

## Ideals containing the squares of the variables

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Abstract: We study the Betti numbers of graded ideals containing the squares of the variables, in a polynomial ring. We prove the lex-plus-powers conjecture for such ideals.

### 1. Introduction

Throughout the paper  $S = k[x_1, \dots, x_n]$  is a polynomial ring over a field  $k$  graded by  $\deg(x_i) = 1$  for all  $i$ , and  $P = (x_1^2, \dots, x_n^2)$  is the ideal generated by the squares of the variables.

It is well known how the Hilbert function changes when we add  $P$  to a squarefree monomial ideal  $I$ ; this is given by the relation between the  $f$ -vector and the  $h$ -vector, cf. [BH]. It has been an open question how the Betti numbers change. We answer this question in Theorem 2.1, which provides a relation between the Betti numbers of  $I$  and those of  $I + P$ . In Theorem 3.3, we describe a basis of the minimal free resolution of  $I + P$  in the case when  $I$  is Borel.

By Kruskal-Katona's Theorem [Kr,Ka], there exists a squarefree lex ideal  $L$  such that  $L + P$  has the same Hilbert function as  $I + P$ . The ideal  $L + P$  is called lex-plus-squares. It was conjectured by Herzog and Hibi that the graded Betti numbers of  $L + P$  are greater than or equal to those of  $I + P$ . Later, Graham Evans conjectured the more general lex-plus-powers conjecture that, among all graded ideals with a fixed Hilbert function and containing a homogeneous regular sequence in fixed degrees, the lex-plus-powers ideal has greatest graded Betti numbers in characteristic 0. This conjecture is

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very difficult and wide open. Some special cases are proved by G. Evans, C. Francisco, B. Richert, and S. Sabourin [ER,Fr1,Fr2,Ri,RS]. An expository paper describing the current status of the conjecture is [FR]. In 5.1 we prove the following:

**Theorem 1.1.** *Suppose that  $\text{char}(k) = 0$ . Let  $F$  be a graded ideal containing  $P = (x_1^2, \dots, x_n^2)$ . Let  $L$  be the squarefree lex ideal such that  $F$  and the lex-plus-squares ideal  $L + P$  have the same Hilbert function. The graded Betti numbers of the lex-plus-squares ideal  $L + P$  are greater than or equal to those of  $F$ .*

The methods used in [Bi,Hu,Pa,CGP] to show that the lex ideal has greatest Betti numbers are not applicable; see Examples 3.10 and 3.11. We use the technique of compression. Compression was introduced by Macaulay [Ma], and used by [Ma, CL,MP1,MP2,Me1,Me2] to study Hilbert functions. It is not known how Betti numbers behave under compression, but it is reasonable to expect that they increase. We address this in Section 4.

The proof of Theorem 1.1 consists of the following steps:

- In Section 5 (the proof of Theorem 5.1.), we reduce to the case of a squarefree Borel ideal (plus squares); this is not immediate because a generic change of variables does not preserve  $P$ .
- In Section 3, we reduce to the case of a squarefree  $\{x_1, \dots, x_{n-1}\}$ -compressed Borel ideal (plus squares).
- In Section 4, we deal with squarefree  $\{x_1, \dots, x_{n-1}\}$ -compressed Borel ideals.

Given the intricacy of the proof in the Borel case (Section 3 and 4), we think that the following particular case of the lex-plus-powers conjecture is of interest:

**Conjecture 1.2.** *The lex-plus-powers ideal has greatest graded Betti numbers among all Borel-plus-powers monomial ideals with the same Hilbert function.*

A refinement of the lex-plus-powers conjecture is to study consecutive cancellations in Betti numbers. In view of the result in [Pe], it is natural to ask:

**Problem 1.3.** *Under the assumptions of Theorem 1.1, is it true that the Betti numbers of  $L + P$  and those of  $F$  differ by consecutive cancellations?*

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## 2. Squarefree monomial ideals plus squares

A monomial ideal is called *squarefree* if it is generated by squarefree monomials. If  $I$  is squarefree, then  $I + P$  is called *squarefree-plus-squares*.

For a monomial  $m$ , let  $\max(m)$  be the index of the lex-last variable dividing  $m$ , that is,  $\max(m) = \max\{i \mid x_i \text{ divides } m\}$ .

The ring  $S$  is standardly graded by  $\deg(x_i) = 1$  for each  $i$ . In addition,  $S$  is  $\mathbf{N}^n$ -graded by setting the multidegree of  $x_i$  to be the  $i$ 'th standard vector in  $\mathbf{N}^n$ . Usually we say that  $S$  is *multigraded* instead of  $\mathbf{N}^n$ -graded, and we say *multidegree* instead of  $\mathbf{N}^n$ -degree. For every vector  $\mathbf{a} = (a_1, \dots, a_n) \in \mathbf{N}^n$  there exists a unique monomial of degree  $\mathbf{a}$ , namely  $\mathbf{x}^{\mathbf{a}} = x_1^{a_1} \cdots x_n^{a_n}$ . If an element  $g$  (say in a module) has  $\mathbf{N}^n$ -degree  $\mathbf{a}$ , then we say that it has *multidegree*  $\mathbf{x}^{\mathbf{a}}$  and denote  $\deg(g) = \mathbf{x}^{\mathbf{a}}$ . Denote by  $S(-\mathbf{x}^{\mathbf{a}})$  the free  $S$ -module generated by one element in multidegree  $\mathbf{x}^{\mathbf{a}}$ . Every monomial ideal is multihomogeneous, so it has a multigraded minimal free resolution. Thus, the minimal free resolutions of  $S/I$  and  $S/(I + P)$  are graded and multigraded. We will use both gradings.

