# Monomial regular sequences

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Abstract: We study Hilbert functions of ideals containing a regular sequence of monomials.

## 1 Introduction

Let $B = k[x_1, \cdots, x_n]$ be a polynomial ring over a field $k$, and $I \subset B$ a homogeneous ideal. One of the most important invariants of $I$ is its Hilbert function, $\mathrm{Hilb}_I^B(d) = \dim_k I_d$. We study Hilbert functions of ideals containing a regular sequence of monomials.

Macaulay [Ma] showed in 1927 that all Hilbert functions over $B = k[x_1, \cdots, x_n]$ are attained by lex ideals. Over what quotients of $B$ is this true? Let $M$ be a monomial ideal of $B$. We say that $M$ is *Lex-Macaulay* if every Hilbert function over $B/M$ is attained by a lex ideal of $B/M$. Clements and Lindström [CL] proved that $(x_1^{e_1}, \cdots, x_n^{e_n})$ is Lex-Macaulay for $e_1 \leq \cdots \leq e_n$. In section 4 we classify the monomial regular sequences which are Lex-Macaulay: Theorem 4.8 says that a regular sequence of monomials is Lex-Macaulay if and only if it has the form $(x_1^{e_1}, \cdots, x_{r-1}^{e_{r-1}}, x_r^{e_r-1}y)$, with $e_1 \leq \cdots \leq e_r$ and $y = x_i$ for some $i \geq r$.

Eisenbud, Green, and Harris [EGH1, EGH2] made the following conjecture motivated by applications in algebraic geometry:

**Conjecture 1.1 (Eisenbud-Green-Harris).** *Let $N$ be any homogeneous ideal containing a regular sequence in degrees $e_1 \leq \cdots \leq e_r$. There is a lex ideal $L$ such that $N$ and $L + (x_1^{e_1}, \cdots, x_r^{e_r})$ have the same Hilbert function.*

In the original conjecture, $r = n$. The conjecture is wide open. We prove it for ideals containing a regular sequence of monomials in Theorem 3.10.

The main tool in our proofs is $S$-compression, which we introduce in Section 3. It generalizes the notion of $x_i$-compression used in [CL, MP1, MP2].

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## 2 Preliminaries

We give some definitions and notation that are used throughout.

Let $k$ be a field and $B = k[x_1, \cdots, x_n]$ be graded by $\deg x_i = 1$. We order the variables by $x_1 > \cdots > x_n$, and denote by $>_{\text{Lex}}$ the graded lexicographic order on the monomials, $x_1^{\alpha_1} \ldots x_n^{\alpha_n} >_{\text{Lex}} x_1^{\beta_1} \ldots x_n^{\beta_n}$ if $\deg x_1^{\alpha_1} \ldots x_n^{\alpha_n} > \deg x_1^{\beta_1} \ldots x_n^{\beta_n}$, or if $\deg x_1^{\alpha_1} \ldots x_n^{\alpha_n} = \deg x_1^{\beta_1} \ldots x_n^{\beta_n}$ and for some index $j$ we have $\alpha_j > \beta_j$ and $\alpha_i = \beta_i$ for all $i < j$.

A monomial $x_1^{\alpha_1} \ldots x_n^{\alpha_n}$ has *exponent vector* $\alpha = (\alpha_1, \ldots, \alpha_n)$, and is sometimes written $\mathbf{x}^\alpha$. We denote by $|\alpha|$ the degree of $\mathbf{x}^\alpha$, $|\alpha| = \sum_{i=1}^n \alpha_i$.

A *d-vector space* is a $k$-vector subspace of $B$ spanned by homogeneous polynomials in degree $d$. A *d-monomial space* is a $d$-vector space spanned by monomials. The dimension of such a space $M_d$ will be denoted $|M_d|$. Let $\mathbf{m}$ be the monomial space generated by the variables, $\mathbf{m} = (x_1, \cdots, x_n)_1$.

