude that X_{i_1, \dots, i_ℓ} yields a minimal resolution.

For (3), set $\mathcal{I} = \bigcap_{j=1}^{\ell} X_{i_j}$ and $\mathcal{P} = \Gamma_{[i_1, i_2-1]} \times \cdots \times \Gamma_{[i_\ell, i_1-1]}$. We will show $\mathcal{I} = \mathcal{P}$.

To show $\mathcal{P} \subseteq \mathcal{I}$, let Λ be a facet of \mathcal{P} . We may write $\Lambda = \prod_j \Gamma_{i_j}(f_j)$, with each $f_j \in \mathbf{p}_j$. Observe that the poset on $\{x_{i_j}, \dots, x_{i_{j+1}-1}\}$ defined by the covering relations $x_{i_j} < x_{i_j+1} < \cdots < x_{i_{j+1}-1}$ is a sub-poset of Q_{i_j} for all j . Setting $f = \prod_j f_j$, we immediately have, for all $t \in \{1, \dots, \ell\}$, $f \in \text{Borel}_{Q_{i_t}}(m)$ and thus $\Gamma_{i_j}(f_j) \subseteq \Gamma_{i_t}(f)$. We conclude $\Lambda \subseteq \Gamma_{i_t}(f) \in \Gamma_{[i_t, i_{t+1}-1]}$. Since t is arbitrary, we have $\Lambda \in \mathcal{I}$ as desired.

Now we show $\mathcal{I} \subseteq \mathcal{P}$. It follows from (2) that $I_{i_1} \cap \cdots \cap I_{i_\ell} = \mathbf{p}_1 \mathbf{p}_2 \cdots \mathbf{p}_\ell$ is an equigenerated ideal and its minimal generators are of degree d , so $G(I_{i_1} \cap \cdots \cap I_{i_\ell}) = \bigcap_{j=1}^{\ell} G(I_{i_j})$. This implies that vertices of $\bigcap_{j=1}^{\ell} X_{i_j}$ are labeled with monomials in

$\bigcap_{j=1}^{\ell} G(I_{i_j})$. Now we show that for each vertex f of $\bigcap_{j=1}^{\ell} X_{i_j}$, $\bigcap_{j=1}^{\ell} \Gamma_{Q_{i_j}}(f)$ is contained in $\prod_{j=1}^{\ell} \Gamma_{[i_j, i_j-1]}$. Since $f \in \mathbf{p}_1 \mathbf{p}_2 \dots \mathbf{p}_{\ell}$, it can be written as

$$f = f_1 \cdots f_{\ell},$$

where for each j , $f_j \in \text{Borel}_{Q_{i_j}}(x_{i_j}^{d_{i_j}} \cdots x_{i_{j+1}-1}^{d_{i_{j+1}-1}})$. For each j , let $Z_j := \Gamma_{Q_{i_j}}(f_j) \subset \Gamma_{[i_j, i_j-1]}$. Then

$$\bigcap_{j=1}^{\ell} \Gamma_{Q_{i_j}}(f) = Z_1 \times \cdots \times Z_{\ell}$$

which is contained in $\Gamma_{[i_1, i_2-1]} \times \Gamma_{[i_2, i_3-1]} \times \cdots \times \Gamma_{[i_{\ell}, i_1-1]}$. This proves (3). \square

We have now shown that the box complexes resolving the I_i satisfy the hypotheses of Lemma 3.4. We conclude that their union resolves I and obtain the following:

Theorem 5.9. *For any generator m of $I = \mathbf{m}^d$, the complex*

$$X = \bigcup_{i=1}^n X_i$$

supports a minimal resolution of I .

We now simplify the union in Theorem 5.9. We need a lemma.

Lemma 5.10. *Let $m = x_1^{d_1} x_2^{d_2} \cdots x_n^{d_n}$.*

(i) *If $d_i = 0$, then $I_i \supseteq I_{i+1}$.*

(ii) $I = \sum_{u \in \text{supp}(m)} I_{u+1}$.

Proof. For (i), assume that $d_i = 0$. On one hand,

$$\begin{aligned} \mathbf{p}_1 &= \mathbf{m}_{[i, i]}^{d_i} = \mathbf{m}_{[i, i]}^0 = (1), \\ \mathbf{p}_2 &= \mathbf{m}_{[i+1, i+1]}^{d_{i+1}} \mathbf{m}_{[i+1, i+2]}^{d_{i+2}} \cdots \mathbf{m}_{[i+1, i-1]}^{d_{i-1}}, \end{aligned}$$

so it follows from Theorem 5.8 that

$$I_i \cap I_{i+1} = \mathbf{p}_1 \mathbf{p}_2 = \mathbf{p}_2.$$

On the other hand,

$$I_{i+1} = \mathbf{m}_{[i+1, i+1]}^{d_{i+1}} \mathbf{m}_{[i+1, i+2]}^{d_{i+2}} \cdots \mathbf{m}_{[i+1, i-1]}^{d_{i-1}} \mathbf{m}_{[i+1, i]}^{d_i} = \mathbf{p}_2 \mathbf{m}_{[i+1, i]}^0 = \mathbf{p}_2.$$

Thus $I_i \cap I_{i+1} = I_{i+1}$, or equivalently, $I_i \supseteq I_{i+1}$.

To prove (ii), assume that $\text{supp}(m) = \bigcup_{t=1}^{\ell} [i_t, j_t]$ where $i_t \leq j_t$, $j_t + 1 < i_{t+1}$. It follows from (i) that $I_j \supset I_{j_t+1}$, for all $j_t < j \leq i_{t+1}$. By Lemma 5.5,

$$I = I_1 + \cdots + I_n = \sum_{t=1}^{\ell} \sum_{u=i_t}^{j_t} I_{u+1} = \sum_{u \in \text{supp}(m)} I_{u+1}.$$

This completes the proof. \square

Consequently, we immediately obtain the following corollary.

Corollary 5.11. *If $d_i = 0$ and j is the smallest index such that $j > i$ and $d_j \neq 0$, then $X_i \supseteq X_\ell$ for all $\ell \in [i, j]$.*

Theorem 5.12. *For any generator m of $I = \mathbf{m}^d$, the complex*

$$X = \bigcup_{u \in \text{supp}(m)} X_{u+1}$$

supports a minimal resolution of I .

Example 5.13. The complexes in Figure 12 are minimal resolutions of $(a, b, c)^4$, centered around ab^2 and around b^2c^2 . Observe that the complex on the left contains boxes of all three orientations, while the complex on the right has no yellow boxes (because a does not divide b^2c^2).

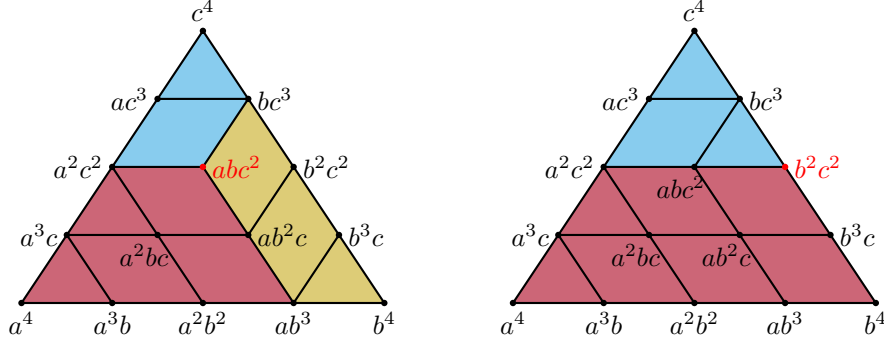


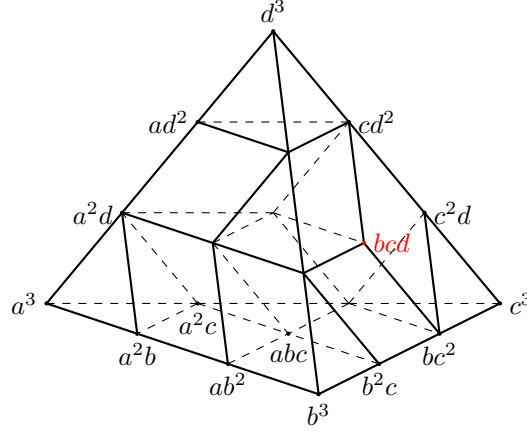
FIGURE 12. Minimal linear resolutions of $(a, b, c)^4$.

Example 5.14. The complex in Figure 13 supports a minimal resolution of $I = (a, b, c, d)^3$ (centered around bcd).