For a subset  $\sigma \subseteq \{1, \dots, n\}$ , let  $|\sigma|$  denote the number of elements in  $\sigma$ . We will abuse notation to sometimes identify a subset with the squarefree monomial supported on it, so  $\sigma$  may stand for  $\prod_{j \in \sigma} x_j$ . It will always be clear from context what is meant.

By  $S(-2\sigma)$  we denote the free  $S$ -module generated in multidegree  $\prod_{i \in \sigma} x_i^2$ .

**Theorem 2.1.** *Let  $I$  be a squarefree monomial ideal.*

(1) Set  $F_i = \bigoplus_{|\sigma|=i} S/(I : \sigma)(-2\sigma)$ , where  $\sigma \subseteq \{1, \dots, n\}$ . We have the long exact sequence

$$(2.2) \quad 0 \rightarrow F_n \xrightarrow{\varphi_n} \cdots \xrightarrow{\varphi_2} F_1 \xrightarrow{\varphi_1} F_0 = S/I \rightarrow S/(I + P) \rightarrow 0$$

with maps  $\varphi_i$  the Koszul maps for the sequence  $x_1^2, \dots, x_n^2$ .

(2) Each of the ideals  $(I : \sigma)$  in (1) is a squarefree monomial ideal.

(3)  $S/(I + P)$  is minimally resolved by the iterated mapping cones from (2.2).

(4) For the graded Betti numbers of  $S/(I + P)$  we have

$$b_{p,s}(S/(I + P)) = \sum_{i=0}^n \left( \sum_{|\sigma|=i} b_{p+i,s+2i}(S/(I : \sigma)) \right).$$

*Proof:* Since the ideal  $I$  is squarefree, it follows that  $(I : \sigma^2) = (I : \sigma)$  is squarefree.

We have the exact Koszul complex  $\mathbf{K}$  for the sequence  $x_1^2, \dots, x_n^2$ :

$$\begin{aligned} 0 \rightarrow \bigoplus_{|\sigma|=n} S \xrightarrow{\varphi_n} \dots \rightarrow \bigoplus_{|\sigma|=i} S \xrightarrow{\varphi_i} \bigoplus_{|\sigma|=i-1} S \rightarrow \dots \\ \dots \rightarrow \bigoplus_{|\sigma|=1} S = \bigoplus_{j=1}^n S \xrightarrow{\varphi_1} \bigoplus_{|\sigma|=0} S = S \rightarrow S/P \rightarrow 0. \end{aligned}$$

We can write  $\mathbf{K} = \mathbf{K}' \oplus \mathbf{K}''$ , where  $\mathbf{K}'$  consists of the components of  $\mathbf{K}$  in all multidegrees  $m \notin I$ , and  $\mathbf{K}''$  consists of the components of  $\mathbf{K}$  in all multidegrees  $m \in I$ . Both  $\mathbf{K}'$  and  $\mathbf{K}''$  are exact. We will show that (2.2) coincides with  $\mathbf{K}'$ . Consider  $\mathbf{K}$  as an exterior algebra on basis  $e_1, \dots, e_n$ . The multidegree of the variable  $e_j$  is  $x_j^2$ . Let  $f = me_{j_1} \wedge \dots \wedge e_{j_i}$  be an element in  $\mathbf{K}_i$  and  $m$  be a monomial. The multidegree of  $f$  is  $mx_{j_1}^2 \dots x_{j_i}^2$ . We have that  $f \in \mathbf{K}'$  if and only if  $mx_{j_1}^2 \dots x_{j_i}^2 \notin I$ , if and only if  $mx_{j_1} \dots x_{j_i} \notin I$ , if and only if  $m \notin (I : x_{j_1} \dots x_{j_i})$ . Therefore, we have the vector space isomorphism

$$\begin{aligned} \mathbf{K}'_i &\rightarrow \bigoplus_{|\sigma|=i} S/(I : \sigma) \\ me_{j_1} \wedge \dots \wedge e_{j_i} &\mapsto m \in S/(I : x_{j_1} \dots x_{j_i}). \end{aligned}$$

This proves (1).

We will prove (3) by induction on  $n - i$ . Denote by  $K_i$  the kernel of  $\varphi_i$ . We have the short exact sequence

$$0 \rightarrow K_i \rightarrow \bigoplus_{|\sigma|=i} S/(I : \sigma) \rightarrow K_{i-1} \rightarrow 0.$$

Each of the ideals  $(I : \sigma)$  is squarefree. By Taylor's resolution, it follows that the Betti numbers of  $\bigoplus_{|\sigma|=i} S/(I : \sigma)$  are concentrated in squarefree multidegrees. On the other

hand, the entries in the matrix of the map  $\varphi_i$  are squares of the variables. Therefore, there can be no cancellations in the mapping cone. Hence, the mapping cone yields a minimal free resolution of  $K_{i-1}$ .