A $d$-monomial space $L_d$ is *lex* or a *lex segment* if, whenever $\mathbf{x}^\alpha$ and $\mathbf{x}^\beta$ are degree $d$ monomials with $\mathbf{x}^\alpha$ lexicographically greater than $\mathbf{x}^\beta$ and $\mathbf{x}^\beta \in L_d$, we have $\mathbf{x}^\alpha \in L_d$ as well. A monomial ideal $L$ is *lex* if $L_d = \{f : f \in L \text{ and } \deg f = d\}$ is a lex segment for all $d$.

A $d$-monomial space $N_d$ is *strongly stable* or *Borel* if, whenever $i < j$ and $gx_j \in N_d$, we have $gx_i \in N_d$ as well. A monomial ideal $N$ is *strongly stable* or *Borel* if $N_d$ is Borel for all $d$.

A *monomial regular sequence* is a set of monomials $\{f_1, \cdots, f_r\}$ satisfying $\gcd(f_i, f_j) = 1$ for all $i \neq j$.

## 3 $S$-compression

Compressed ideals were used heavily in [CL, MP1, MP2]. We introduce $S$-compressed ideals, which are more general.

**Definition 3.1.** Let $S$ be a subset of $\{x_1, \cdots, x_n\}$, and let $M_d$ be a $d$-monomial vector space. Denote by $\oplus_S$ a direct sum over exponent vectors $\alpha = (\alpha_1, \cdots, \alpha_n)$ such that $\alpha_i = 0$ whenever $x_i \in S$. Then $M_d$ may be written uniquely in the form

$$M_d = \bigoplus_S \mathbf{x}^\alpha V_\alpha,$$

where $V_\alpha$ is a $(d - |\alpha|)$-monomial space in the ring $k[S]$. We say that $M_d$ is *$S$-compressed* if each $V_\alpha$ is lex in $k[S]$.

If $L_\alpha$ is the lex monomial space in degree $(d - |\alpha|)$ with $|L_\alpha| = |V_\alpha|$, and $L_d = \oplus_S \mathbf{x}^\alpha L_\alpha$, we say that $L_d$ is the *$S$-compression* of $M_d$.

*Remark.* What [CL,MP1,MP2] called $i$-compressed ideals (or monomial spaces) are $\{x_1, \cdots, \hat{x_i}, \cdots, x_n\}$-compressed in this notation. (Here the $\hat{x_i}$ means that $x_i$ is omitted.) This reversal is for simplicity in several proofs below, and so that $S$-compressed spaces will remain $S$-compressed after new variables are added to the ring $B$.

**Example 3.2.** If $S$ is a one-element set, every monomial space is $S$-compressed.

**Example 3.3.** Let $M = (a^3, a^2b, a^2c, ab^2, abc, abd, b^3, b^2c)$. Then $M$ is not $\{a, b\}$-compressed because $V_d = (ab)$ is not lex in $k[a, b]$, but $M$ is $S$-compressed for every other two-element set $S$.

**Lemma 3.4.** *Let $M_d$ be $S$-compressed. Then $\mathbf{m}M_d$ is $S$-compressed as well.*

**Definition 3.5.** A monomial ideal $M$ is *$S$-compressed* if it is $S$-compressed in every degree $d$.

It is easy to prove the following two propositions:

**Proposition 3.6.** *A lex ideal (or lex monomial space) is $S$-compressed for every $S$.*

**Proposition 3.7.** *Suppose $S \supset T$. Then every $S$-compressed ideal (or $S$-compressed monomial space) is $T$-compressed.*

**Example 3.8.** Borel ideals are $S$-compressed for every two-element set $S$.

Due to the following lemma, $S$-compressed ideals are useful in the study of Hilbert functions:

**Lemma 3.9.** *Let $M_d$ be a $d$-monomial vector space, and let $K_d$ be its $S$-compression. Then $|\mathbf{m}K_d| \leq |\mathbf{m}M_d|$.*

*Proof.* Let $\mathbf{n}$ be the monomial space generated by the variables of $S$. Denote by $\beta \uparrow \alpha$ the property that $\mathbf{x}^\beta$ divides $\mathbf{x}^\alpha$ and $\deg \mathbf{x}^\beta = \deg \mathbf{x}^\alpha - 1$.

