LEXIFYING IDEALS

Jeffrey Mermin *Irena Peeva*

Department of Mathematics, Cornell University, Ithaca, NY 14853, USA.

Abstract: This paper is on monomial quotients of polynomial rings over which Hilbert functions are attained by lexicographic ideals.

1. Introduction

Let $B = k[x_1, \dots, x_n]$ be a polynomial ring over a field k graded by $\deg(x_i) = 1$ for all i.

What are the possible Hilbert functions of graded ideals in B? This question was answered by Macaulay [Ma], who showed that for every graded ideal there exists a lexicographic ideal with the same Hilbert function. Lexicographic ideals are highly structured: they are defined combinatorially and it is easy to derive the inequalities characterizing their possible Hilbert functions. Macaulay's Theorem also plays an important role in the study of graded B-ideals; for example,

- Hartshorne's [Ha] proof that the Hilbert scheme is connected uses lexicographic ideals in an essential way.
- The homological properties of lexicographic ideals are combinatorially tractable [EK]. This leads to results by Bigatti, Hulett, Pardue, showing that the lexicographic ideals have extremal Betti numbers.

Let M be a monomial ideal. We say that a graded ideal in B/M is *lexifiable* if there exists a lexicographic ideal in B/M with the same Hilbert function. We call M and B/M *Macaulay-Lex* if every graded ideal in B/M is lexifiable. The following results are well known: Macaulay's Theorem [Ma] says that 0 is a Macaulay-Lex ideal, Kruskal-Katona's Theorem [Ka, Kr] says that $(x_1^2, \dots, x_n^2)$ is a Macaulay-Lex ideal, and Clements-Lindström's Theorem [CL] says that $(x_1^{e_1}, \dots, x_n^{e_n})$ is a Macaulay-Lex ideal if $e_1 \le \dots \le e_n \le \infty$. These theorems are well-known and have many applications in Commutative Algebra, Combinatorics, and Algebraic Geometry.

1991 Mathematics Subject Classification: 13F20.
*Keywords and Phrases:*Hilbert function, lexicographic ideals.

It is easy to construct examples like Example 2.13, where problems occur in the degrees of the minimal generators of M. This motivated us to slightly weaken the definition: Let q be the maximal degree of a minimal monomial generator of M; we call M and B/M *pro-lex* if every graded ideal generated in degrees $\geq q$ in B/M is lexifiable. There exist examples of non pro-lex rings; see Example 3.14. The main goal in this paper is to open a new direction of research along the lines of the following problem.

Problem 1.1. *Find classes of pro-lex monomial ideals.*

Theorem 5.1 shows that if M is Macaulay-Lex and N is lexicographic, then $M + N$ is Macaulay-Lex. Theorem 4.1 shows that if M is Macaulay-Lex, then it stays Macaulay-Lex after we add extra variables to the ring B. In Section 3 we prove:

Theorem 1.2. *Let $P = (x_1^{e_1}, \cdots, x_n^{e_n})$, with $e_1 \leq e_2 \leq \cdots \leq e_n \leq \infty$ (here $x_i^\infty = 0$), and M be a compressed monomial ideal in B/P generated in degrees $\leq p$. If $n = 2$, assume that M is (B/P)-lex. Set $\Upsilon = B/(M + P)$. Then Υ is pro-lex above p, that is, for every graded ideal Γ in Υ generated in degrees $\geq p$ there exists an Υ-lex ideal Θ with the same Hilbert function.*

In the case when $M = P = 0$, Theorem 1.2 is Macaulay's Theorem [Ma]; in the case when $M = 0$, Theorem 1.2 is Clements-Lindström's Theorem [CL]. Examples 3.13 and 3.14 show that there are obstructions to generalizing Theorem 1.2.

We make use of ideas of Bigatti [Bi], Clements and Lindström [CL], and Green [Gr]. Our proofs are algebraic, and we avoid computations using generic forms (used in [Gr]) and combinatorial counting (used in [CL]). In Section 2 we introduce definitions and notation used throughout the paper.

Acknowledgments. We thank Christopher Francisco, Mike Stillman, and Steven Sinnott for helpful discussions.

2. Lexification

The notation in this section will be used throughout the paper. We introduce several definitions.

Let k be a field and $B = k[x_1, \ldots, x_n]$ be graded by $\deg(x_i) = 1$ for all i. We denote by B_d the k-vector space spanned by all monomials of degree d. Denote $\mathbf{m} = (x_1, \ldots, x_n)_1$ the k-vector space spanned by the variables. We order the variables lexicographically by $x_1 > \ldots > x_n$, and we denote by $\succ_{lex}$ the homogeneous lexicographic order on the monomials. We say that an ideal is *p-generated* if it has a system of generators of degree p.

A monomial $x_1^{a_1} \ldots x_n^{a_n}$ has *exponent vector* $\mathbf{a} = (a_1, \ldots, a_n)$, and is sometimes denoted by $\mathbf{x}^{\mathbf{a}}$. An ideal is called *monomial* if it can be generated by monomials; such an ideal has a unique minimal system of monomial generators.

Notation 2.1. Let M be a monomial ideal. Set $\Upsilon = B/M$. Vector spaces in Υ (and sometimes ideals) are denoted by greek letters. For example, we denote by C_d a subspace of B_d, and we denote by τ_d a subspace of Υ_d.

Definition 2.2. A monomial is a product of powers of the variables, so it can be considered as an element in either B or Υ. We say that a monomial is an Υ-*monomial* if it does not vanish in Υ, that is, it is not in M. We say that a monomial is an Υ_d-*monomial* if it is an Υ-monomial of degree d. Furthermore, we say that τ_d is an Υ_d-*monomial space* if it can be spanned by Υ_d-monomials. We denote by $\{\tau_d\}$ the set of Υ_d-monomials contained in τ_d. The cardinality of this set is $|\tau_d| = \dim_k \tau_d$. By $\mathbf{m}\tau_d$ we mean the k-vector subspace $\Big(\mathbf{m}\,(\tau_d)\Big)_{d+1}$ of Υ_{d+1}.