(4) follows from (3). □

The Hilbert function of a graded finitely generated module  $T$  is

$$\text{Hilb}_T(i) = \dim_k(T_i).$$

For squarefree ideals, we consider also the squarefree Hilbert function,  $\text{sHilb}$ , that counts only squarefree monomials. It is well-known that if  $I$  and  $J$  are squarefree ideals, then  $S/I$  and  $S/J$  have the same Hilbert function if and only if  $S/(I+P)$  and  $S/(J+P)$  have the same Hilbert function; thus,  $I$  and  $J$  have the same Hilbert function if and only if they have the same squarefree Hilbert function.

**Proposition 2.3.** *Let  $I$  and  $J$  be squarefree monomial ideals with the same Hilbert function. Fix an integer  $1 \leq p \leq n$ . The graded modules  $\bigoplus_{|\sigma|=p} (I : \sigma)$  and  $\bigoplus_{|\sigma|=p} (J : \sigma)$  have the same Hilbert function and the same squarefree Hilbert function.*

*Proof:* We consider squarefree Hilbert functions. Set  $I(p) = \bigoplus_{|\sigma|=p} (I : \sigma)$ . Let  $\tau$  be a

squarefree monomial of degree  $d$  in  $(I : \nu)$ . Then  $\nu\tau \in I_{d+p}$ . If  $|\nu \cap \tau| = s$ , choose  $\mu$  so that  $\mu = \text{lcm}(\nu, \tau)$  is a squarefree monomial in  $I_{d+p-s}$ .

Let  $\mu$  be a squarefree monomial in  $I_{d+p-s}$ . We can choose  $\nu$  in  $\binom{d+p-s}{p}$  ways so that  $|\nu| = p$  and  $\nu$  divides  $\mu$ . For each so chosen  $\nu$ , we can choose  $\tau$  in  $\binom{p}{s}$  ways so that  $|\nu \cap \tau| = s$  and  $\tau$  divides  $\mu$ . Therefore, the monomial  $\mu$  contributes  $\binom{d+p-s}{p} \binom{p}{s}$  monomials in  $(I(p))_d$ . For such a monomial, we say that it is coming from  $I_{d+p-s}$ , or that its source is  $I_{d+p-s}$ .

Suppose that one element in  $(I(p))_d$  can be obtained in two different ways by this procedure. Since we have the same element in  $(I(p))_d$ , it follows that  $\nu$  and  $\tau$  are fixed. But then,  $\mu$  and  $s$  are uniquely determined. Hence, both the source and  $\mu$  are uniquely determined. Therefore, one and the same element in  $(I(p))_d$  cannot be obtained in two different ways by the above procedure.

For a vector space  $Q$  spanned by monomials, we denote by  $\text{sdim}(Q)$  the number of squarefree monomials in  $Q$ . We have shown that

$$(2.4) \quad \text{sdim} \left( \bigoplus_{|\sigma|=p} (I : \sigma) \right)_d = \sum_{x=0}^n \binom{d+p-s}{p} \binom{p}{s} \text{sdim}(I_{d+p-s}).$$

The same formula holds for  $J$  as well. Now, the proposition follows from the well-known fact that for each  $j \geq 0$  we have that  $\text{sdim}(I_j) = \text{sdim}(J_j)$  since  $I$  and  $J$  are squarefree ideals with the same Hilbert function.  $\square$

Let  $I$  be a squarefree monomial ideal, and  $\Delta$  be its Stanley-Reisner simplicial complex. Let  $\sigma \subseteq \{1, \dots, n\}$ . We remark that it is well-known that the Stanley-Reisner simplicial complex of  $(I : \sigma)$  is  $\text{star}_\Delta(\sigma) = \{\tau \in \Delta \mid \tau \cup \sigma \in \Delta\}$ .

### 3. Squarefree Borel ideals plus squares

A squarefree monomial ideal  $N$  is *squarefree Borel* if, whenever  $mx_j \in N$  is a squarefree monomial, and  $i < j$  and  $mx_i$  is squarefree, we have  $mx_i \in N$  as well. A squarefree monomial ideal  $L$  is *squarefree lex* if, whenever  $m \in L$  is a squarefree monomial and  $m'$  is a squarefree monomial lexicographically greater than  $m$ , we have  $m' \in L$  as well.

If  $N$  is squarefree Borel, then by Kruskal-Katona's Theorem [Kr,Ka], there exists a squarefree lex ideal  $L$  with the same Hilbert function.

#### Lemma 3.1.

- (1) Let  $N$  be a squarefree Borel ideal. For any  $\sigma \subseteq \{1, \dots, n\}$ , the ideal  $(N : \sigma)$  is squarefree Borel in the ring  $S/(\{x_i \mid i \in \sigma\})$ .
- (2) Let  $L$  be a squarefree lex ideal. For any  $\sigma \subseteq \{1, \dots, n\}$ , the ideal  $(L : \sigma)$  is squarefree lex in the ring  $S/(\{x_i \mid i \in \sigma\})$ .

The ideals  $(N : \sigma)$  and  $(L : \sigma)$  are generated by monomials in the smaller ring  $S/(\{x_i \mid i \in \sigma\})$ , so we may view them as ideals of  $S/(\{x_i \mid i \in \sigma\})$ .

For a monomial ideal  $M$ , we denote by  $\text{gens}(M)$  the set of monomials that generate  $M$  minimally.