Write $M = \bigoplus_S \mathrm{x}^\alpha N_\alpha$,and $K = \bigoplus_S \mathrm{x}^\alpha L_\alpha$. Now we have $\mathbf{m}K = \bigoplus_S \mathrm{x}^\alpha L'_\alpha$ and $\mathbf{m}M = \bigoplus_S \mathrm{x}^\alpha N'_\alpha$, where

$$N'_\alpha = \mathbf{n}N_\alpha + \sum_{\beta\uparrow\alpha} N_\beta$$
$$L'_\alpha = \mathbf{n}L_\alpha + \sum_{\beta\uparrow\alpha} L_\beta.$$

$|\mathbf{n}L_\alpha| \leq |\mathbf{n}N_\alpha|$ by Macaulay's Theorem, and $|N_\beta| = |L_\beta|$ for all $\beta$ by definition, so we have:

$$|L'_\alpha| = \max\{\max_{\beta\uparrow\alpha}\{|L_\beta|\}, |\mathbf{n}L_\alpha|\} \leq \max\{\max_{\beta\uparrow\alpha}\{|N_\beta|\}, |\mathbf{n}N_\alpha|\} \leq |N'_\alpha|,$$

the first equality because the $L_\beta$ and $\mathbf{n}L_\alpha$ are lex segments, so their sum is the longest segment. This proves the lemma. $\square$

Using Lemma 3.9 we prove Conjecture 1.1 for ideals containing a regular sequence of monomials:

**Theorem 3.10.** *Let $R = (f_1 \cdots, f_r)$ be a regular sequence of monomials with $\deg f_i = e_i$ and $e_1 \leq \cdots \leq e_r$. Let $N$ be any homogeneous ideal containing $R$. Then there is a lex ideal $L$ such that $N$ and $L + (x_1^{e_1}, \cdots, x_r^{e_r})$ have the same Hilbert function.*

*Proof.* Set $P = (x_1^{e_1}, \cdots, x_r^{e_r})$. It suffices to show that, for any $d$-vector space $N_d$ containing $R_d$, there is a lex monomial space $L_d$ such that $|L_d + P_d| = |N_d|$ and $|\mathbf{m}L_d + P_{d+1}| \leq |\mathbf{m}N_d|$.

Reorder the variables so that $x_i$ divides $f_i$ for all $i$. By Gröbner basis theory, we may assume (after taking an initial ideal if necessary) that $N_d$ is a monomial space. Set $N(0) = N_d$. For each $i \leq r$, let $S_i$ be the set of variables dividing $f_i$, and let $N(i)$ be the $S_i$-compression of $N(i-1)$. Then $N(i)$ contains $x_i^{e_i}$ if $d \geq e_i$. Furthermore, $|N(r)| = |N_d|$ and $|\mathbf{m}N(r)| \leq |\mathbf{m}N_d|$ by Lemma 3.9. By Clements-Lindström's theorem, there is a lex space $L_d$ such that $|L_d + P_d| = |N(r)|$ and $|\mathbf{m}L_d + P_{d+1}| \leq |\mathbf{m}N(r)|$. □

# 4 Lex-Macaulay monomial regular sequences

Throughout this section, let $M$ be a monomial ideal of $B$. We recall the following definitions and results from [MP1]:

**Definition 4.1.** [MP1] $M$ is *Lex-Macaulay* if every Hilbert function in the quotient $B/M$ is attained by a lex ideal. Equivalently, $M$ is Lex-Macaulay if, for every $d$ and every $d$-monomial vector space $V_d$, there exists a lex space $L_d$ such that $|L_d + M_d| = |V_d + M_d|$ and $|\mathbf{m}L_d + M_{d+1}| \leq |\mathbf{m}V_d + M_{d+1}|$.

**Theorem 4.2.** [MP1] *If $M$ is Lex-Macaulay as an ideal of $B$, then it is Lex-Macaulay as an ideal of $B[y]$.*

**Proposition 4.3.** [MP1] *If $M$ is Lex-Macaulay and $L$ is lex, then $L + M$ is Lex-Macaulay.*

Now, we will characterize the monomial regular sequences which are Lex-Macaulay.