Definition 2.3. Let L be a monomial ideal in Υ minimally generated by Υ-monomials $l_1, \ldots, l_r$. We say that L is Υ-*lex*, (Υ-*lexicographic*), if the following property is satisfied:

$$\left.\begin{array}{r} m \text{ is an } \Upsilon\text{-monomial} \\ m \succ_{lex} l_i \ \text{ and } \deg(m) = \deg(l_i), \text{ for some } 1 \le i \le r \end{array}\right\} \implies m \in L\,.$$

The Υ_d-*lex-segment* $\lambda_{d,p}$ of length p in degree d is defined as the k-vector space spanned by the lexicographically first (greatest) p monomials in Υ_d. We say that λ_d is a *lex-segment in* Υ_d if there exists a p such that $\lambda_d = \lambda_{d,p}$. For a Υ_d-monomial space τ_d, we say that $\lambda_{d,|\tau_d|}$ is its Υ_d-*lexification*.

For simplicity, we sometimes say lex instead of Υ-lex if it is clear over which ring we work.

Example 2.4. The ideal $(a^2,\, ab,\, b^2)$ is lex in the ring $k[a,b,c,d]/(ac,\, ad)$, and its generators span a lex-segment. The k-vector space spanned by a^2, ab, b^2 is the lexification of the k-vector space spanned by b^2, c^2, cd. However, the ideal is not lex in $k[a,b,c,d]$.

Proposition 2.5. *If τ_d is an Υ_d-lex-segment, then $\mathbf{m}\tau_d$ is an Υ_{d+1}-lex-segment.*

Definition 2.6. A monomial m' is said to be *in the big shadow of* a monomial m if $m' = \dfrac{x_i\, m}{x_j}$ for some x_j dividing m and some $i \le j$. A monomial ideal in Υ is Υ-*Borel* if it contains all Υ-monomials in the big shadows of its minimal Υ-monomial generators. Ideals that are B-Borel are usually called strongly stable or 0-Borel fixed. We say that a monomial space τ_d is Υ_d-*Borel* if it contains all Υ_d-monomials in the big shadows of its monomial generators.

Proposition 2.7. *If τ_d is Υ_d-Borel, then $\mathbf{m}\tau_d$ is Υ_{d+1}-Borel.*

Proposition 2.8. *If τ_d is an Υ_d-lex-segment, then it is Υ_d-Borel.*

Notation 2.9. Let Γ be a graded ideal in Υ. It decomposes as a direct sum of its components $\Gamma = \oplus_{d\geq 0}\,\Gamma_d$. Its *Hilbert function* $\mathrm{Hilb}_\Gamma^\Upsilon :\ \mathbf{N}\cup 0\ \to \mathbf{N}\cup 0$ is defined by

$$\mathrm{Hilb}_\Gamma^\Upsilon(d) = \dim_k(\Gamma_d) \quad \text{ for all } d \geq 0\,.$$

We use the following notation

$$|\Gamma_d|^\Upsilon \;=\; \mathrm{Hilb}_\Gamma^\Upsilon(d)\,;$$

and for simplicity, we write $|\Gamma_d|$ if it is clear over which ring we work.

Definition 2.10. We say that an Υ_d-monomial space τ_d is Υ_d*-lexifiable* if its lexification λ_d has the property that $|\mathbf{m}\lambda_d| \leq |\mathbf{m}\tau_d|$. The monomial ideal M and the quotient ring $\Upsilon = B/M$ are called *d-pro-lex*, if every Υ_d-monomial space is Υ_d-lexifiable.

Definition 2.11. We say that a graded ideal R in Υ is *lexifiable* if there exists an Υ-lex ideal with the same Hilbert function as R. The monomial ideal M and the quotient ring $\Upsilon = B/M$ are called *Macaulay-Lex* if every graded ideal in Υ is lexifiable.

Example 2.12. This example shows that the order of the variables can make a difference. The ideal (ab) is not lexifiable in the ring $k[a,b]/(ab^2)$ for the lex order with $a > b$, but it is lexifiable for the lex order with $b > a$.

Example 2.13. The ideal (ab) is not lexifiable in the ring $k[a,b]/(a^2b,\, ab^2)$ in any lex order.

It is easy to construct many examples like Example 2.13. This observation suggests that in order to obtain positive results we need to slightly relax Definition 2.11:

Definition 2.14. Let q be the maximal degree of a minimal monomial generator of M. The monomial ideal M and the quotient ring $\Upsilon = B/M$ are called *pro-lex* if every graded ideal generated in degrees $\geq q$ in Υ is lexifiable.

In the examples we usually denote the variables by a, b, c, d for simplicity.

3. Compression

The following definition generalizes a definition introduced by Clements and Lindström [CL], who used it over a quotient of a polynomial ring modulo pure powers of the variables.

Definition 3.1. Let E be a monomial ideal in B. A $(B/E)_d$-monomial space τ_d is called *i-compressed* (or i-compressed in $(B/E)_d$) if we have the disjoint union

$$\{\tau_d\} = \coprod_{0\leq j\leq d} x_i^{d-j}\{\sigma_j\}$$

and each σ_j is a lex-segment in $\big(B/(E,x_i)\big)_j$. We say that a k-vector space τ_d is $(B/E)_d$-*compressed* (or compressed) if it is a $(B/E)_d$-monomial space and is i-compressed for all $1 \leq i \leq n$. A monomial ideal W in B/E is called *compressed* if W_d is compressed for all $d \geq 0$.

Example 3.2. The ideal

$$(a^3,\ a^2b,\ a^2c,\ ab^2,\ abc,\ b^3,\ b^2c)$$

is compressed in the ring $k[a,b,c]$.

Lemma 3.3. *If τ_d is i-compressed in $(B/E)_d$, then $\mathbf{m}\tau_d$ is i-compressed in $(B/E)_{d+1}$. If τ_d is $(B/E)_d$-lex, then it is $(B/E)_d$-compressed.*

Definition 3.4. A B-monomial ideal K is called *compressed-plus-powers* if $K = M + P$, where $P = (x_1^{e_1}, \cdots, x_n^{e_n})$ with $e_1 \leq e_2 \leq \cdots \leq e_n \leq \infty$ and the monomial ideal M is compressed in B/P. Sometimes, when we need to be more precise, we say that K is *compressed-plus-P*. Furthermore, we say that K is *lex-plus-P* if M is lex in B/P.