**Construction 3.2.** If  $N$  is squarefree Borel, then the minimal free resolution of  $S/N$  is the squarefree Eliahou-Kervaire resolution [AHH] with basis denoted

$$\{1\} \cup \left\{ (h, \alpha) \mid h \in \text{gens}(N), \alpha \subset \{1, \dots, n\}, h\alpha \text{ is squarefree, } \max(\alpha) < \max(h) \right\}.$$

The basis element  $(h, \alpha)$  has homological degree  $\deg(\alpha) + 1$ , degree  $\deg(h) + \deg(\alpha)$ , and multidegree  $h\alpha$ ; the basis element 1 has homological degree 0 and degree 0. In order to describe a basis of the minimal free resolution of  $S/(N + P)$  we introduce

EK-triples. For  $\sigma \subseteq \{1, \dots, n\}$ , we say that  $(\sigma, h, \alpha)$  is an *EK-triple* if  $(h, \alpha)$  is a basis element in the minimal free resolution of  $S/(N : \sigma)$ . By Lemma 3.1, it follows that  $(\sigma, h, \alpha)$  is an EK-triple  $(\sigma, h, \alpha)$  if and only if:

- $h \in \text{gens}(N : \sigma)$
- $\alpha = \{j_1, \dots, j_t\}$  is an increasing sequence of numbers in the set  $\{i \mid i \notin \sigma\}$ , such that  $1 \leq j_1 < \dots < j_t < \max(h)$
- $\sigma h \alpha$  is squarefree.

By Theorem 2.1, Lemma 3.1, and Construction 3.2, it follows that:

**Theorem 3.3.** *Let  $N$  be a squarefree Borel ideal. The minimal free resolution of  $S/(N + P)$  has basis consisting of  $\{1\}$  and the EK-triples. An EK-triple  $(\sigma, h, \alpha)$  has homological degree  $|\sigma| + |\alpha|$  and degree  $2|\sigma| + |\alpha| + |h|$ ; it has multidegree  $\sigma^2 h \alpha$ . In particular, for all  $p, s \geq 0$ , the graded Betti number  $b_{p,s}(S/(N + P))$  is equal to the number of EK-triples such that  $p = |\sigma| + |\alpha|$  and  $s = 2|\sigma| + |\alpha| + |h|$ .*

We will prove:

**Theorem 3.4.** *Let  $N$  be a squarefree Borel and  $L$  be the squarefree lex with the same Hilbert function, (equivalently, let  $N + P$  and  $L + P$  have the same Hilbert function). For all  $p, s$ , the graded Betti numbers satisfy*

$$b_{p,s}(S/(L + P)) \geq b_{p,s}(S/(N + P)).$$

For the proof, we need the notion of compression. Compression of ideals was introduced by Macaulay [Ma], and was used by Clements-Lindström [CL], Macaulay [Ma], Mermin [Me1,Me2], Mermin-Peeva [MP1,MP2] to study Hilbert functions.

**Definition 3.5.** Let  $I$  be a squarefree monomial ideal. We denote by  $\bar{I}$  the monomial ideal in  $S/P$  generated by the squarefree monomials generating  $I$ .

Let  $\mathcal{A}$  be a subset of the variables  $\{x_1, \dots, x_n\}$ ; its complement is  $\mathcal{A}^c = \{x_1, \dots, x_n\} \setminus \mathcal{A}$ . Denote by  $\oplus_f$  the direct sum over all squarefree monomials  $f$  in the variables in  $\mathcal{A}^c$ . The vector space  $\bar{I}$  may be written uniquely in the form

$$\bar{I} = \bigoplus_f f \bar{V}_f,$$

where  $V_f$  is an ideal in the ring  $k[\mathcal{A}] = k[x_i \mid x_i \in \mathcal{A}]$ .

We say that  $\bar{I}$ , or  $I$ , is  $\mathcal{A}$ -compressed if each  $V_f$  is squarefree lex in  $k[\mathcal{A}]$ .

Furthermore, we say that  $I$  is  $j$ -compressed if it is  $\mathcal{A}$ -compressed for all subsets  $\mathcal{A}$  of size  $j$ . In this case,  $I$  is  $i$ -compressed for every  $i \leq j$ .

We remark, that in this paper we consider squarefree-compression, that is, we think of the squarefree monomial ideals as ideals in the quotient ring  $S/P$  and consider compression there, but, for simplicity, we say “compression” instead of “squarefree-compression”.

Denote by  $W_f$  the squarefree lex ideal of  $k[\mathcal{A}]$  with the same Hilbert function as  $V_f$ . Set  $\bar{C} = \bigoplus_f f \bar{W}_f$ .

We will prove that  $\bar{C}$  is an ideal: If  $x_i \in \mathcal{A}$ , then  $x_i \bar{C} \subseteq \bar{C}$  since each  $W_f$  is an ideal in  $k[\mathcal{A}]$ . Choose an  $x_i \notin \mathcal{A}$ , and fix  $f$ . Then either  $x_i f = 0$  or  $\bar{V}_{x_i f} \supseteq \bar{V}_f$ . Hence, either  $x_i f = 0$  or  $\bar{W}_{x_i f} \supseteq \bar{W}_f$ . Therefore,  $x_i \bar{C} \subseteq \bar{C}$ .

We say that the ideal  $C$  is the  $\mathcal{A}$ -compression of  $I$ .