6. A MINIMAL RESOLUTION OF THE IDEAL OBTAINED BY REMOVING $m = x_1^{d_1} \cdots x_n^{d_n}$ FROM THE GENERATORS OF $I = (x_1, \dots, x_n)^d$.

In this section, we will use the polytopal resolution for $I = (x_1, \dots, x_n)^d$ established in Theorem 5.9 to describe the minimal resolution of the pinched power ideal \hat{I} obtained by removing m . The technique is to modify the resolution of I (centered on m) obtained in the previous section by replacing all boxes containing m with their (topological) union, a polytope that does not contain m as a vertex. Essentially the same operation appears in [Bi], where Biermann uses it to construct a cellular complex supporting the resolution of the complement of an n -cycle.

As in the previous section, the resolution of \hat{I} depends strongly on the support of m . For example, if $m = x_j^d$ is a pure power, then \hat{I} is a Borel-with-holes ideal [CE] and so is resolved by a truncation of the Eliahou-Kervaire resolution (or by a truncation of the complex of boxes).

FIGURE 13. A minimal resolution of $(a, b, c, d)^3$.

Notation 6.1. We require a tremendous amount of notation. The general philosophy is to adapt notation from the previous section, and add a hat when m has been removed. Fix a monomial m of degree d , and write $m = x_1^{d_1} \cdots x_n^{d_n}$. We denote:

- $I = (x_1, \dots, x_n)^d$, the d^{th} power of the maximal ideal, and X , the polytopal complex supporting the resolution of I centered at m , as constructed in the previous section.
- \hat{I} the ideal obtained by removing m from the generators of $I = (x_1, \dots, x_n)^d$.
- Y the star of m in X , that is, the subcomplex of X generated by all facets containing m .
- $J = \mathbf{m}_1^{d_1} \mathbf{m}_2^{d_2} \cdots \mathbf{m}_n^{d_n}$, the ideal supported by Y .
- \hat{J} the ideal obtained by removing m from the generators of J .
- \hat{X}_i the complex obtained by removing all faces containing m from X_i .
- $Y_i = \Gamma_i(m)$ (so $Y = \bigcup Y_i$), the collection of boxes containing m inside X .
- \hat{Y} the polytope obtained by forgetting the interior structure (i.e., the faces containing m) of Y . That is, \hat{Y} has a single $(n-1)$ -dimensional face, which is the (interior of the) union of the Y_i ; for $d < n-1$ its d -dimensional faces are the d -dimensional faces of the Y_i which don't contain m .
- \hat{I}_i the ideal obtained by removing m from the generators of I_i . Then \hat{I}_i is a Q_i -Borel ideal and \hat{X}_i is the complex-of-boxes resolving \hat{I}_i .
- \hat{J}_i the ideal obtained by removing m from the generators of J_i , and \hat{Y}_i the complex obtained by deleting all faces of Y_i containing m .
- $\hat{X} = \hat{Y} \cup \bigcup_{i=1}^n \hat{X}_i$, the complex obtained from X by replacing Y with \hat{Y} .

Remark 6.2. We have $Y = \bigcup_{u \in \text{supp}(m)} Y_{u+1}$ and $\hat{X} = \hat{Y} \cup \bigcup_{u \in \text{supp}(m)} \hat{X}_{u+1}$.

Remark 6.3. The defining half-spaces of \hat{Y} are

$$\left\{ \begin{array}{l} x_1 + \cdots + x_n = d, \\ d_a + \cdots + d_b - 1 \leq x_a + \cdots + x_b \leq d_a + \cdots + d_b + 1, \\ 0 \leq x_a + \cdots + x_b \leq d \end{array} \right\}$$

(for all cyclic intervals $[a, b]$).

Observe that all $Y_i = \Gamma_{Q_i}(m)$ and J_i satisfy the hypotheses of Lemma 3.4, so their union resolves J .

Theorem 6.4. *The complex Y supports the minimal resolution of J .*

Example 6.5. The complex in Figure 14 supports a minimal resolution of $J = (bcd, abd, acd, abc, bd^2, cd^2, c^2d, b^2d, ac^2, bc^2, b^2c)$.

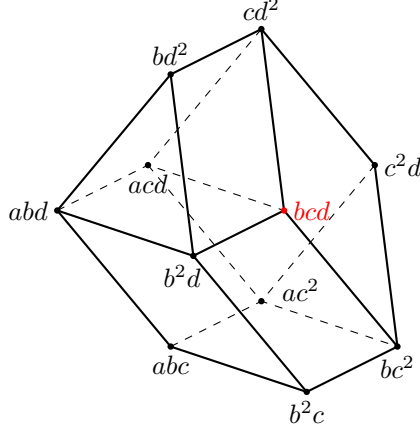


FIGURE 14. A minimal resolution of $J = (b, c)(c, d)(d, a, b)$, inside $I = (a, b, c, d)^3$.

Lemma 6.6. *The complex \hat{Y} supports a minimal resolution of \hat{J} .*

To prove Lemma 6.6, we will show that $\hat{Y}_{\leq \alpha}$ is acyclic for all multidegrees α . Indeed, we will make use of a lemma of Nagel and Reiner [NR] to show that $\hat{Y}_{\leq \alpha}$ is homotopic to Y_α , which is acyclic by Corollary 6.4 and Theorem 2.8.

Lemma 6.7. [NR, Lemma 6.4] *Let \mathcal{C} be a polytopal complex and v a vertex in \mathcal{C} that lies in a unique facet P , and assume that P has strictly positive dimension.*

Then the vertex-induced subcomplex $\mathcal{C} \setminus \{v\}$, obtained by deleting v and all faces that contain it, is homotopy equivalent to \mathcal{C} .

Proof of Lemma 6.6. Let $\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ be a monomial. We will show that $\hat{Y}_{\leq \alpha}$ is acyclic.

Assume that $m = x_{i_1}^{d_{i_1}} \cdots x_{i_s}^{d_{i_s}}$, where $i_1 < \cdots < i_s$ and $\text{supp}(m) = \{i_1, \dots, i_s\}$.

We first observe that if $m = x_{i_1}^{d_{i_1}} \cdots x_{i_s}^{d_{i_s}}$ does not divide α , then $\hat{Y}_{\leq \alpha} = Y_{\leq \alpha}$ which is acyclic because Y supports a minimal resolution by Corollary 6.4.

Next observe that, if α is a multiple of $(x_1 \cdots x_n)m$, the least common multiple of the generators of J , then $\hat{Y}_{\leq \alpha} = \hat{Y}$, which is homeomorphic to Y . Thus, since

$Y = Y_{\leq \alpha}$ and Y supports a minimal resolution by Corollary 6.4, we have that $\widehat{Y}_{\leq \alpha}$ is acyclic.

Finally, assume that α is divisible by m but not by $(x_1 \cdots x_n)m$. Then, without loss of generality, we may assume that

$$\alpha = (x_{u_1} \cdots x_{u_t})(x_{v_1} \cdots x_{v_\ell})m,$$

where $A = \{x_{u_1}, \dots, x_{u_t}\} \subset \text{supp}(m)$, $B = \{x_{v_1}, \dots, x_{v_\ell}\}$ and $\text{supp}(m)$ are disjoint, and $A \cup B$ is a proper subset of $\{x_1, \dots, x_n\}$. Denote $B_j = B \cap \{x_{i_j}, \dots, x_{i_{j+1}}\}$.

If $A = \text{supp}(m)$, let $R' = k[A, B]$ and $m' = \prod_{x_i \in A \cup B} x_i^{d_i}$. Construct Y' and J' analogously to Y and J , but for this smaller collection of variables. Then $Y_{\leq \alpha} = m' \times Y' \cong Y'$ is acyclic because Y' supports the minimal resolution of J' .

Next, consider the case that A is a proper subset of $\text{supp}(m)$. We claim that m is contained in a unique facet of $Y_{\leq \alpha}$. By Remark 2.22, we may assume that m is squarefree. Let $\mathcal{P}_{m, \alpha}$ be the box

$$\mathcal{P}_{m, \alpha} = \left(\prod_{i_j \in A} (\{x_{i_j}, x_{i_{j+1}}\} \cup B_j) \right) \times \left(\prod_{i_j \in \text{supp}(m) \setminus A} (\{x_{i_{j+1}}\} \cup B_j) \right).$$

We claim that $\mathcal{P}_{m, \alpha}$ is the unique facet of $Y_{\leq \alpha}$ containing m . First, observe that $m \in \mathcal{P}_{m, \alpha}$ since one can choose $x_{i_{j+1}}$ from every interval in the product. Next, observe that the multidegree of $\mathcal{P}_{m, \alpha}$ is $\gcd(\alpha, (x_1 \cdots x_n)m)$, which is divisible by α . Thus $\mathcal{P}_{m, \alpha}$ is a face of $Y_{\leq \alpha}$.