Notation 3.5. Throughout this section we use the following notation and make the following assumptions:

- $P = (x_1^{e_1}, \cdots, x_n^{e_n})$ with $2 \leq e_1 \leq e_2 \leq \cdots \leq e_n \leq \infty$.
- The ideal $K = M + P$ is a compressed-plus-P monomial ideal in B; here M is compressed in B/P.
- If $n = 2$ we assume in addition that K is lex-plus-P.
- We assume that M is p-generated.
- Set $\Upsilon = B/K$.
- d is a degree such that $d \geq p$.

  For a $(B/P)_d$-monomial space A_d set

$$t_i(A_d) = \Big|\, \{\, m \in \{A_d\} \mid \max(m) \leq i \,\} \,\Big|$$

$$s_i(A_d) = \Big|\, \{\, m \in \{A_d\} \mid \max(m) = i \ \text{ and } \ x_i^{e_i - 1} \text{ divides } m \,\} \,\Big|$$

$$r_{i,j}(A_d) = \Big|\, \{\, m \in \{A_d\} \mid \max(m) \leq i \ \text{ and } \ x_i^{j} \text{ does not divide } m \,\} \,\Big|.$$

The formula in the following lemma is a generalization of a formula introduced by Bigatti [Bi], who used it for B-Borel ideals.

Lemma 3.6. *Let A_d be a $(B/P)_d$-monomial space.*

(1) If A_d is compressed and $n \geq 3$, then A_d is $(B/P)_d$-Borel.

(2) If A_d is $(B/P)_d$-Borel, then

$$\Big|\{\mathbf{m}A_d\}\Big| = \sum_{i=1}^{n} t_i(A_d) - s_i(A_d) = \sum_{i=1}^{n} r_{i,e_i-1}(A_d)\,.$$

Proof: First, we prove (1). Let $m \in \{A_d\}$ and m' be a $(B/P)_d$–monomial in its big shadow. Hence $m' = \dfrac{x_i\,m}{x_j}$ for some x_j dividing m and some $i \le j$. There exists an index $1 \le q \le n$ such that $q \ne i, j$. Note that that m and m' have the same q-exponents. Since A_d is q-compressed and $m' \succ_{lex} m$, it follows that $m' \in \{A_d\}$. Therefore, A_d is $(B/P)_d$–Borel.

Now, we prove (2). We will show that $\{\mathbf{m}A_d\}$ is equal to the set

$$\coprod_{i=1}^{n} x_i\,\{\, m \in \{A_d\}\,|\,\max(m) \le i\,\} \setminus \coprod_{i=1}^{n} x_i\,\{\, m \in \{A_d\}\,|\,\max(m) = i \text{ and } x_i^{e_i-1} \text{ divides } m\,\}.$$

Denote by $\mathcal{P}$ the set above. Let $w \in A_d$. For $j \ge \max(w)$ we have that $x_j w \in \mathcal{P}$. Let $j < \max(w)$. Then $v = x_j\,\dfrac{w}{x_{\max(w)}} \in A_d$. So, $x_j w = x_{\max(w)} v \in \mathcal{P}$. □

Lemma 3.7 is a generalization of a result by M. Green [Gr], who proved a particular case of it it over a polynomial ring (in the case $M = 0$). Green's proof is entirely different than ours; he makes a computation with generic linear forms. It is not clear how to apply his computation to the case $M \ne 0$.

Lemma 3.7. *Let τ_d be an n-compressed Borel Υ_d-monomial space, and let λ_d be a lex-segment in Υ_d with $|\{\lambda_d\}| \le |\{\tau_d\}|$. Let L_d and T_d be the $(B/P)_d$-monomial spaces such that $\{L_d\} = \{\lambda_d\}\coprod\{M_d\}$ and $\{T_d\} = \{\tau_d\}\coprod\{M_d\}$. For each $1 \le i \le n$ and each $1 \le j \le e_i$ we have*

$$r_{i,j}(L_d) \le r_{i,j}(T_d)\,.$$

Proof: Set $R = B/P$. By Lemma 3.6, M_d is R_d-Borel. Therefore, both L_d and T_d are R_d-Borel and n-compressed.

First, we consider the case $i = n$. Clearly, $r_{n,e_n}(L_d) = |L_d| = |T_d| = r_{n,e_n}(T_d)$ (if $e_n = \infty$, then we consider $r_{n,d+1}$ here). We induct on j decreasingly. Suppose that $r_{i,j+1}(L_d) \le r_{i,j+1}(T_d)$ holds by induction.

If $\{T_d\}$ contains no monomial divisible by x_n^j then

$$r_{i,j}(L_d) \le r_{i,j+1}(L_d) \le r_{i,j+1}(T_d) = r_{i,j}(T_d)\,.$$

Suppose that $\{T_d\}$ contains a monomial divisible by x_n^j. Denote by $e = x_1^{b_1}\dots x_n^{b_n}$, with $b_n \ge j$, the lex-smallest monomial in T_d that is divisible by x_n^j. Let $0 \le q \le j-1$. Since T_d is R_d-Borel, it follows that $c_q = x_{n-1}^{b_n-q}\dfrac{e}{x_n^{b_n-q}} \in T_d$. This is the lex-smallest monomial that is lex-greater than e and x_n divides it at power q. Let the monomial $a = x_1^{a_1}\dots x_{n-1}^{a_{n-1}}x_n^q \in R_d$ be lex-greater than e. Since T_d is n-compressed and a is lex-greater (or equal) than c_q, it follows that $a \in T_d$.