We need the following lemmas:

**Lemma 3.6.** *Let  $I$  be a squarefree ideal. If  $I$  is  $(2i - 2)$ -compressed for some  $i \geq 1$ , then  $I$  is  $(2i - 1)$ -compressed.*

*Proof:* Let  $v \in I$  be a squarefree monomial. Suppose that  $u$  is a squarefree monomial of the same degree such that  $u > v$ . Set  $w = \gcd(u, v)$ , so that we can write  $u = u'w$ ,  $v = v'w$  with  $\gcd(u', v') = 1$ . Suppose  $|u'v'| \leq 2i - 1$ . Denote by  $\mathcal{B}$  the set of variables that appear in exactly one of the monomials  $u$  and  $v$ . Since  $u$  and  $v$  have the same degree, it follows that the number of variables in  $\mathcal{B}$  is even. Since  $|\mathcal{B}| \leq 2i - 1$ , we have  $|\mathcal{B}| \leq 2i - 2$ . Hence,  $I$  is  $\mathcal{B}$ -compressed. Therefore,  $v \in I$  implies that  $u \in I$ .  $\square$

**Lemma 3.7.** *Let  $N$  be a squarefree Borel ideal. Its  $\{x_1, \dots, x_{n-1}\}$ -compression  $J$  is a squarefree Borel ideal.*

*Proof:* We have to show that  $\bar{J}$  is a squarefree Borel ideal. Consider the disjoint unions

$$\bar{N} = x_n \bar{V}_{x_n} \cup \bar{V}_1 \quad \text{and} \quad \bar{J} = x_n \bar{W}_{x_n} \cup \bar{W}_1.$$

Set  $\mathbf{n} = (x_1, \dots, x_{n-1})$ . Since  $\bar{N}$  is squarefree Borel, it follows that  $\mathbf{n} \bar{V}_{x_n} \subseteq \bar{V}_1$ . By Kruskal-Katona's Theorem [Kr, Ka] it follows that  $\mathbf{n} \bar{W}_{x_n} \subseteq \bar{W}_1$ . If  $x_n m$  is a monomial in  $x_n \bar{W}_{x_n}$ , then for each  $1 \leq i < n$  we have that  $x_i m \in \bar{W}_1$ . If  $x_j$  divides a monomial  $m \in \bar{W}_1$  (respectively,  $\bar{W}_{x_n}$ ), then for each  $1 \leq i \leq j$  we have that  $\frac{x_i m}{x_j} \in \bar{W}_1$



(respectively,  $\bar{W}_{x_n}$ ) since  $\bar{W}_1$  (respectively,  $\bar{W}_{x_n}$ ) is squarefree lex. Thus,  $\bar{J}$  is squarefree Borel.  $\square$

**Lemma 3.8.** *Let  $N$  be a squarefree Borel ideal and  $J$  be its  $\{x_1, \dots, x_{n-1}\}$ -compression. For all  $p, s$ , the graded Betti numbers satisfy*

$$b_{p,s}(S/(J+P)) \geq b_{p,s}(S/(N+P)).$$

*Proof:* Set  $A = S/x_n$ . We assume, by induction on the number of variables, that Theorem 3.4 holds over the polynomial ring  $A$ .

Consider the EK-triples  $(\sigma, h, \alpha)$  for  $N$  in degree  $(p, s)$ . Let  $c_{p,s}(N)$  be the number of triples such that  $x_n$  divides  $\sigma$ ; let  $d_{p,s}(N)$  be the number of triples such that  $x_n$  divides  $h$ ; let  $e_{p,s}(N)$  be the number of triples such that  $x_n$  does not divide  $\sigma h \alpha$ . Since  $x_n$  cannot divide  $\alpha$  by Construction 3.2, it follows by Theorem 3.3 that

$$b_{p,s}(S/(N+P)) = c_{p,s}(N) + d_{p,s}(N) + e_{p,s}(N).$$

Similarly, we introduce the numbers  $c_{p,s}(J)$ ,  $d_{p,s}(J)$ ,  $e_{p,s}(J)$  and get

$$b_{p,s}(S/(J+P)) = c_{p,s}(J) + d_{p,s}(J) + e_{p,s}(J).$$

We will show that the above introduced numbers for  $J$  are greater than or equal to the corresponding numbers for  $N$ .

As in the proof of Lemma 3.7, we write  $\bar{N} = \bar{V}_1 \oplus x_n \bar{V}_{x_n}$  and  $\bar{J} = \bar{W}_1 \oplus x_n \bar{W}_{x_n}$ .

First, we consider the number  $c_{p,s}(N)$ . Note that  $\bar{V}_{x_n} \supset \bar{V}_1$ . Therefore,  $(\bar{N} : x_n \tau) = (\bar{V}_{x_n} : \tau)$  for  $x_n \notin \tau$ . Hence, the EK-triples counted by  $c_{p,s}(N)$  correspond bijectively to the EK-triples for  $V_{x_n}$  of degree  $(p-1, s-2)$  by the correspondence  $(x_n \tau, h, \alpha) \iff (\tau, h, \alpha)$ . Thus,

$$c_{p,s}(N) = b_{p-1,s-2}(A/(V_{x_n} + (x_1^2, \dots, x_{n-1}^2))).$$

By Lemma 3.7,  $J$  is squarefree Borel, so we get the same formula for  $J$ . By the construction of compression, the ideal  $W_{x_n}$  is the squarefree lex ideal in the polynomial ring  $A$  with the same Hilbert function as  $V_{x_n}$ . Since Theorem 3.4 holds over the ring  $A$  by the induction hypothesis, we conclude that  $c_{p,s}(J) \geq c_{p,s}(N)$ .