It remains to show that every face of $Y_{\leq \alpha}$ containing m is contained in $\mathcal{P}_{m, \alpha}$. The least common multiple of vertices of every such face is of the form

$$\alpha' = \left(\prod_{x_u \in A'} x_u \right) \left(\prod_{x_v \in B'} x_v \right) m,$$

where $A' \subset A$ and $B' \subset B$. Denote $B'_j = B' \cap \{x_{i_j}, \dots, x_{i_{j+1}}\}$. Every such face must have the form

$$\left(\prod_{i_j \in A'} (\{x_{i_j}, x_{i_{j+1}}\} \cup B'_j) \right) \times \left(\prod_{i_j \in \text{supp}(m) \setminus A'} (\{x_{i_{j+1}}\} \cup B'_j) \right),$$

which implies that the face is contained in $\mathcal{P}_{m, \alpha}$.

If $\alpha \neq m$, then $\mathcal{P}_{m, \alpha}$ has positive dimension and satisfies the hypotheses of Lemma 6.7, so $\widehat{Y}_{\leq \alpha}$ is acyclic. If $\alpha = m$, then $\widehat{Y}_{\leq \alpha}$ is the empty complex, which is acyclic. \square

Example 6.8. The complex at the center in Figure 11 supports a minimal resolution of

$$(a, b)(b, c)(c, d)(d, a) \setminus \{abcd\}$$

and the complex in Figure 15 supports a minimal resolution of

$$(b, c)(c, d)(d, a, b) \setminus \{bcd\}$$

in $k[a, b, c, d]$.

We are now ready to prove that \widehat{I} is minimally resolved by \widehat{X} .

Theorem 6.9. *The pinched power ideal \widehat{I} has polytopal minimal resolution supported on the complex \widehat{X} .*

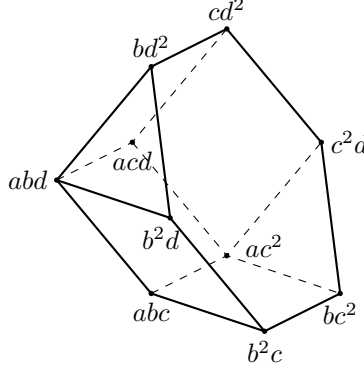


FIGURE 15. A minimal resolution of $(b, c)(c, d)(d, a, b) \setminus \{bcd\}$, inside $I = (a, b, c, d)^3$ with $m = bcd$.

Proof. Let α be given; we will verify that $\widehat{X}_{\leq \alpha}$ is acyclic. As in the proof of Lemma 6.6, we show that in every case $\widehat{X}_{\leq \alpha}$ is homeomorphic to $X_{\leq \alpha}$.

If α is not divisible by m , then $\widehat{X}_{\leq \alpha} = X_{\leq \alpha}$. If α is divisible by $\deg(\widehat{Y}) = (x_1 \dots x_n)m$, then $\widehat{X}_{\leq \alpha}$ contains \widehat{Y} and $X_{\leq \alpha}$ contains Y , so the two complexes are homeomorphic.

Otherwise, we may assume without loss of generality that α has full support.

In this case, following the proof of Lemma 6.6, m is contained in a unique facet of $Y_{\leq \alpha}$; since m is not contained in any face of $X \setminus Y$, it is thus contained in a unique facet of $X_{\leq \alpha}$. By Lemma 6.7, $X_{\leq \alpha}$ is homotopic to $\widehat{X}_{\leq \alpha}$. \square

Example 6.10. The complex in Figure 16 supports a minimal resolution of $(a, b, c, d)^3 \setminus \{bcd\}$ in $k[a, b, c, d]$.

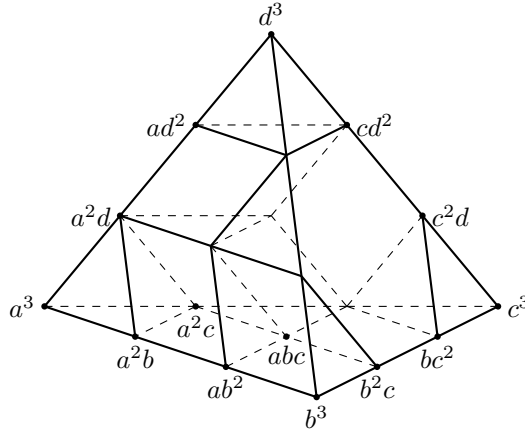


FIGURE 16. A minimal resolution of $\widehat{I} = (a, b, c, d)^3 \setminus \{bcd\}$ in $k[a, b, c, d]$.

7. BETTI NUMBERS

In this section, we will use the polytopal structure of the resolution in Theorem 6.9 to compute the graded Betti numbers of \widehat{I} .

From the construction in Theorem 6.9, the difference between the minimal resolutions of I and \widehat{I} is the same as the difference between the resolutions of J and \widehat{J} , namely the faces containing m in Y . Thus, since the Betti numbers of I are known (see, for example, [FMS1, Example 6.5]), computing the Betti numbers of \widehat{I} is equivalent to understanding the combinatorics of Y and \widehat{Y} . Doing so is a tremendously long, and not particularly enlightening, exercise. Instead, we will take advantage of the mapping cone construction.

Theorem 7.1. *Let $I = (x_1, \dots, x_n)^d$ and m be a degree d monomial which is not a pure power. Then there is a short exact sequence*

$$0 \rightarrow \frac{S}{\widehat{I}:m}(m^{-1}) \xrightarrow{\times m} \frac{S}{\widehat{I}} \rightarrow \frac{S}{I} \rightarrow 0.$$

(Here the (m^{-1}) represents a Serre twist of multidegree m .) Furthermore, $\frac{S}{\widehat{I}:m} \cong \frac{S}{(x_1, \dots, x_n)}$ is minimally resolved by the Koszul complex.

The minimal resolution of $\frac{S}{I}$ is very well known; see for example [FMS1, Example 6.5] or [BR].

Theorem 7.2. *The minimal resolution of $\frac{S}{I}$ is linear. The multigraded Betti numbers are $\beta_{0,0}(\frac{S}{I}) = 1$, $\beta_{i+1,i+d}(\frac{S}{I}) = \sum_{j=1}^d \binom{j+d-2}{d-1} \binom{j-1}{i}$ for $i < n$, and $\beta_{p,q}(\frac{S}{I}) = 0$ for all other p, q .*

$\frac{S}{I}$ is (non-minimally) resolved by the mapping cone resolution arising from the Koszul complex on (x_1, \dots, x_n) and the resolution of $\frac{S}{I}$. Since we know the two resolutions on the outside, understanding the resolution of $\frac{S}{I}$ is equivalent to understanding the cancellation in the mapping cone.

Proposition 7.3. *The cancellation in the mapping cone is precisely the non-linear strand in the resolution of $\frac{S}{I}$. That is:*

- (1) *The linear strand of $\frac{S}{I}$ is not cancelled in the mapping cone.*
- (2) *If $(p, q) \neq (i+1, i+d)$, and $\beta_{p,q}(\frac{S}{I}) \neq 0$, then all generators for $\beta_{p,q}(\frac{S}{I})$ are cancelled in the mapping cone.*

Proof. For (1), observe that $\text{Tor}_{i+1}(\frac{S}{\widehat{I}:m}(m^{-1}), k)$ is concentrated in degree $i+d+1$, while the linear strand of $\text{Tor}_{i+1}(\frac{S}{I}, k)$ is concentrated in degree $i+d$. Since the degrees are unequal, there can be no cancellation.

For (2), observe that $\text{Tor}_{i+1}(\frac{S}{I}, k)$ is concentrated in degree $i+d$. Thus any generators for $\text{Tor}_{i+1}(\frac{S}{I}, k)$ in different degrees must cancel. \square

To count the non-linear strand of \widehat{I} , it suffices to determine the non-linear faces of \widehat{Y} . The Betti number $\beta_{p,p+d}$ is counted by the nonlinear faces of dimension p .