For a monomial u, we denote by $x_n \notin u$ the property that x_n^j does not divide u. By what we proved above, it follows that

$$\Big|\{u \in \{T_d\}\,|\, x_n \notin u,\ u \succ_{lex} e\}| = |\{u \in \{R_d\}\,|\, x_n \notin u,\ u \succ_{lex} e\}\Big|. \tag{3.8}$$

Therefore,

$$\begin{aligned} r_{i,j}(L_d) &= |\{u \in \{L_d\}\,|\, x_n \notin u,\ u \succ_{lex} e\}| + |\{u \in \{L_d\}\,|\, x_n \notin u,\ u \prec_{lex} e\}| \\ &\le |\{u \in \{R_d\}\,|\, x_n \notin u,\ u \succ_{lex} e\}| + |\{u \in \{L_d\}\,|\, x_n \notin u,\ u \prec_{lex} e\}| \\ &\le |\{u \in \{R_d\}\,|\, x_n \notin u,\ u \succ_{lex} e\}| + |\{u \in \{L_d\}\,|\ u \prec_{lex} e\}| \\ &\le |\{u \in \{R_d\}\,|\, x_n \notin u,\ u \succ_{lex} e\}| + |\{u \in \{T_d\}\,|\ u \prec_{lex} e\}| \\ &= |\{u \in \{R_d\}\,|\, x_n \notin u,\ u \succ_{lex} e\}| + |\{u \in \{T_d\}\,|\, x_n \notin u,\ u \prec_{lex} e\}| \\ &= |\{u \in \{T_d\}\,|\, x_n \notin u,\ u \succ_{lex} e\}| + |\{u \in \{T_d\}\,|\, x_n \notin u,\ u \prec_{lex} e\}| \\ &= r_{i,j}(T_d)\,; \end{aligned}$$

for the third inequality we used the fact that λ_d is a lex-segment in Υ_d with $|\{\lambda_d\}| \le |\{\tau_d\}|$; for the equality after that we used the definition of e; for the next equality we used (3.8). Thus, we have the desired inequality in the case $i = n$.

In particular, we proved that

$$t_{n-1}(L_d) = r_{n,1}(L_d) \le r_{n,1}(T_d) = t_{n-1}(T_d)\,. \tag{3.9}$$

Finally, we prove the lemma for all $i < n$. Both $\{\tau_d/x_n\}$ and $\{\lambda_d/x_n\}$ are lex-segments in Υ_d/x_n since τ_d is n-compressed. By (3.9) the inequality $t_{n-1}(L_d) \le t_{n-1}(T_d)$ holds, and it implies the inclusion $\{\tau_d/x_n\} \supseteq \{\lambda_d/x_n\}$. The desired inequalities follow since

$$\begin{aligned} r_{i,j}(T_d) &= r_{i,j}\big(T_d/(x_{i+1},\dots,x_n)\big) = r_{i,j}\big(\{\tau_d/(x_{i+1},\dots,x_n)\}\coprod\{M_d/(x_{i+1},\dots,x_n)\}\big) \\ r_{i,j}(L_d) &= r_{i,j}\big(L_d/(x_{i+1},\dots,x_n)\big) = r_{i,j}\big(\{\lambda_d/(x_{i+1},\dots,x_n)\}\coprod\{M_d/(x_{i+1},\dots,x_n)\}\big) \end{aligned}$$ □

Lemma 3.10. *Let υ_d be a Υ_d-monomial space. There exists a compressed monomial space τ_d in Υ_d such that $|\tau_d| = |\upsilon_d|$ and $|\mathbf{m}\tau_d| \le |\mathbf{m}\upsilon_d|$.*

Proof: Suppose that υ_d is not i-compressed. Set $z = x_i$. Since M is z-compressed in B/P, we have the disjoint union

$$\{M_d\} = \coprod_{0\le j\le d} z^{d-j}\{N_j\}\,,$$

where each N_j is a $(B/(z,P))_j$-lex-segment.

We also have the disjoint union

$$\{\upsilon_d\} = \coprod_{0\le j\le d} z^{d-j}\{\nu_j\}$$

where each ν_j is a monomial space in $B/(z,P,N_j)$. Let γ_j be the lexification of the space ν_j in $B/(z,P,N_j)$. Consider the Υ_d-monomial space τ_d defined by

$$\{\tau_d\} = \coprod_{0\le j\le d} z^{d-j}\{\gamma_j\}\,.$$

Clearly, $|\tau_d| = |\upsilon_d|$.

Consider the $(B/P)_d$-monomial spaces V_d and T_d such that

$$\{V_d\} = \{\upsilon_d\}\coprod\{M_d\} \quad \text{and} \quad \{T_d\} = \{\tau_d\}\coprod\{M_d\}\,.$$

Set $R = B/P$. The short exact sequence of k-vector subspaces of $(B/P)_{d+1}$

$$0 \to \mathbf{m}M_d \to \mathbf{m}T_d \longrightarrow \mathbf{m}T_d/\mathbf{m}M_d \;=\; \mathbf{m}\tau_d/\big(\mathbf{m}\tau_d \cap \mathbf{m}M_d\big) \to 0$$

shows that $|\mathbf{m}\tau_d| = |\mathbf{m}T_d| - |\mathbf{m}M_d|$ (here we mean $|\mathbf{m}\tau_d|^{\Upsilon} = |\mathbf{m}T_d|^{B/P} - |\mathbf{m}M_d|^{B/P}$). Similarly, the short exact sequence of k-vector subspaces of $(B/P)_{d+1}$

$$0 \to \mathbf{m}M_d \to \mathbf{m}V_d \longrightarrow \mathbf{m}V_d/\mathbf{m}M_d \;=\; \mathbf{m}\upsilon_d/\big(\mathbf{m}\upsilon_d \cap \mathbf{m}M_d\big) \to 0$$

shows that $|\mathbf{m}\upsilon_d| = |\mathbf{m}V_d| - |\mathbf{m}M_d|$. Therefore, the desired inequality $|\mathbf{m}\tau_d| \le |\mathbf{m}\upsilon_d|$ is equivalent to the inequality

$$|\mathbf{m}T_d| \le |\mathbf{m}V_d|\,.$$

We will prove the latter inequality.