Now we consider the number  $e_{p,s}(N)$ . The EK-triples counted by  $e_{p,s}(N)$  are exactly the EK-triples for  $V_1$ . Hence,  $e_{p,s}(N) = b_{p,s}(A/(V_1 + (x_1^2, \dots, x_{n-1}^2)))$ . The

same equality holds for  $e_{p,s}(J)$ . By the construction of compression, the ideal  $W_1$  is the squarefree lex ideal in the polynomial ring  $A$  with the same Hilbert function as  $V_1$ . Since Theorem 3.4 holds over the ring  $A$  by the induction hypothesis, we conclude that  $e_{p,s}(J) \geq e_{p,s}(N)$ .

It remains to consider  $d_{p,s}(N)$ . Since  $N$  is squarefree Borel, it follows that  $\bar{V}_1 \supseteq \mathbf{n}\bar{V}_{x_n}$ . Therefore, for each degree  $j$

$$\{\text{gens}(N : \sigma)_j \text{ that are divisible by } x_n\} = \{(\bar{x}_n(V_{x_n} : \sigma))_j\} \setminus \{(x_n(\bar{V}_1 : \sigma))_j\}.$$

Hence, for each degree  $j$ , the number of minimal monomial generators of degree  $j$  of  $(N : \sigma)$  that are divisible by  $x_n$  is

$$\dim_k(\bar{V}_{x_n} : \sigma)_{j-1} - \dim_k(\bar{V}_1 : \sigma)_{j-1}.$$

For each such minimal monomial generator  $h$ , we have that  $\max(h) = n$ . Since  $\alpha$  is prime to  $\sigma$  and  $\text{supp}(h)$ , by Construction 3.2 we see that there are  $\binom{n - |h| - |\sigma|}{|\alpha|} = \binom{n - j - |\sigma|}{p - |\sigma|}$  possibilities for  $\alpha$  in the EK-triples. By Theorem 3.3, we conclude that

$$d_{p,s}(N) = \sum_{n \notin \sigma} \binom{n - j - |\sigma|}{p - |\sigma|} \left( \dim_k(\bar{V}_{x_n} : \sigma)_{s-p-|\sigma|-1} - \dim_k(\bar{V}_1 : \sigma)_{s-p-|\sigma|-1} \right).$$

As the ideal  $J$  is squarefree Borel by Lemma 3.6, the same formula holds for  $J$ . By the construction of compression,  $\bar{V}_1$  and  $\bar{W}_1$  have the same Hilbert function, as do  $\bar{V}_{x_n}$  and  $\bar{W}_{x_n}$ . By the displayed formula for  $d_{p,s}$  above and Proposition 2.3, the number  $d_{p,s}$  depends only on these Hilbert functions. Therefore,  $d_{p,s}(J) = d_{p,s}(N)$ .  $\square$

**Main Lemma 3.9.** *Let  $N$  be a squarefree Borel  $\{x_1, \dots, x_{n-1}\}$ -compressed ideal. Suppose that  $N$  is not squarefree lex. There exists a squarefree Borel ideal  $T$  such that:*

- $T$  has the same Hilbert function as  $N$
- $T$  is lexicographically greater than  $N$  (here “lexicographically greater” means that for each  $d \geq 0$  we order the monomials in  $N_d$  and  $T_d$  lexicographically, and then compare the two ordered sets lexicographically)

◦ for all  $p, s$ , the graded Betti numbers satisfy

$$b_{p,s}(S/(T + P)) \geq b_{p,s}(S/(N + P)).$$

The proof of Lemma 3.9 is long and very technical. We present it in the next section.

*Proof of Theorem 3.4:* Let  $N$  be a squarefree Borel ideal. By Lemma 3.7, we can assume that  $N$  is  $\{x_1, \dots, x_{n-1}\}$ -compressed. Lemma 3.9 implies that we can replace  $N$  by a squarefree Borel ideal which is lexicographically greater.

We proceed in this way until we reach the squarefree lex ideal  $L$ . This process is finite since there exist only finitely many squarefree Borel ideals with a fixed Hilbert function.  $\square$

**Example 3.10.** It is natural to ask if Green's Theorem can be used, as in [CGP], in order to obtain a short proof of Theorem 3.4. Unfortunately, in the example  $N = (ab, ac, bc)$ ,  $L = (ab, ac, ad, bcd)$  in  $k[a, b, c, d]$ , one of the inequalities needed for the proof does not hold. Thus, the short proof in [CGP] cannot be generalized to cover Theorem 3.4.

Furthermore, the inequality

$$\sum_{|\sigma|=i} b_{r,s}(S/(N : \sigma)) \leq \sum_{|\sigma|=i} b_{r,s}(S/(L : \sigma))$$

may not hold. For example, it fails for  $S = k[a, b, c, d, e]$  and

$$N = (abc, abd, acd, bcd)$$

and  $i = 2$ . In this case we have  $L = (abc, abd, abe, acd, bcde)$ . Computer computation gives

$$\sum_{|\sigma|=2} b_{1,2}(S/(N : \sigma)) = 12 \quad \text{while} \quad \sum_{|\sigma|=2} b_{1,2}(S/(L : \sigma)) = 11,$$

and

$$\sum_{|\sigma|=2} b_{2,2}(S/(N : \sigma)) = 6 \quad \text{while} \quad \sum_{|\sigma|=2} b_{2,2}(S/(L : \sigma)) = 5.$$

**Example 3.11.** Let  $N$  be squarefree Borel and  $L$  be squarefree lex with the same Hilbert function (equivalently, let  $N + P$  and  $L + P$  have the same Hilbert function). It is natural to ask:

*Question:* Are the graded Betti numbers of  $S/(L + (x_1^2, \dots, x_i^2))$  greater or equal to those of  $S/(N + (x_1^2, \dots, x_i^2))$ , for each  $i$ ?