Proposition 7.4. *The number and dimension of the non-linear faces of \widehat{Y} depends on the cardinality of $\text{supp}(m)$. In particular,*

- (1) *If $|\text{supp}(m)| = 1$, then \widehat{Y} has no non-linear faces.*

- (2) If $|\text{supp}(m)| = s \neq 1$, then the non-linear faces of \widehat{Y} of dimension p are in bijection with the subsets of $\{1, \dots, n\} \setminus \text{supp}(m)$ with cardinality $n - 1 - p$. Each such face has degree $d + p + 1$.

Proof. If $|\text{supp}(m)| = 1$, then, without loss of generality, $m = x_1^d$ and $\widehat{Y} = [x_1^d] \times \{x_2, \dots, x_n\}$.

Otherwise, the faces of \widehat{Y} which are not faces of Y are precisely the intersections of (the geometric realization of) Y with the coordinate hyperplane $\langle x_j = 0 : j \in A \rangle$ for each subset A such that $A \cap \text{supp}(m) = \emptyset$. Each such intersection has dimension $n - 1 - |A|$ and multidegree $m \prod_{j \notin A} x_j$. \square

Corollary 7.5. *If $|\text{supp}(m)| = s > 1$, then the graded Betti numbers of $\frac{S}{I}$ are given by:*

$$\begin{aligned} \beta_{0,0}\left(\frac{S}{I}\right) &= 1 \\ \beta_{p,d+p-1}\left(\frac{S}{I}\right) &= \beta_{p,d+p-1}\left(\frac{S}{I}\right) - \binom{n}{p} + \binom{n-s}{n-p} \\ \beta_{p,d+p}\left(\frac{S}{I}\right) &= \binom{n-s}{n-p-1} \\ \beta_{p,q}\left(\frac{S}{I}\right) &= 0 \quad \text{otherwise.} \end{aligned}$$

The Poincaré polynomial $P_{\frac{S}{I}}(t, u) = \sum \beta_{p,q}\left(\frac{S}{I}\right) t^p u^q$ is given by

$$P_{\frac{S}{I}}(t, u) = P_{\frac{S}{I}}(t, u) - tu^d(1 + tu)^n + (1 + t)t^{s-1}u^{s+d}(1 + tu)^{n-s}.$$

Proof. The Betti numbers of the non-minimal mapping cone resolution for $\frac{S}{I}$ are obtained by adding the corresponding Betti numbers for $\frac{S}{I}$ and $\frac{S}{I:m}(m^{-1})$. The Betti numbers for the minimal resolution are then obtained by subtracting the cancellations. By Proposition 7.3, these cancellations affect $\beta_{p,d+p}\left(\frac{S}{I}\right)$ and $\beta_{p+1,d+p}\left(\frac{S}{I:m}\right)(m^{-1})$. The size of the cancellation is given by Proposition 7.4. Rearranging the algebra yields formulas for $\beta\left(\frac{S}{I}\right)$. \square

Example 7.6. Let $I = (a, b, c, d)^4$. The Betti table for $\frac{S}{I}$ is

	0	1	2	3	4
total:	1	35	84	70	20
0 :	1
1 :
2 :
3 :	.	35	84	70	20

The Betti tables for $\frac{S}{I}$ for various choices of m are given below.

Figure 17 yields a minimal resolution. (Note that the grey “hexagon”, which has the deleted monomials on its interior, is not actually a hexagon; for example, in Figure 17, it has nine vertices and edges.)

We speculate that a similar construction will work in general for these ideals.

Question 8.3. Let I be an almost-linear ideal, built by deleting several disjoint simplices $\mathcal{S}_{d,f}$. For each $\mathcal{S}_{d,f}$, let $\mathcal{T}_{d,f}$ be the smallest polytope with lattice points as vertices and containing every point of $\mathcal{S}_{d,f}$ on its interior. Under what conditions on the collection of $\mathcal{S}_{d,f}$ is I minimally resolved by a polytopal complex consisting of the $\mathcal{T}_{d,f}$ and oriented boxes $\Gamma_i(m)$?

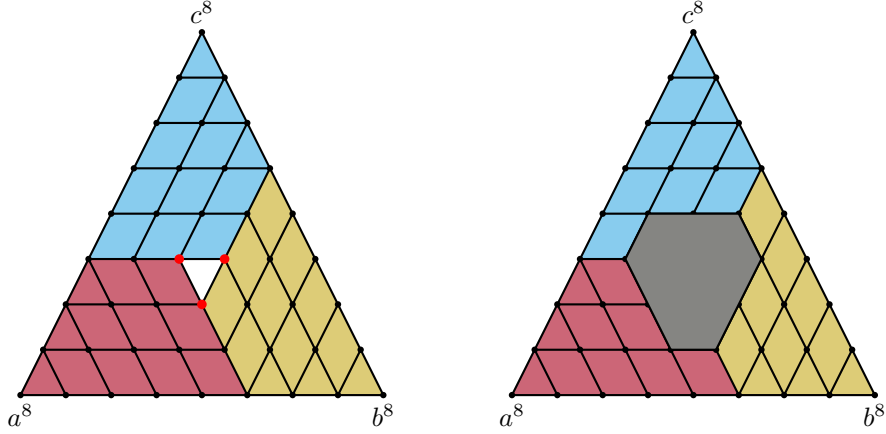


FIGURE 17. A polytopal complex supporting the minimal resolution of $(a, b, c)^8 \setminus \mathcal{S}_{8, a^3 b^3 c^3}$.

Question 8.2 is well understood in three variables, where every monomial ideal is known to have a polytopal resolution ([Mi]). The situation is considerably wilder in four or more variables; it's not obvious that polytopal resolutions are the right target.

We refine Question 8.2 by making the following definitions:

Definition 8.4. Let I be an equigenerated monomial ideal. We say that a minimal resolution of I is:

- A *box resolution* if it is supported on a polytopal complex in which every cell is an admissible box, in the sense of Definition 2.18.
- A *cyclic box resolution* if it is supported on a polytopal complex in which every cell is a box of the form $\Gamma_i(m)$, for some monomial m and some cyclic ordering Q_i .
- A *box-plus resolution* if it is supported on a polytopal complex in which every cell is either some $\Gamma_i(m)$ or a copy of \widehat{Y} .

Example 8.5. The complex of boxes is a cyclic box resolution, for any Borel ideal.

All the resolutions of $\frac{S}{I}$ constructed earlier in the paper are cyclic box resolutions. The resolutions of $\frac{S}{\widehat{I}}$ in Figures 4.11, 10, 11, 15, and 16 are all box-plus resolutions, and will become cyclic box resolutions if we replace their copy of \widehat{Y} with Y .

Another example of a box-plus resolution (in three variables) appears in Figure 18.

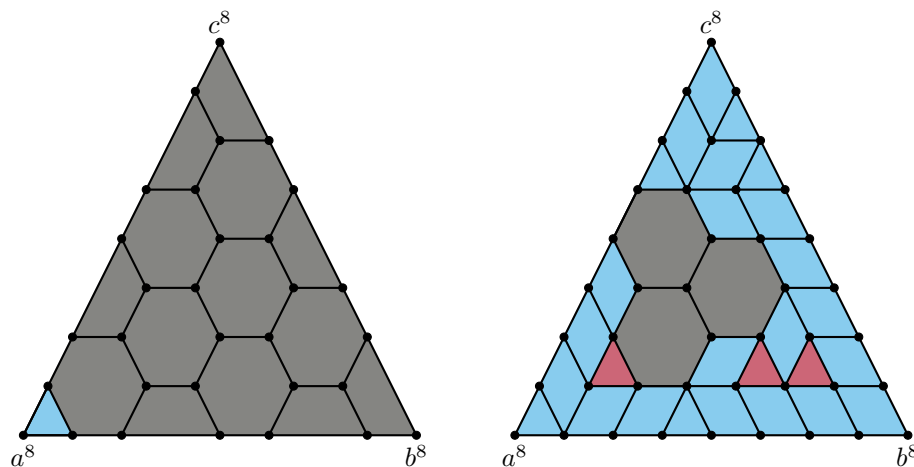


FIGURE 18. The complex on the left supports a box-plus resolution. The complex on the right does not, because the red triangles are admissible boxes but not equal to $\Gamma_i(m)$ for any i and m .

Example 8.6. Let \widehat{I} be the ideal of $k[a, b, c]$ generated by all degree-five monomials except a^3bc , ab^3c , and abc^3 . Then \widehat{I} does not admit a box-plus resolution: We would need the hexagons around the three omitted generators, as in figure 8, creating the isolated triangular region in the center, which is too small to be tiled by any Γ_i . (In fact, because this triangle has negative orientation, it is not a box and cannot be tiled by boxes.) However, the complex in this picture does support a non-minimal resolution of $\frac{S}{\widehat{I}}$. Deleting any one of the three sides of this triangle yields a polytopal minimal resolution that is not box-plus.