We have the disjoint unions

$$\{V_d\} = \coprod_{0\le j\le d} z^{d-j}\{U_j\} \quad \text{and} \quad \{T_d\} = \coprod_{0\le j\le d} z^{d-j}\{F_j\}\,, \quad \text{where}$$

$$\{U_j\} = \{\nu_j\}\coprod\{N_j\} \quad \text{and} \quad \{F_j\} = \{\gamma_j\}\coprod\{N_j\} \quad \text{in the ring } B/(z,P)\,.$$

Note that each F_j is a $(B/(z,P))_j$-lex-segment. Furthermore, we have the disjoint unions

$$\{\mathbf{m}V_d\} = \coprod_{0\le j\le d} z^{d-j+1}\{U_j + \mathbf{n}U_{j-1}\}$$

$$\{\mathbf{m}T_d\} = \coprod_{0\le j\le d} z^{d-j+1}\{F_j + \mathbf{n}F_{j-1}\}\,,$$

where $\mathbf{n} = \mathbf{m}/z$. We will show that

$$|F_j + \mathbf{n}F_{j-1}| = \max\Big\{|F_j|,\, |\mathbf{n}F_{j-1}|\Big\} \leq \max\Big\{|U_j|,\, |\mathbf{n}U_{j-1}|\Big\} \leq |U_j + \mathbf{n}U_{j-1}|\,.$$

The first equality above holds because both F_j and $\mathbf{n}F_{j-1}$ are $(B/(z,P))_j$-lex-segments, so $F_j + \mathbf{n}F_{j-1}$ is the longer of these two lex-segments. The last inequality is obvious. It remains to prove the middle inequality. Using the short exact sequences of k-vector subspaces of $(B/P)_j$

$$0 \to \mathbf{n}N_{j-1} \to \mathbf{n}F_{j-1} \longrightarrow \mathbf{n}F_{j-1}/\mathbf{n}N_{j-1} \;=\; \mathbf{n}\gamma_{j-1}/\big(\mathbf{n}\gamma_{j-1} \cap \mathbf{n}N_{j-1}\big) \to 0$$

$$0 \to \mathbf{n}N_{j-1} \to \mathbf{n}U_{j-1} \longrightarrow \mathbf{n}U_{j-1}/\mathbf{n}N_{j-1} \;=\; \mathbf{n}\nu_{j-1}/\big(\mathbf{n}\nu_{j-1} \cap \mathbf{n}N_{j-1}\big) \to 0$$

we get $|\mathbf{n}\gamma_{j-1}| = |\mathbf{n}F_{j-1}| - |\mathbf{n}N_{j-1}|$ and $|\mathbf{n}\nu_{j-1}| = |\mathbf{n}U_{j-1}| - |\mathbf{n}N_{j-1}|$. Therefore, the desired inequality $|\mathbf{n}F_{j-1}| \leq |\mathbf{n}U_{j-1}|$ is equivalent to the inequality $|\mathbf{n}\gamma_{j-1}| \leq |\mathbf{n}\nu_{j-1}|$. The latter inequality holds since by construction γ_{j-1} is the lexification of ν_{j-1}, so $|\gamma_{j-1}| = |\nu_{j-1}|$ and by induction on the number of variables we can apply Theorem 3.11 to the ring $B/(z,P,N_j)$.

Thus, $|F_j + \mathbf{n}F_{j-1}| \leq |U_j + \mathbf{n}U_{j-1}|$. Multiplication by z^{d-j+1} is injective if $d-j+1 \leq e_i - 1$ and is zero otherwise, therefore we conclude that

$$\Big|z^{d-j+1}(F_j + \mathbf{n}F_{j-1})\Big| \leq \Big|z^{d-j+1}(U_j + \mathbf{n}U_{j-1})\Big|\,.$$

This implies the desired inequality $|\mathbf{m}T_d| \leq |\mathbf{m}V_d|$.

Note that $\{\tau_d\}$ is greater lexicographically than $\{\upsilon_d\}$. If τ_d is not compressed, we can apply the argument above. After finitely many steps in this way, the process must terminate because at each step we construct a lex-greater monomial space. Thus, after finitely many steps, we reach a compressed monomial space. □

Theorem 3.11. *Let υ_d be a Υ_d-monomial space and λ_d be its lexification in Υ_d. Then $|\,\mathbf{m}\lambda_d\,| \leq |\,\mathbf{m}\upsilon_d\,|$.*

Proof: The theorem clearly holds if $n = 1$. Suppose that $n = 2$. An easy calculation shows that the theorem holds, provided we do not have $e_2 \leq d + 1 < e_1$. By the assumption on the ordering of the exponents, this does not hold and we are fine.

Suppose that $n \geq 3$. First, we apply Lemma 3.10 to reduce to the compressed case. We obtain a compressed Υ_d-monomial space τ_d such that $|\tau_d| = |\upsilon_d|$ and $|\mathbf{m}\tau_d| \leq |\mathbf{m}\upsilon_d|$. Let L_d and T_d be the $(B/P)_d$-monomial spaces such that $\{L_d\} = \{\lambda_d\} \cup \{M_d\}$ and $\{T_d\} = \{\tau_d\} \cup \{M_d\}$, where the disjoint unions take place in B/P. Both L_d and T_d are $(B/P)_d$-compressed. We apply Lemma 3.6 to conclude that

$$\Big|\,\{\mathbf{m}T_d\}\,\Big| = \sum_{i=1}^{n} t_i(T_d) - \sum_{i=1}^{n} s_i(T_d) \quad \text{and} \quad \Big|\,\{\mathbf{m}L_d\}\,\Big| = \sum_{i=1}^{n} t_i(L_d) - \sum_{i=1}^{n} s_i(L_d)\,.$$

Finally, we apply Lemma 3.7 and conclude that $\big|\{\mathbf{m}L_d\}\big| \le \big|\{\mathbf{m}T_d\}\big|$. This inequality and short exact sequences, as in the proof of Lemma 3.10, imply the desired $|\mathbf{m}\lambda_d| \le |\mathbf{m}\upsilon_d|$. □

Equivalently, we obtain the following theorem, stated in the introduction:

Theorem 1.2. *Let $P = (x_1^{e_1}, \cdots, x_n^{e_n})$, with $e_1 \le e_2 \le \cdots \le e_n \le \infty$ (here $x_i^\infty = 0$), and M be a compressed monomial ideal in B/P generated in degrees $\le p$. If $n = 2$, assume that M is (B/P)-lex. Set $\Upsilon = B/(M+P)$. Then Υ is pro-lex above p, that is, for every graded ideal Γ in Υ generated in degrees $\ge p$ there exists an Υ-lex ideal Θ with the same Hilbert function.*

Proof: We can assume that Γ is a monomial ideal by Gröbner basis theory. For each $d \ge p$, let λ_d be the lexification of Γ_d. By Theorem 3.11, it follows that $\Theta = \oplus_{d\ge p}\lambda_d$ is an ideal. By construction, it is a lex-ideal and has the same Hilbert function as Γ in all degrees greater than or equal to p. □

Remark 3.12. In the case when $M = P = 0$, Theorem 1.2 is the well-known Macaulay's Theorem [Ma]. In the case $M = 0$, Theorem 1.2 is the Clements-Lindström's Theorem [CL].