This question is closely related to a result proved by Charalambous and Evans [CE]. Let  $M$  be a squarefree Borel ideal. Set  $P(i) = (x_1^2, \dots, x_i^2)$  and  $P(0) = 0$ . By [CE], for each  $0 \leq i < n$ , the mapping cone of the short exact sequence

$$0 \rightarrow S/((M + P(i)) : x_{i+1}) \rightarrow S/(M + P(i)) \rightarrow S/(M + P(i+1)) \rightarrow 0$$

yields a minimal free resolution of  $S/(M + P(i+1))$ .

The following example gives a negative answer to the above question. Let  $A = k[a, b, c, d, e, f]$  and  $T$  be the ideal generated by the squarefree cubic monomials. The ideal  $N = (ab, ac, ad, bc, bd) + T$  is squarefree Borel. The ideal  $L = (ab, ac, ad, ae, af) + T$  is squarefree lex. The ideals  $N$  and  $L$  have the same Hilbert function. The graded Betti numbers of  $S/(L + (x_1^2))$  are not greater or equal to those of  $S/(N + (x_1^2))$ . For example,

$$b_{5,7}(S/(L + (x_1^2))) = 0 \quad \text{and} \quad b_{5,7}(S/(N + (x_1^2))) = 1.$$

#### 4. Proof of the Main Lemma 3.9

Throughout this section, we make the following assumptions:

**Assumptions 4.1.**  $N$  is a squarefree Borel  $\{x_1, \dots, x_{n-1}\}$ -compressed ideal in  $S = k[x_1, \dots, x_n]/P$ , and is not squarefree lex.

**Construction 4.2.** Since every squarefree Borel ideal in two variables is squarefree lex, it follows that the ideal  $N$  is  $\mathcal{B}$ -compressed for every set  $\mathcal{B}$  of two variables. Let  $r \geq 2$  be maximal such that  $N$  is  $(2r-2)$ -compressed. By Lemma 3.6, we have that  $N$  is  $(2r-1)$ -compressed. There exists a set  $\mathcal{A}$  of  $2r$  variables such that  $N$  is not  $\mathcal{A}$ -compressed. Choose  $w$  lex-first such that there exist variables  $w > y_1 > \dots > y_r > z_2 > \dots > z_r$  for which  $N$  is not  $\{w, y_1, \dots, y_r, z_2, \dots, z_r\}$ -compressed. Then choose  $\{y_1 > \dots > y_r\}$

lex-last such that there exist  $z_2, \dots, z_r$  with  $y_r > z_2 > \dots > z_r$  such that  $N$  is not  $\{w, y_1, \dots, y_r, z_2, \dots, z_r\}$ -compressed. Finally, choose  $z_2 > \dots > z_r$  lex-first such that  $N$  is not  $\{w, y_1, \dots, y_r, z_2, \dots, z_r\}$ -compressed. Set  $\mathcal{A} = \{w, y_1, \dots, y_r, z_2, \dots, z_r\}$ . We make this choice so that we can show in Lemma 4.6 that the  $\mathcal{A}$ -compression of  $N$  is still Borel.

If  $z_r \neq x_n$ , then  $N$  is  $\mathcal{A}$ -compressed because  $\mathcal{A} \subseteq \{x_1, \dots, x_{n-1}\}$ . Therefore,  $z_r = x_n$ .

Following the notation of Definition 3.5, write

$$\bar{N} = \bigoplus_f f \bar{N}_f.$$

Each  $\bar{N}_f$  is an ideal in  $k[\mathcal{A}]$ . For simplicity we will write  $N, N_f$  instead of  $\bar{N}, \bar{N}_f$ , that is, we will abuse notation and regard  $N$  (resp.  $N_f$ ) as both a squarefree ideal of  $S$  (resp.  $k[\mathcal{A}]$ ) and an ideal of  $S/P$  (resp.  $k[\mathcal{A}]/(P \cap k[\mathcal{A}])$ ).

Our assumptions imply the existence of a squarefree monomial  $f$  such that  $N_f$  is not squarefree lex.

**Notation 4.3.** In this section,  $f$  stands for a squarefree monomial in  $k[\mathcal{A}^c]$ .

Set  $\mathbf{y} = y_1 \dots y_r$  and  $\mathbf{z} = z_2 \dots z_r$ .

We denote by  $a, c$  variables in  $k[\mathcal{A}^c]$  (usually, these are variables dividing  $f$ ).

We denote by  $m, u, v$  squarefree monomials.

The following lemma gives some properties of  $N_f$ :

**Lemma 4.4.** *Suppose that  $f$  is a squarefree monomial such that the ideal  $N_f$  is not squarefree lex.*

- (1) *The vector space  $(N_f)_j$  is lex for every degree  $j \neq r$ .*
- (2) *The vector space  $(N_f)_r$  contains precisely the monomials  $\{m \mid m \geq \mathbf{y}, m \neq w\mathbf{z}\}$ , that is,  $(N_f)_r$  is spanned by the initial squarefree lex segment ending at  $\mathbf{y}$  with one gap at  $w\mathbf{z}$ .*

**Remark.** The proof of Lemma 4.4 uses only that  $N$  is squarefree Borel and  $(2r - 2)$ -compressed, and that  $N_f$  is not lex. Thus, lemma 4.4 holds for every ideal  $Y$  satisfying these properties.