Let $R = (f_1, \cdots, f_r)$ be a monomial regular sequence (that is, $\gcd(f_i, f_j) = 1$ for all $i \neq j$), and order these monomials so that $i < j$ if $\deg f_i < \deg f_j$, or if $\deg f_i = \deg f_j$ and $f_i >_{\text{Lex}} f_j$. Set $e_i = \deg f_i$, and suppose throughout that $e_i > 1$.

**Lemma 4.4.** *Suppose that $R$ is Lex-Macaulay. Then $x_i^{e_i - 1}$ divides $f_i$.*

*Proof.* By induction we have $x_j^{e_j - 1}$ divides $f_j$ for $j < i$, and by Proposition 4.3 $R+(x_1, \cdots, x_{i-1})$ is Lex-Macaulay, so we may assume without loss of generality that $i = 1$. Let $g$ be any monomial in degree $e_1 - 1$ dividing $f_1$. Then, since $R$ is Lex-Macaulay, we have $|(x_1^{e_1-1})_{e_1}| \leq |(g)_{e_1}| \leq n - 1$, i.e., $x_1^{e_1-1}$ divides $f_1$. □

**Lemma 4.5.** *Suppose that $R$ is Lex-Macaulay, and write $f_i = x_i^{e_i-1} y_i$. Suppose that $y_i \neq x_i$. Then $i = r$.*

*Proof.* We may assume as in the proof of Lemma 4.4 that $i = 1$. Suppose $r \neq 1$ and $y_1 \neq x_1$. Then set $g = x_1^{e_1-1}x_2^{e_2-1}$. Then $|(g)_{e_1+e_2-1}| = n-2$, while $|(x_1^{e_1+e_2-2})_{e_1+e_2-1}| = n-1$, contradicting the assumption that $R$ is Lex-Macaulay. $\square$

Thus, all Lex-Macaulay monomial regular sequences may be written in the form $(x_1^{e_1}, \cdots, x_{r-1}^{e_{r-1}}, x_r^{e_r-1}y)$. We will show conversely that all such sequences are Lex-Macaulay.

**Lemma 4.6.** *Suppose $r \neq n$, and let $R = (x_1^{e_1}, \cdots, x_{r-1}^{e_{r-1}}, x_r^{e_r-1}x_n)$. Let $N_d$ be a $d$-monomial vector space containing $R_d$. Then there exists a $d$-monomial vector space $K_d$ containing $Q_d$, where $Q = (x_1^{e_1}, \cdots, x_{r-1}^{e_{r-1}}, x_r^{e_r-1}x_{n-1})$ such that $|K_d| = |M_d|$ and $|\mathbf{m}K_d| \leq |\mathbf{m}M_d|$.*

*Proof.* Set $S = \{x_n, x_{n-1}\}$, and take $K_d$ to be the $S$-compression of $M_d$. Apply Lemma 3.9, and note that $Q$ is the $S$-compression of $R$. $\square$

**Lemma 4.7.** *Suppose $r \neq n$, and let $R = (x_1^{e_1}, \cdots, x_{r-1}^{e_{r-1}}, x_r^{e_r-1}x_n)$ and $Q = (x_1^{e_1}, \cdots, x_{r-1}^{e_{r-1}}, x_r^{e_r-1}x_{n-1})$. Let $L_d$ be a lex segment in $B$. Then there exists a lex $d$-monomial space $T_d$ such that $|T_d + R_d| = |L_d + Q_d|$ and $|\mathbf{m}T_d + R_{d+1}| = |\mathbf{m}L_d + Q_{d+1}|$.*

*Proof.* Let $T_d$ be the smallest lex segment such that $T_d + Q_d = L_d + Q_d$.

We claim that $T_d$ satisfies the property: If $g$ is a monomial such that $gx_r^{e_r-1}x_{n-1} \in T_d$, then $gx_r^{e_r-1}x_n \in T_d$. We will prove this claim. Observe that $gx_r^{e_r-1}x_{n-1} \in Q_d$, and that $gx_r^{e_r-1}x_n$ is the successor of $gx_r^{e_r-1}x_{n-1}$ in the graded lex order. By construction, if $gx_r^{e_r-1}x_{n-1} \in T_d$, we must have a monomial $v \in T_d$ which comes lexicographically after $gx_r^{e_r-1}x_{n-1}$ and $v \notin Q_d$. Then $v$ is lexicographically after $gx_r^{e_r-1}x_n$ as well, and since $T_d$ is a lex segment, we have $gx_r^{e_r-1}x_n \in T_d$, proving the claim.