It is natural to ask which monomial ideals support which type of resolution. In particular, the difference between a box resolution and a cyclic box resolution seems important. For example, if $n = 4$ and the generating degree is large, then almost every cell of the Eliahou-Kervaire resolution has the form $\{f\} \times \{a, d\} \times \{b, d\} \times \{c, d\}$ for some f . These cells are admissible boxes, but are not equal to any $\Gamma_i(m)$.

On the other hand, we can make some substantial statements about cyclic box resolutions and box-plus resolutions by rephrasing these objects in terms of staircase diagrams, as we do below.

Staircase diagrams have a long history in the study of monomial ideals; see for example [MS, Chapters 3,4,6] or [Ba]. Normally these are studied as a Stanley

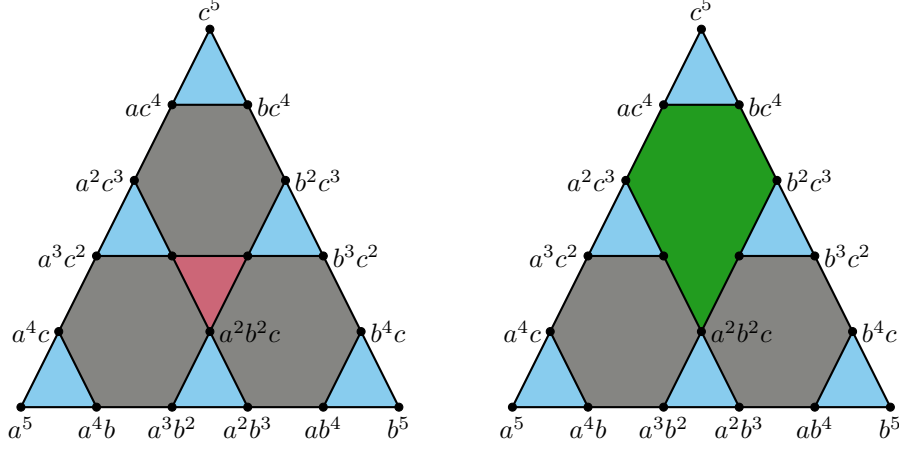


FIGURE 19. Let \widehat{I} be the ideal generated by all degree five monomials except a^3bc , ab^3c , and abc^3 . The complex on the left supports a non-minimal resolution of \widehat{I} . The complex on the right supports a minimal resolution, but is not box-plus.

decomposition adjacent to the Newton polytope, (so that corners correspond to generators or syzygies,) but we will use them somewhat differently to parametrize resolutions.

Remark 8.7. We note that the staircase diagrams discussed below feel similar to the tropical pictures found in [DJS]. While we are far from experts in tropical geometry, we hope that the subject might have something interesting to say about these questions.

Definition 8.8. A *staircase diagram* (see Figure 20 for an example of a staircase diagram) in \mathbb{R}^n is a nonempty infinite n -dimensional cubical complex $\mathcal{S} \subset \mathbb{R}^n$ satisfying:

- (1) Every vertex is a lattice point.
- (2) Every vertex is the outermost corner of a d -dimensional cube. More precisely, if $\mathbf{v} = (a_1, \dots, a_n)$ is a vertex of \mathcal{S} , then the hypercube $[a_1 - 1, a_1] \times [a_2 - 1, a_2] \times \dots \times [a_n - 1, a_n]$ is a facet of \mathcal{S} .

The *visible surface* of \mathcal{S} is the $(n - 1)$ -dimensional cubical complex consisting of the faces of \mathcal{S} visible from infinitely far into the first orthant in the direction $(1, 1, 1, \dots, 1)$.

Construction 8.9. Fix a degree d monomial m and a staircase diagram \mathcal{S} . To the lattice point (a_1, a_2, \dots, a_n) , we associate the monomial

$$m \cdot (x_1^{a_1 - a_2} x_2^{a_2 - a_3} \dots x_n^{a_n - a_1}) = m \left(\frac{x_1}{x_n} \right)^{a_1} \left(\frac{x_2}{x_1} \right)^{a_2} \dots \left(\frac{x_n}{x_{n-1}} \right)^{a_n}$$

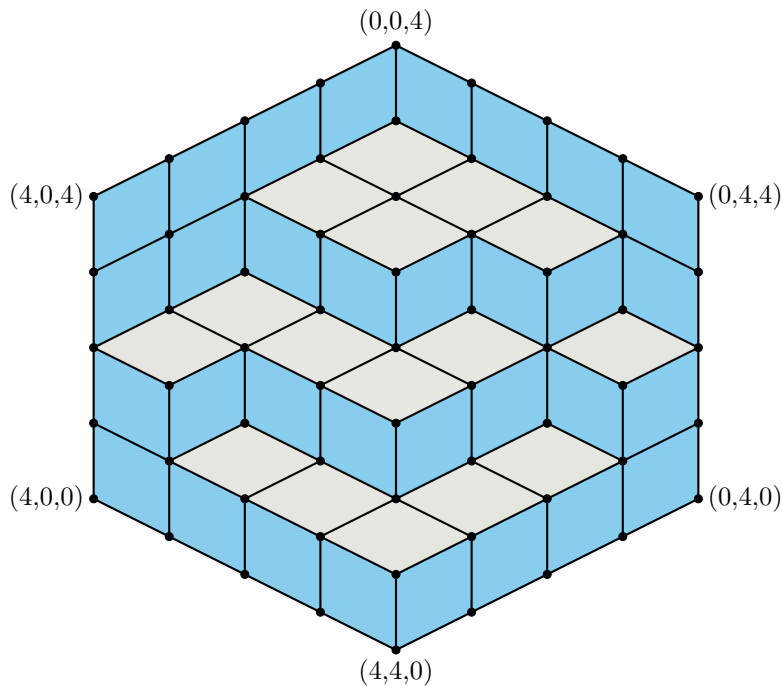


FIGURE 20. An example of a staircase diagram.

in $k[x_1, \dots, x_n, x_1^{-1}, \dots, x_n^{-1}]$. (Note that this means the origin is associated to m .) Label every vertex of the visible surface of \mathcal{S} by the associated monomial. We refer to the resulting labeled staircase diagram as \mathcal{S}_m .

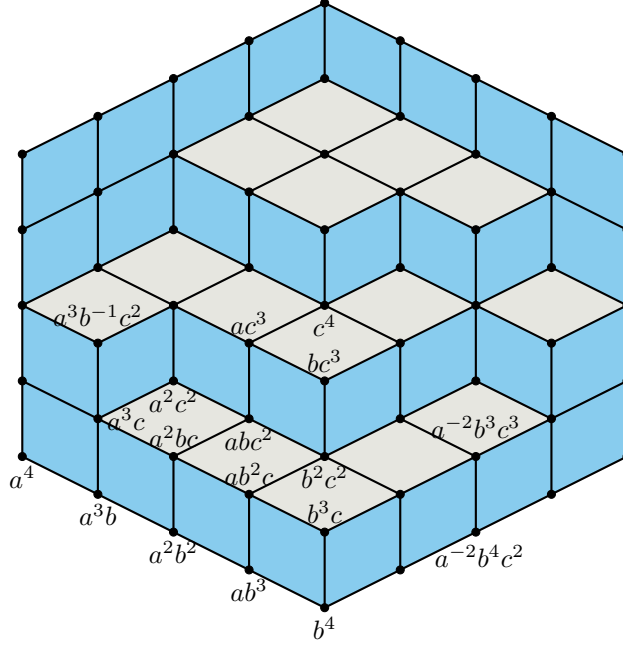
We illustrate Construction 8.9 via the following example.

Example 8.10. Consider the staircase diagram given in Figure 20, and set $m = c^4 \in k[a, b, c]$. Then $(2, 2, 2)$ is labeled with $m = m \cdot a^0 b^0 c^0$. Meanwhile, $(3, 2, 2)$ is labeled with $c^4 \cdot (a^1 b^0 c^{-1}) = ac^3$. Similarly, we can obtain labels of all lattice points (see Figure 21 for labels of some lattice points of the staircase diagram).

Remark 8.11. Observe that for any lattice point v , the labels on v and $v + (1, 1, \dots, 1)$ are equal. In fact, the labeling in Construction 8.9 provides a bijection among the set of lattice points on the visible surface of \mathcal{S} , the set of lattice points in \mathbb{R}^n (modulo translation by $(1, 1, \dots, 1)$), and the set of degree d monomials in the ring $k[x_1, \dots, x_n, x_1^{-1}, \dots, x_n^{-1}]$. (Indeed, this bijection is given by orthogonal projection along the vector $(1, 1, \dots, 1)$ onto the degree- d hyperplane $\sum x_i = d$.)