Example 3.13. It is natural to ask if a compressed ideal is Macaulay-Lex. This example shows that the answer is negative. Take $P = 0$. The ideal

$$M = (\, a^3,\ a^2b,\ a^2c,\ ab^2,\ abc,\ b^3,\ b^2c\,)$$

is compressed (and Borel) in the ring $k[a, b, c]$. The ideal (a^2, ab, b^2) in $k[a, b, c]/M$ is not lexifiable. □

Example 3.14. It is natural to ask if Theorem 1.2 holds in the case when M is a B-Borel ideal. It does not. Take $P = 0$. The ideal

$$M = (\, a^3,\, a^2b,\, a^2c,\, a^2d,\, ab^2,\, abc,\, abd,\, b^3,\, b^2c\,)$$

is Borel in the ring $k[a, b, c]$. However it is not pro-lex because the ideal (b^2d) is not lexifiable in $k[a, b, c]/M$. □

4. Adding new variables

Theorem 4.1. *If B/M is Macaulay-Lex then $B[y]/M$ is Macaulay-Lex.*

In this section, $W = B[y]/M$, $\mathbf{m}$ is the k-vector space spanned by the variables in B (as in Section 2), and $\mathbf{q}$ is the k-vector space spanned by $\mathbf{m}$ and y.

Lemma 4.2. *Let V_d be a W_d-monomial space, and let T_d be its y-compression. Then $|T_d| = |V_d|$ and $|\mathbf{q}T_d| \le |\mathbf{q}V_d|$.*

Proof: The proof is based on the same idea as the proof of Lemma 3.10. We write $\{V_d\} = \coprod_{0\le j\le d} y^{d-j}\{U_j\}$ and $T_d = \coprod_{0\le j\le d} y^{d-j}\{F_j\}$, where the F_j are B/M-lex satisfying $|F_j| = |U_j|$. Then, as in the proof of Lemma 3.10, we have the disjoint unions

$$\{\mathbf{q}V_d\} = \coprod_{0\le j\le d} y^{d-i+1}\{U_j + \mathbf{m}U_{j-1}\}$$

$$\{\mathbf{q}T_d\} = \coprod_{0\le j\le d} y^{d-i+1}\{F_i + \mathbf{m}F_{j-1}\},$$

and we have the inequalities

$$|F_i + \mathbf{m}F_{j-1}| = \max\{|F_j|, |\mathbf{m}F_{j-1}|\} \le \max\{|U_j|, |\mathbf{m}U_{j-1}|\} \le |U_j + \mathbf{m}U_{j-1}|,$$

where the middle inequality holds because B/M is Macaulay-Lex. Since multiplication by y is injective, we get

$$|y^{d-i+1}(F_i + \mathbf{m}F_{j-1})| \le |y^{d-i+1}(U_j + \mathbf{m}U_{j-1})|.$$

□

Lemma 4.3. *Let T_d be a y-compressed W_d-monomial space. Then either T_d is W_d-lex, or there exists a W_d-monomial space F_d, such that F_d is strictly lexicographically greater than T_d, $|F_d| = |T_d|$, and $|\mathbf{q}F_d| \le |\mathbf{q}T_d|$.*

Proof: Let r be as large as possible among the numbers for which we can write

$$T_d = y^{d-r}P \oplus \left(\bigoplus_{i>r} y^{d-i}L_i\right)$$

with P a lex segment of W_d. Such an r always exists, as we can if necessary take $r = 0$.

If $r = d$, then T_d is W_d-lex and we are done. If not, then $yP + L_{r+1}$ is not lex in W. Let m be the lex-greatest monomial of W_{r+1} such that $m \notin yP + L_{r+1}$. We consider two cases depending on whether y divides m or not.

Suppose that y divides m. Let u be the lex-least monomial of $yP + L_{r+1}$. Since P is lex and y does not divide m, it follows that y does not divide u. Let Q be the k-vector space spanned by $\{Q\}$, defined by

$$\{Q\} = \Big(\{yP\} \cup \{L_{r+1}\} \cup \{m\}\Big) \setminus \{u\}\,.$$

Set

$$F_d = y^{d-r-1}Q \oplus \left(\bigoplus_{i>r+1} y^{d-i}L_i\right).$$

Now, $\{F_d\} \setminus y^{d-r-1}m = \{T_d\} \setminus y^{d-r-1}u$. Hence, $\{F_d\}$ is strictly lexicographically greater than T_d. We will compare $\{\mathbf{q}F_d\}$ and $\{\mathbf{q}T_d\}$. The set $\{\mathbf{m}y^{d-r-1}m\}$ is contained in $\{\mathbf{q}T_d\}$, so we have $\mathbf{q}F_d \setminus (\mathbf{q}F_d \cap \mathbf{q}T_d) \subseteq \{y^{d-r}m\}$. Furthermore, we will show that $y^{d-r}u \notin \{\mathbf{q}F_d\}$. Suppose the opposite. Hence, there exists a q such that $y^{d-r}u = x_q\left(y^{d-r}\frac{u}{x_q}\right)$, where $\frac{u}{x_q} \in P$. But $y\frac{u}{x_q} \in yP$ is lex-smaller than u; this contradicts the choice of u. Hence $\{\mathbf{q}T_d\} \setminus (\mathbf{q}F_d \cap \mathbf{q}T_d) \supseteq \{y^{d-r}u\}$. Therefore, we have the desired inequality $|\mathbf{q}F_d| \leq |\mathbf{q}T_d|$. Thus, the lemma is proved in this case.