Set $A = T_d + (x_1^{e_1}, \cdots, x_{r-1}^{e_{r-1}})_d$ and $B$ and $C$ such that $T_d + Q_d = A \oplus B$ and $T_d + R_d = A \oplus C$. Then if $\{B\}$ and $\{C\}$ are the sets of monomials of $B$ and $C$, respectively, we have:

$$\begin{aligned}
\{B\} &= \{gx_r^{e_r-1}x_{n-1} : gx_r^{e_r-1}x_{n-1} \notin A\} \\
\{C\} &= \{gx_r^{e_r-1}x_n : gx_r^{e_r-1}x_n \notin A\} \\
&= \{gx_r^{e_r-1}x_n : gx_r^{e_r-1}x_{n-1} \in B\}.
\end{aligned}$$

In particular, multiplication by $\frac{x_n}{x_{n-1}}$ is a bijection from $\{B\}$ to $\{C\}$. Thus $|\mathbf{m}B| = |\mathbf{m}C|$ and $|\mathbf{m}B \cap \mathbf{m}A| = |\mathbf{m}C \cap \mathbf{m}A|$, so we have $|\mathbf{m}(T_d + R_d)| = |\mathbf{m}(T_d + Q_d)| = |\mathbf{m}(L_d + Q_d)|$, the first equality by inclusion-exclusion, the second by construction.

Now $|\mathbf{m}T_d + R_{d+1}| = |\mathbf{m}(T_d + R_d)|$ unless $R$ has minimal monomial generators in degree $d+1$ which are not in $\mathbf{m}T_d$; likewise $|\mathbf{m}L_d + Q_{d+1}| = |\mathbf{m}(L_d + Q_d)|$ unless $Q$ has minimal monomial generators in degree $d+1$ which are not in $\mathbf{m}L_d$ and hence not in $\mathbf{m}T_d$. Since $x_r^d x_{n-1} \in \mathbf{m}T_d$ if and only if $x_r^d x_n \in \mathbf{m}T_d$, we obtain $|\mathbf{m}T_d + R_{d+1}| = |\mathbf{m}L_d + Q_{d+1}|$ as desired. $\square$

**Theorem 4.8.** *Let $R$ be a regular sequence of monomials. Then $R$ is Lex-Macaulay if and only if $R = (x_1^{e_1}, \cdots, x_{r-1}^{e_{r-1}}, x_r^{e_r-1}y)$, with $e_1 \leq \cdots \leq e_r$ and $y = x_i$ for some $i \geq r$.*

*Proof.* If $R$ is Lex-Macaulay, apply Lemmas 4.4 and 4.5.

Conversely, suppose $R$ has the form above. By Theorem 4.2, we may assume $y = x_n$. If $n = r$, this is Clements-Lindström's Theorem; otherwise, we induct on $n-r$. Set $Q = (x_1^{e_1}, \cdots, x_{r-1}^{e_{r-1}}, x_r^{e_r-1}x_{n-1})$. Choose a degree $d$, and let $N_d$ be any $d$-monomial space containing $R_d$. By Lemma 4.6, there is a $d$-monomial space $K_d$ containing $Q_d$ with $|K_d| = |N_d|$ and $|\mathbf{m}K_d| \leq |\mathbf{m}N_d|$. $Q$ is Lex-Macaulay by induction, so there is a monomial space $L_d$ containing $Q_d$ with $|L_d| = |K_d|$ and $|\mathbf{m}L_d| \leq |\mathbf{m}K_d|$. By Lemma 4.7, there is a monomial space $T_d$ containing $R_d$ with $|T_d| = |L_d|$ and $|\mathbf{m}T_d| = |\mathbf{m}L_d|$. Thus $R$ is Lex-Macaulay. □

# References


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