Remark 8.12. The labeling of the visible surface of \mathcal{S}_m can be achieved by the following inductive procedure:

First, choose an arbitrary vertex v on the visible surface of \mathcal{S} , and label v as described in Construction 8.9. Then label the remaining vertices of the visible surface inductively as follows:

FIGURE 21. An example of \mathcal{S}_m .

If w is already labeled by a monomial m , and u is adjacent to w , then $u - w = \pm e_j$ for some j . We label u with $m \frac{x_j}{x_{j-1}}$ if the coefficient on e_j is positive, and with $m \frac{x_{j-1}}{x_j}$ if the coefficient is negative.

Construction 8.13. Fix a labeled staircase diagram \mathcal{S}_m , and let T_m be its orthogonal projection onto the degree d hyperplane $\mathcal{H} := \{\mathbf{v} = (a_1, \dots, a_n) : a_1 + \dots + a_n = d\}$. Observe that T_m is an (infinite) polytopal complex that tiles the hyperplane.

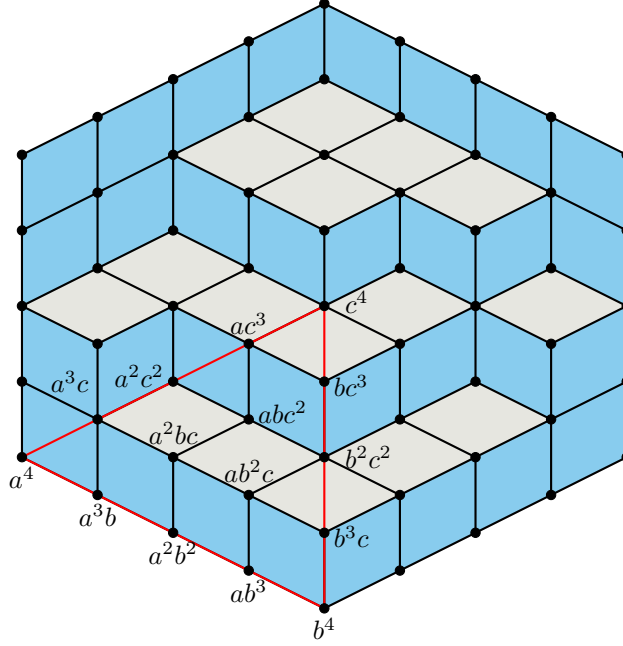
Consider the convex hull \mathcal{F}_m in \mathcal{H} of all vertices whose labels are in $k[x_1, \dots, x_n]$ (i.e., the vertices with no negative exponents), and view \mathcal{F}_m as a simplex in the obvious way. Make \mathcal{F}_m into a polytopal complex by setting the i -dimensional faces equal to the i -dimensional intersections between faces of \mathcal{F}_m and faces of T_m .

Example 8.14. In Figure 22, the simplex \mathcal{H} is bounded by the red triangle. Observe that the red line cuts across some diagonals of faces in \mathcal{S} , so \mathcal{H} contains some triangular faces (e.g., $\{ab^2c, b^3c, b^2c^2\}$) in addition to the many square faces arising from the projection.

Proposition 8.15. Fix a staircase, a vertex, and a monomial. Then Construction 8.13 produces a cyclic box resolution.

Proof. Fix a labeled staircase \mathcal{S}_m . The cells arising from Construction 8.13 are clearly boxes, but we need to verify that the facets have the form $\Gamma_i(\mu)$ for some μ . We also need to show that Construction 8.13 in fact supports a resolution.

We first show that every facet has the form $\Gamma_i(\mu)$. Let a facet F be given; then F corresponds to an $n - 1$ -dimensional face of \mathcal{S} , which by abuse we also refer to as F . Let \mathbf{v} be the outermost vertex of F (that is, the vertex maximizing the dot product

FIGURE 22. An example of T_m .

$\mathbf{v} \bullet (1, 1, \dots, 1)$), and observe that, without loss of generality F is the intersection of \mathcal{F}_m with the convex hull of the 2^{n-1} points $\mathbf{v} - \sum_{j \in \sigma} e_j$, the sum taken over all subsets $\sigma \subset \{2, \dots, n\}$.

Let \mathbf{v} be labeled by μ , and observe that the vertex $\mathbf{v} - \sum_{j \in \sigma} e_j$ is labeled by $\mu \cdot \prod_{j \in \sigma} \frac{x_j - 1}{x_j}$. Thus the convex hull of the $\mathbf{v} - \sum_{j \in \sigma} e_j$ has the form $\frac{\mu}{x_2 \dots x_n} \times [x_1, x_2] \times [x_2, x_3] \times \dots \times [x_{n-1}, x_n]$. If $\text{supp}(\mu)$ contains $\{2, \dots, n\}$, this convex hull is $\Gamma_1(\mu)$. If not, we obtain $\Gamma_1(\mu)$ upon intersecting with \mathcal{F}_m .

To verify that the construction supports a resolution, fix a multidegree $\alpha = x_1^{\alpha_1} \dots x_n^{\alpha_n}$; we will show that $(\mathcal{F}_m)_{\leq \alpha}$ is contractible.

If any $\alpha_i < 0$, $(\mathcal{F}_m)_{\leq \alpha}$ is empty and hence contractible.

We may assume without loss of generality that $0 \leq \alpha_i \leq d$.

Let \mathcal{G}_α be the polytope described by the half-spaces $\{a_j \geq 0\}_j$ and $\{a_j \leq \alpha_j\}_j$, and observe that \mathcal{G}_α is contractible. We will show that $\mathcal{F}_{\leq \alpha}$ is homotopic to \mathcal{G}_α .

Observe that every facet Γ of \mathcal{F}_m is equal to $\Gamma_i(\mu)$ for some monomial μ . Writing $\mu = \prod x_i^{t_i}$, we see that Γ is the set of all points (a_1, \dots, a_n) satisfying the equations

$$\begin{pmatrix} t_i & \leq & a_i & \leq & t_i + 1 \\ \max(0, t_{i-1} - 1) & \leq & a_{i-1} & \leq & t_{i-1} \\ \max(0, t_j - 1) & \leq & a_j & \leq & t_j + 1 \end{pmatrix}$$

We have that $\Gamma \subset \mathcal{F}_{\leq \alpha}$ if and only if $\Gamma \subset \mathcal{G}_\alpha$. If $\Gamma \cap \mathcal{G}_\alpha$ is a nontrivial proper subset of Γ , set $A = \{j : \alpha_j = t_j\}$, the set of indices such that the hyperplane $(a_j = \alpha_j)$ slices through Γ . Movement of every point on the interior of $\Gamma \cap \mathcal{G}_\alpha$ along the vector $\sum_{j \in A} e_j$ defines a deformation retract from $\Gamma \cap \mathcal{G}_\alpha$ to $\Gamma \cap \mathcal{F}_{\leq \alpha}$. The union of all

these deformations is a deformation retract from \mathcal{G}_α to $\mathcal{F}_{\leq \alpha}$. In particular, $\mathcal{F}_{\leq \alpha}$ is homotopic to \mathcal{G}_α and thus is contractible. \square

Proposition 8.16. *Every cyclic box resolution arises from a staircase diagram via Construction 8.13.*

Proof. Let X be a cyclic box resolution. We will build a corresponding staircase diagram, following the intuition of Remark 8.12.

Choose an arbitrary vertex m of X and an arbitrary point P in \mathbb{Z}^n , and label P with m . Then assign the remaining vertices of X to points in \mathbb{Z}^n recursively as follows: If f has already labeled a point Q and g is connected to f by an edge of X , then for some monomial m and index i , f and g are both vertices of $\Gamma_i(m)$. It follows that there exist indices j and j' such that either $g = f \frac{x_j}{x_{j'}}$ or $g = f \frac{x_{j'}}{x_j}$ and, cyclically, $i \leq j < j' \leq i-1$. If $g = f \frac{x_j}{x_{j'}}$, assign the label g to $Q + e_{j+1} + e_{j+2} + \cdots + e_{j'}$. If $g = f \frac{x_{j'}}{x_j}$, assign the label g to $Q - e_{j+1} - e_{j+2} - \cdots - e_{j'}$.