It remains to consider the case when m is not divisible by y. In this case, m is the lex-greatest monomial not divisible by y that is lex-smaller than all the monomials in $\{L_{r+1}\}$. Set $z = x_{\max(m)}$. In our construction we will use the set

$$N = \left\{ u \in yP \;\middle|\; u \prec_{lex} m \text{ and } \left(\frac{z}{y}\right)^{e_u} u \neq 0 \text{ in } B/M \right\},$$

where e_u is the largest power of y dividing u. We will show that $N \neq \emptyset$ because $\frac{y}{z}m \in N$. Since m is the lex-greatest monomial missing in $m \notin yP + L_{r+1}$, it follows that there exists a monomial $ym' \in yP$ that is lex-smaller than m. Therefore, m' is (non-strictly) lex-smaller than $\frac{m}{z}$. As $m' \in P$ and P is lex, it follows that $\frac{m}{z} \in P$. Thus, $\frac{y}{z}m \in N$ as desired.

We will need three of the properties of N:

Claim.

(1) m is (non-strictly) lex-greater than all the monomials in $\dfrac{z}{y}N$.

(2) $\dfrac{z}{y}N \cap \{L_{r+1}\} = \emptyset$.

(3) $\dfrac{z}{y}N \cap \{yP\} \subseteq N$.

We will prove the claim. (3) is clear. (2) follows from (1) and the fact that in the considered case m is the lex-greatest monomial not divisible by y that is lex-smaller than all the monomials in $\{L_{r+1}\}$. We will prove (1). Write

$$m = x_1^{a_1}x_2^{a_2}\dots z^{a_z} \quad \text{and} \quad u = x_1^{b_1}x_2^{b_2}\dots z^{b_z}wy^{b_y}\,,$$

where w is not divisible by $x_1, \dots, z$ or by y. Suppose that $\frac{z}{y}u = x_1^{b_1}x_2^{b_2}\dots z^{b_z+1}wy^{b_y-1}$ is lex-greater than m. On the other hand, m is lex-greater than u. It follows that $a_j = b_j$ for $j < \max(m)$ and $b_z < a_z \leq b_z + 1$. Since the monomials have the same degree, it follows that $a_z = b_z + 1$, $w = 1$, and $b_y = 1$. Hence $m = \frac{z}{y}u$. The claim is proved.

Let Q be the k-vector space such that

$$\{Q\} = \Big(\{yP + L_{r+1}\} \setminus N\Big) \cup \frac{z}{y}N\,.$$

By the claim above, it follows that we have the disjoint union $\{Q\} = \{L_{r+1}\} \coprod yP \setminus N \coprod \frac{z}{y}N$.

Clearly, $|Q| = |L_{r+1} \oplus yP|$.

We consider the set

$$F_d = y^{d-r-1}Q \oplus \left(\bigoplus_{i>r+1} y^{d-i}L_i \right).$$

It is clear that $|F_d| = |T_d|$. Since $y^{d-r-1}m \in F_d$, we see that F_d is strictly lexicographically greater than T_d. We will show that the inequality $|\mathbf{q}F_d| \leq |\mathbf{q}T_d|$ holds. Set $U = L_{r+1} \oplus yP$ and $V = \oplus_{i>r+1} y^{d-i}L_i$.

Since

$$|\mathbf{q}Q| - |\mathbf{q}U| = -\left| \left\{ \ t \in \mathbf{q}N \setminus (\mathbf{q}N \cap \mathbf{q}(U \setminus N) \ \Big| \ \frac{z}{y}t = 0 \ \right\} \right| \leq 0$$

it follows that $|\mathbf{q}Q| \leq |\mathbf{q}U|$. Furthermore, we have

$$\begin{aligned}
|\mathbf{q}F_d| &= |\mathbf{q}\, y^{d-r-1}Q| + |\mathbf{q}V| - |\mathbf{q}V \cap \mathbf{q}y^{d-r-1}Q| \\
&= |\mathbf{q}\, y^{d-r-1}Q| + |\mathbf{q}V| - \left| y^{d-r-1}\big(L_{r+2} \cap \mathbf{m}\big\{v \in Q \big| y \text{ does not divide } v\big\}\big)\right| \\
&\leq |\mathbf{q}\, y^{d-r-1}U| + |\mathbf{q}V| - \left| y^{d-r-1}\big(L_{r+2} \cap \mathbf{m}\big\{v \in Q \big| y \text{ does not divide } v\big\}\big)\right| \\
&\leq |\mathbf{q}\, y^{d-r-1}U| + |\mathbf{q}V| - \left| y^{d-r-1}\big(L_{r+2} \cap \mathbf{m}\big\{v \in U \big| y \text{ does not divide } v\big\}\big)\right| \\
&= |\mathbf{q}T_d|;
\end{aligned}$$

the first inequality holds because multiplication by y is injective, the second holds by set containment. □

Proof of Theorem 4.1: Let V_d be a W_d-monomial space. If V_d is not W-lex, apply Lemmas 4.2 and 4.3 to obtain a y-compressed W_d-monomial space F_d which is strictly greater lexicographically than V_d and satisfies $|F_d| = |V_d|$ and $|\mathbf{q}F_d| \leq |\mathbf{q}V_d|$. If F_d is not W-lex, we can apply the lemmas again. After finitely many steps, the process must terminate in a lexicographic monomial space. Hence W is d-pro-lex for all degrees $d \geq 0$, and so is Macaulay-Lex. □

5. Lexicographic quotients

Theorem 5.1. *If M is Macaulay-Lex and N is a B/M-lex ideal, then $M + N$ is Macaulay-Lex.*

The theorem follows immediately from the following result:

Proposition 5.2. *Fix a degree $d \geq 1$. If M is $(d-1)$-pro-lex and N is a B/M-lex ideal, then $M + N$ is $(d-1)$-pro-lex.*

Proof: Throughout this proof, for a monomial space $\bar{V}$ in $B/(M + N)$, we denote by V the k-vector space spanned by $\{\bar{V}\}$ in B/M.