Our staircase diagram is then the cubical complex built by taking the shadow of all labeled vertices (that is, all points $A = (a_1, \dots, a_n) \in \mathbb{R}^n$ such that, for some labeled vertex $Q = (q_1, \dots, q_n)$, we have $a_i \leq q_i$ for all i). \square

Remark 8.17. The map from staircases to cyclic box resolutions is surjective but not injective, as illustrated by Figure 23.

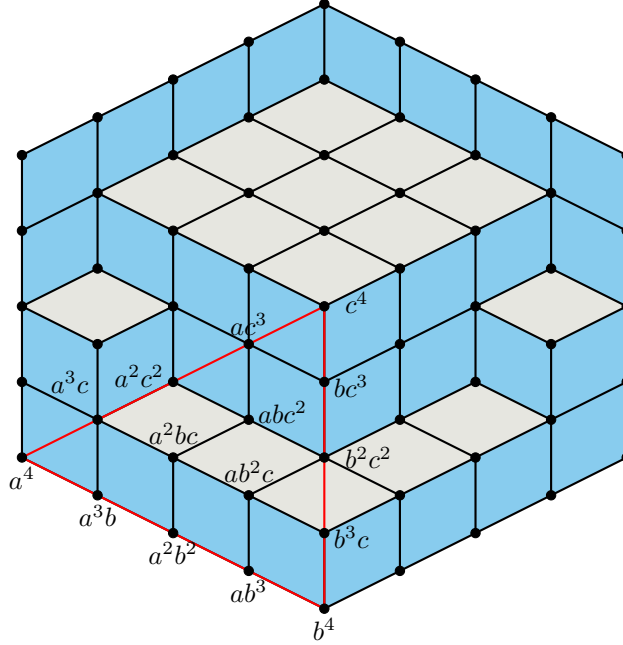


FIGURE 23. This staircase diagram is different from the one given in Figure 20, because the face containing c^4 is perpendicular to the x_1 -axis instead of the x_3 -axis. Nevertheless, the two diagrams support the same resolution for $I = (a, b, c)^4$.

Definition 8.18. Fix a labeled staircase diagram \mathcal{S} and a monomial m . We say that \mathcal{S} has an *exterior corner* at m if there exists a vertex v on the visible surface of \mathcal{S} such that:

- v is labeled with m .
- For all i , $v - e_i$ is on the visible surface of \mathcal{S} .

Proposition 8.19. *Fix a monomial ideal I , equigenerated in degree d . Then I has a minimal box-plus resolution if and only if there exists a labeled staircase diagram with exterior corners at the degree- d monomials not contained in I .*

Proof. First, suppose that \hat{I} admits a box-plus resolution, supported on the polytopal complex \hat{X} . Then the facets of \hat{X} are either boxes of the form $\Gamma_i(m)$ or copies of \hat{Y}_m , centered at various omitted monomials m . Replacing each Y_m with the collection of boxes $\Gamma_i(m)$ (for all i) yields a box-plus resolution whose faces include all $\Gamma_i(m)$. Applying Proposition 8.16 gives us a staircase diagram in which all m are exterior corners, as desired.

In the other direction, suppose \mathcal{S} is a staircase diagram with exterior corners labeled by various monomials m . Applying Proposition 8.15 yields a cyclic box resolution in which $\Gamma_i(m)$ is a facet for all m and all i . For each m , we may replace the collection of $\Gamma_i(m)$ with a copy of $\hat{Y}(m)$; the result is a box-plus resolution. \square

Proposition 8.19 provides a new perspective on Question 8.2, allowing us to make several statements about which “dense” ideals admit box-plus resolutions. We begin with the deletion of exactly two monomials.

Lemma 8.20. *Let \mathbf{v} and \mathbf{v}' be two distinct lattice points in \mathbb{R}^n . Then there exists a staircase diagram with exterior corners at \mathbf{v} and \mathbf{v}' if and only if \mathbf{v} and \mathbf{v}' are incomparable.*

Proof. Take \mathcal{S} equal to the collection of all points \mathbf{w} such that $\mathbf{w} \preceq \mathbf{v}$ or $\mathbf{w} \preceq \mathbf{v}'$. (Here, $\mathbf{w} \preceq \mathbf{v}$ means that every entry of \mathbf{w} is less than or equal to the corresponding entry of \mathbf{v} .) \square

Corollary 8.21. *Let $m = \prod x_i^{a_i}$ and $m' = \prod x_i^{b_i}$ be two monomials of degree d , and let \hat{I} be the ideal generated by all degree- d monomials except m and m' . The following are equivalent:*

- (1) \hat{I} has a minimal box-plus resolution.
- (2) For some (not necessarily distinct) i and j , the cyclic sum $\sum_{t=i}^{t=j} (a_t - b_t)$ is greater than or equal to 2.
- (3) For some (not necessarily distinct) i and j , the cyclic sum $\sum_{t=i}^{t=j} (a_t - b_t)$ is less than or equal to -2 .
- (4) The vector $(a_1 - b_1, a_2 - b_2, \dots, a_n - b_n)$ either contains an entry not in $\{0, \pm 1\}$ or contains two nonzero entries of the same sign separated only by zeroes. (That is, if we were to remove all the zeroes from this vector, there should be either a 2 or two consecutive entries with the same sign.)
- (5) m is not contained in $\Gamma_i(m')$ for any i .
- (6) m' is not contained in $\Gamma_i(m)$ for any i .

Proof. First observe that (2) and (3) are equivalent since $\sum_{t=1}^n (a_t - b_t) = \deg(m) - \deg(m') = 0$. Observe also that (2) and (3) are equivalent to (4): Given (2), choose i and j with $j - i$ minimal. If $j = i$, we have $a_i - b_i \geq 2$. Otherwise, we have

$a_i - b_i = a_j - b_j = 1$ and $a_t - b_t = 0$ whenever $i < t < j$. Conversely, given (4), if $a_i - b_i \geq 2$, we may take $j = i$; otherwise, if $a_i - b_i = a_j - b_j = 1$ and $a_t - b_t = 0$ for all $i < t < j$ we have (2) immediately.

Next, observe that (5) and (6) are equivalent since $m' \in \Gamma_i(m)$ means that $m' = m \prod_{t \in \sigma} \frac{x_{t-1}}{x_t}$ for some nontrivial proper $\sigma \subset \{1, \dots, n\}$, which is equivalent to $m = m' \prod_{t \notin \sigma} \frac{x_{t-1}}{x_t}$, i.e., $m \in \Gamma_j(m')$ for some j .

To see that (1) implies (5), it suffices to observe that, if (5) fails, the two copies of \hat{J} centered around m and m' must contain m' and m as vertices, respectively.

The remaining implications are more technical.

(5) implies (2),(3),(4): Suppose (2) fails, so for every i and j we have $\sum_{t=i}^{t=j} (a_t - b_t) \leq 1$. Without loss of generality, assume $a_1 - b_1 = 1$. For all j , set $c_i = \sum_{t=1}^{t=j} (a_t - b_t)$. Observe that $c_j \in \{0, 1\}$ for all j : If $c_j \geq 2$, then (2) would hold, and if $c_j < 0$, then $\sum_{t=j+1}^n (a_t - b_t) > 0$, so the cyclic sum $\sum_{t=j+1}^1 (a_t - b_t)$ is greater than 1, contradicting the failure of (2). Now observe that $m = m' \prod_{c_j=1} \frac{x_j}{x_{j+1}}$. It follows that $m \in \Gamma_1(m')$ (since $c_n = \deg(m) - \deg(m') = 0$).

(2) implies (1): Assume (2) holds, and label the lattice points of \mathbb{R}^n according to construction 8.9 so that the origin (and all points of the form $\mathbf{w}_s = s(1, 1, \dots, 1)$) are labeled by m . It follows that the points

$$\mathbf{v}_s = \left(\sum_{t=1}^{n-1} (b_t - a_t), \sum_{t=2}^{n-1} (b_t - a_t), \dots, \sum_{t=n-1}^{n-1} (b_t - a_t), 0 \right) + s(1, 1, \dots, 1)$$

are all labeled by m' .

Since (2) holds, without loss of generality we have $\sum_{t=n}^{t=j} (a_t - b_t) \geq 2$ for some j . Thus the entry in the $(j+1)^{\text{th}}$ spot of \mathbf{v}_0 is $\sum_{t=j+1}^{n-1} (a_t - b_t) \leq -2$. It follows that \mathbf{v}_0 and \mathbf{w}_{-1} are incomparable (since \mathbf{v}_0 is smaller in the $(j-1)^{\text{th}}$ position and larger in the n^{th} position). In particular, there exists a staircase diagram with exterior corners at \mathbf{v}_0 and \mathbf{w}_{-1} by Lemma 8.20; this diagram supports a minimal box-plus resolution of \hat{I} by Proposition 8.19. \square

Remark 8.22. Lemma 8.21 holds for two monomials but not three. Consider the monomial ideal I of Example 8.6, which is generated by all degree five monomials except a^3bc , ab^3c , and abc^3 . The omitted generators satisfy the conditions of Lemma 8.21 pairwise, but the ideal does not have a box-plus resolution. The problem is that, if we attempt to build the necessary staircase diagram, we end up with a triangle connecting the monomials a^2b^2c , a^2bc^2 , and ab^2c^2 . Since the triangle is an odd cycle, attempting to build a staircase diagram for I by assigning lattice points as in Remark 8.12 will yield an Escher-style infinite staircase around this triangle, forcing all choices of $\mathbf{v} + s(1, 1, 1)$ for the corresponding lattice points.

Question 8.23. Are forced odd cycles like in Remark 8.22 the only obstruction to box-plus resolutions? Or are there more interesting ways for things to go wrong, especially with more than three variables?

Proposition 8.24. Suppose $m_1 = \prod x_i^{a_{1,i}}, \dots, m_s = \prod x_i^{a_{s,i}}$ are monomials of degree d with distinct exponents on x_n , ordered so that $a_{1,n} < a_{2,n} < \dots < a_{s,n}$. Suppose further that all pairs (m_i, m_j) satisfy the conditions of Corollary 8.21. Set \hat{I} equal to the ideal generated by all degree d monomials except m_1, \dots, m_s . Then \hat{I} has a minimal box-plus resolution.

Proof. We will construct a labeled staircase diagram that has exterior corners labeled with each of the m_i .

Label the origin with x_1^d . Given any monomial μ , recall from Remark 8.11 that the collection of vertices labeled by μ forms a line with direction vector $(1, 1, \dots, 1)$. In particular, given μ and an integer t , there is a unique vertex $\mathbf{v}_{t,\mu} = (c_{t,1}, c_{t,2}, \dots, c_{t,n-1}, t)$ with last entry t and label μ .

For each monomial $\mu = x_1^{b_1} \dots x_n^{b_n}$, define a statistic $\phi(\mu)$ to be the maximal length of a chain

$$\mu \geq_{Q_n} m_{i_1} >_{Q_n} m_{i_2} >_{Q_n} \dots >_{Q_n} m_{i_{\phi(\mu)}},$$

where \geq_{Q_n} is the Borel partial order with respect to the order $x_n < x_1 < x_2 < \dots < x_{n-1}$; that is, $m_i \geq_{Q_n} m_j$ if the cyclic inequalities $\sum_{p=n}^r a_{i,p} \geq \sum_{p=n}^r a_{j,p}$ hold for all i .

Let \mathcal{S} be the staircase diagram consisting of all $\mathbf{v}_{\phi(\mu),\mu}$ for monomials μ of degree d (which includes both the m_i and the generators of \widehat{I} , and all lattice points that are below any of the $\mathbf{v}_{\phi(\mu),\mu}$). We claim that \mathcal{S} has exterior corners at $\mathbf{v}_{\phi(m_i),m_i}$.

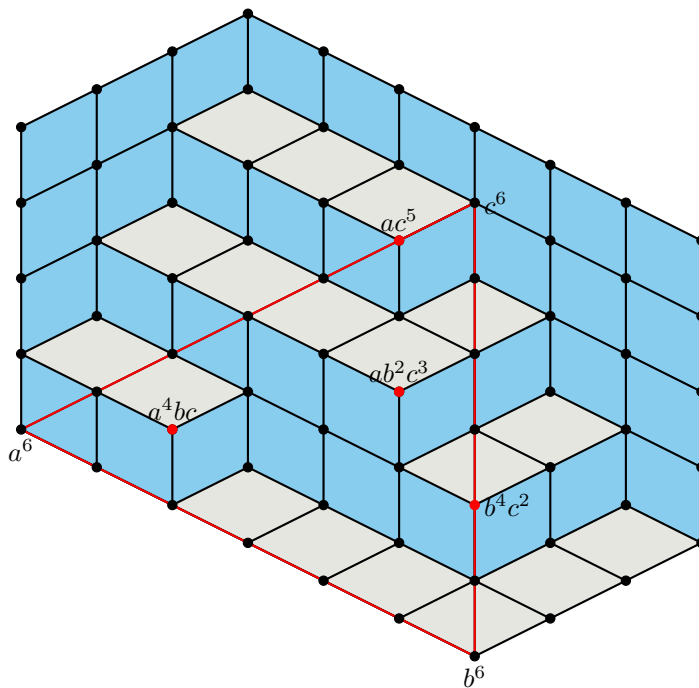


FIGURE 24. This complex supports a minimal resolution for $I = (a, b, c)^6$ with exterior corners at a^4bc , b^4c^2 , ab^2c^3 , and ac^5 . The staircase is constructed by assigning each monomial μ to the vertex $\mathbf{v}_{\phi(\mu),\mu}$, where $\phi(\mu)$ is defined as in the proof of Proposition 8.24.

It suffices to show that, for any nontrivial nonnegative $\mathbf{u} = \sum c_j e_j$, we have $\mathbf{v}_{\phi(m_i),m_i} + \mathbf{u} \notin \mathcal{S}$, for which it is enough to show that $\mathbf{v}_{\phi(m_i),m_i} + \mathbf{u} \neq \mathbf{v}_{\phi(\mu),\mu}$ for any μ . Suppose to the contrary that, for some $\mu = \mu_{\mathbf{u}}$, we have $\mathbf{v}_{\phi(m_i),m_i} + \mathbf{u} = \mathbf{v}_{\phi(\mu_{\mathbf{u}}),\mu_{\mathbf{u}}}$.

First observe that, since $\mathbf{v}_{\phi(m_i), m_i} + \mathbf{u} = \mathbf{v}_{\phi(\mu_{\mathbf{u}}), \mu_{\mathbf{u}}}$, we have $\phi(\mu_{\mathbf{u}}) = \phi(m_i) + c_n$ immediately. If $c_n = 0$, then $\phi(\mu_{\mathbf{u}}) = \phi(m_i)$, so there exists a chain

$$\mu_{\mathbf{u}} \geq m_{j_1} >_{Q_n} m_{j_2} >_{Q_n} \cdots >_{Q_n} m_{j_{\phi(\mu_{\mathbf{u}})}}.$$

On the other hand, $\mu_{\mathbf{u}} = m_i \left(\frac{x_1}{x_n}\right)^{c_1} \left(\frac{x_2}{x_1}\right)^{c_2} \cdots \left(\frac{x_{n-1}}{x_{n-2}}\right)^{c_{n-1}}$, so $m_i >_{Q_n} \mu_{\mathbf{u}}$ and the chain

$$m_i \geq_{Q_n} m_i >_{Q_n} m_{j_1} >_{Q_n} \cdots >_{Q_n} m_{j_{\phi(\mu_{\mathbf{u}})}}$$

has length $\phi(m_i) + 1$, a contradiction.

If $c_n = 1$, there exists a chain

$$\mu_{\mathbf{u}} \geq_{Q_n} m_{j_1} >_{Q_n} m_{j_2} >_{Q_n} \cdots >_{Q_n} m_{j_{\phi(\mu_{\mathbf{u}})}}.$$

By assumption, the x_n -exponent on m_{j_2} is strictly less than the x_n -exponent on m_{j_1} . Since $\mu_{\mathbf{u}} \geq_{Q_n} m_{j_1} >_{Q_n} m_{j_2}$, we conclude that $\mu_{\mathbf{u}-e_n} \geq_{Q_n} m_{j_2}$. Since $c_n = 1$, $m_i \geq_{Q_n} \mu_{\mathbf{u}-e_n}$, and we have the chain

$$m_i \geq_{Q_n} m_i \geq_{Q_n} m_{j_2} >_{Q_n} \cdots >_{Q_n} m_{j_{\phi(\mu_{\mathbf{u}})}}.$$

Since $\phi(m_i) \neq \phi(\mu_{\mathbf{u}})$, the second inequality cannot be strict, i.e., $m_{j_2} = m_i$ and $\mathbf{u} = e_n$. But then $m_{j_1} = m_i \frac{x_n}{x_1}$, violating the conditions of Corollary 8.21.

Finally, if $c_n > 1$, a similar argument shows that $m_{j_{c_n}} = m_i \frac{x_n}{x_1}$. \square

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