Let $\bar{S}_{d-1}$ be a monomial space in $\Big(B/(M+N)\Big)_{d-1}$. Let $\bar{L}_{d-1}$ be the $B/(M+N)$-lexification of $\bar{S}_{d-1}$. Set $\bar{L}_d$ to be the k-vector space spanned by $\mathbf{m}\{\bar{L}_{d-1}\}$ and $\bar{S}_d$ be the k-vector space spanned by $\mathbf{m}\{\bar{S}_{d-1}\}$. We will prove that

$$|\bar{L}_d|^{B/(M+N)} \leq |\bar{S}_d|^{B/(M+N)} .$$

First, we assume that the ideal N has no minimal generators in degree d.

Note that $N_{d-1} + L_{d-1}$ is a B/M-lex-segment. Therefore, $N_{d-1} + L_{d-1}$ is the B/M-lexification of $N_{d-1}+S_{d-1}$ in the ring B/M. Since M is $(d-1)$-pro-lex, the following inequality holds:

$$|N_d + L_d|^{B/M} \leq |N_d + S_d|^{B/M} .$$

On the other hand,

$$|N_d + L_d|^{B/M} = |N_d|^{B/M} + |L_d|^{B/M} - |N_d \cap L_d|^{B/M}$$
$$|N_d + S_d|^{B/M} = |N_d|^{B/M} + |S_d|^{B/M} - |N_d \cap S_d|^{B/M}$$

Therefore, we obtain the inequality

$$|L_d|^{B/M} - |N_d \cap L_d|^{B/M} \leq |S_d|^{B/M} - |N_d \cap S_d|^{B/M} .$$

Note that the left hand-side is equal to $|\bar{L}_d|^{B/(M+N)}$ whereas the right-hand side is equal to $|\bar{S}_d|^{B/(M+N)}$. Thus, we get the desired inequality

$$|\bar{L}_d|^{B/(M+N)} \leq |\bar{S}_d|^{B/(M+N)} .$$

Now, suppose that N has minimal monomial generators in degree d.

If $L_d \subseteq N_d$, then

$$0 = |\bar{L}_d|^{B/(M+N)} \leq |\bar{S}_d|^{B/(M+N)} .$$

Suppose that $L_d \not\subseteq N_d$. Set $Q = \{N_d\} \setminus \{\mathbf{m}N_{d-1}\}$. Since both $\mathbf{m}N_{d-1} + L_d$ and N_d are B/M-lex-segments, it follows that one of them contains the other. Hence $\{L_d\} \supseteq Q$, and therefore

$$|\bar{L}_d|^{B/(M+N)} = |L_d|^{B/(M+(N_{d-1}))} - |Q| .$$

The argument above (for the case when the ideal is $(d-1)$-generated) can be applied to N_{d-1}, and it yields

$$|L_d|^{B/(M+(N_{d-1}))} \leq |S_d|^{B/(M+(N_{d-1}))} .$$

Therefore we have

$$\begin{aligned}|\bar{L}_d|^{B/(M+N)} &= |L_d|^{B/(M+(N_{d-1}))} - |Q| \\ &\le |S_d|^{B/(M+(N_{d-1}))} - |Q| \le |S_d|^{B/(M+(N_{d-1}))} - |Q \cap \{S_d\}| \\ &= |\bar{S}_d|^{B/(M+N)}\,.\end{aligned}$$

□

Macaulay's Theorem [Ma] says that 0 is pro-lex. Hence, Theorem 5.1 applied to $M = 0$ yields the following:

Corollary 5.3. *If U is a B-lex ideal then it is Macaulay-Lex.*

Remark 5.4. Following [Sh], we say that a monomial ideal M in B is *piecewise lex* if, whenever $\mathbf{x}^{\mathbf{a}} \in M$, $\mathbf{x}^{\mathbf{b}} \succ_{lex} \mathbf{x}^{\mathbf{a}}$, and $\max(\mathbf{x}^{\mathbf{b}}) \le \max(\mathbf{x}^{\mathbf{a}})$, we have $\mathbf{x}^{\mathbf{b}} \in M$. Shakin [Sh] proved that if M is a piecewise lex ideal in B, then it is Macaulay-Lex. This result can be proved differently using our technique as follows: We induct on n. Let $\mathbf{x}^{\mathbf{a}_1}, \ldots, \mathbf{x}^{\mathbf{a}_r}$ be the minimal monomial generators of M divisible by x_n. So the lex segment L_j ending in $\mathbf{x}^{\mathbf{a}_j}$ must be contained in M. Set $N = M \cap k[x_1, \cdots, x_{n-1}]$. Then N is piecewise lex and so by induction is Macaulay-Lex in $k[x_1, \cdots, x_{n-1}]$. By Theorem 4.1, N is Macaulay-Lex in B. By induction on j, we conclude that $(N+L_1+\ldots+L_{j-1})+L_j$ is Macaulay-Lex by Theorem 5.1. Hence, $M = N+L_1+\ldots+L_r$ is Macaulay-Lex as well. □

References

[Bi] A. Bigatti: Upper bounds for the Betti numbers of a given Hilbert function, *Comm. in Algebra* **21** (1993), 2317–2334.

[CL] G. Clements and B. Lindström: A generalization of a combinatorial theorem of Macaulay, *J. Combinatorial Theory* **7** (1969), 230–238.

[EK] S. Eliahou and M. Kervaire: Minimal resolutions of some monomial ideals, *J. Algebra*, **129** (1990), 1–25.

[Gr] M. Green: Generic initial ideals, in *Six lectures on commutative algebra*, Birkhäuser, Progress in Mathematics **166**, (1998), 119–185.

[Ha] R. Hartshorne: Connectedness of the Hilbert scheme, *Publications Mathématiques IHES* **29** (1966), 5-48.

[Ka] G. Katona: A theorem for finite sets, *Theory of Graphs* (P. Erdös and G. Katona, eds.), Academic Press, New York (1968), 187–207.

[Kr] J. Kruskal: The number of simplices in a complex, *Mathematical Optimization Techniques* (R. Bellman, ed.), University of California Press, Berkeley/Los Angeles (1963), 251–278.

[Ma] F. Macaulay: Some properties of enumeration in the theory of modular systems, *Proc. London Math. Soc.* **26** (1927), 531–555.

[Sh] D. A. Shakin: Piecewise lexsegment ideals, *Mat. Sbornik* **194** (2003), 1701-1724.