*Proof:* (1) Let  $u > v$  be two squarefree monomials of degree  $j$  in the variables in  $\mathcal{A}$ . Let  $m = \gcd(u, v)$ , and  $u = mu', v = mv'$ . It follows that  $\deg(u') = \deg(v') \leq r$ .

Suppose that  $\deg(u') < r$ . Denote by  $\mathcal{B}$  the set of variables that appear in exactly one of the two monomials. The number of variables in  $\mathcal{B}$  is an even number  $< 2r$ . Therefore,  $N$  is  $\mathcal{B}$ -compressed. If  $fv = fmv' \in N$ , then  $fu = fmu' \in N$ . Hence, if  $(N_f)_j$  is not squarefree lex, then  $\deg(u') = \deg(v') = r$ , so  $m = 1$  and  $j = r$ .

(2) Let  $u > v$  be two squarefree monomials of degree  $r$  in the variables in  $\mathcal{A}$  such that  $v \in N_f$  but  $u \notin N_f$ . The above argument shows that  $\deg(u) = \deg(v) = r$ , and  $uv = wyz$ . Since  $u > v$ , we conclude that  $w$  divides  $u$ .

Suppose that  $u$  is divisible by some  $y_i$ . Hence  $v$  is divisible by some  $z_j$ . As  $N$  is squarefree Borel,  $fv \in N$  implies that  $f \frac{vy_i}{z_j} \in N$ . The ideal  $N$  is  $(\mathcal{A} \setminus \{y_i, z_j\})$ -compressed. Therefore,  $fy_i m \in N$  for every squarefree monomial  $m \in k[\mathcal{A}]$  such that  $m > \frac{v}{z_j}$ . We obtain the contradiction that  $fu \in N$ . Hence,  $u$  is not divisible by any of the variables  $y_i$ .

It follows that  $u = wz$  and  $v = \mathbf{y}$ . □

**Construction 4.5.** Denote by  $T$  the  $\mathcal{A}$ -compression of  $N$ .

The following lemma gives some properties of  $T$ :

**Lemma 4.6.**

- (1)  $(T_f)_j = (N_f)_j$  for  $j \neq r$  and every  $f$ .
- (2) The sets of monomials in  $(T_f)_r$  and in  $(N_f)_r$  differ only in that  $(T_f)_r$  contains  $wz$  instead of  $\mathbf{y}$ , in the case when  $(N_f)_r$  is not squarefree lex. Note that  $wz > \mathbf{y}$  are consecutive monomials in the lexicographic order in  $k[\mathcal{A}]$ .
- (3) Denote by  $\mathcal{F}$  the set of minimal, with respect to divisibility, monomials  $f$  in the variables  $\mathcal{A}^c$  such that  $N_f$  is not squarefree lex. We have that

$$\begin{aligned} \text{gens}(N) \setminus \text{gens}(T) &= \{f\mathbf{y} \mid f \in \mathcal{F}\}, \\ \text{gens}(T) &\supseteq \left\{ \text{gens}(N) \setminus \{f\mathbf{y} \mid f \in \mathcal{F}\} \right\} \cup \{fwz \mid f \in \mathcal{F}\} \\ &\quad \cup \{f\mathbf{y}z_j \mid f \in \mathcal{F}, 2 \leq j \leq r, \max(f) < \max(z_j)\}. \end{aligned}$$

- (4) The ideal  $T$  has the same Hilbert function as  $N$ .
- (5) The ideal  $T$  is lexicographically greater than  $N$ .

(6) The ideal  $T$  is squarefree Borel.

**Remark.** It is possible to show that  $\mathcal{F}$  is the set of all  $f$  such that  $N_f$  is not lex.

*Proof:* (1) and (2) hold by Lemma 4.4, and (4) and (5) hold by the construction of compression.

(3) Denote by  $\{T\}$  and  $\{N\}$  the sets of monomials in  $T$  and in  $N$ , respectively. By (1) and (2), we have that

$$\{N\} \setminus \{T\} = \{f\mathbf{y} \mid N_f \text{ is not squarefree lex}\}.$$

It follows that  $\text{gens}(N) \setminus \text{gens}(T) \supseteq \{f\mathbf{y} \mid f \in \mathcal{F}\}$ .

We will show that equality holds. Suppose that  $m$  is a generator of  $N$  but not of  $T$ , and does not have the form  $f\mathbf{y}$ . Then, by (2),  $m = fw\mathbf{z}x_i$  where  $f\mathbf{y} \in N$ ,  $fw\mathbf{z} \notin N$ .  $N$  is  $(\{x_i\} \cup \mathcal{A} \setminus \{z_r\})$ -compressed, because this set does not contain  $z_r = x_n$ . Thus  $f\mathbf{y} \in N$  implies that  $fw\frac{\mathbf{z}}{z_r}x_i \in N$ , and so  $fw\mathbf{z}x_i \notin \text{gens}(N)$ . Hence, any multiple of  $fw\mathbf{z}$  is not a minimal monomial generator of  $N$ . The equality follows.

Now, we will prove that

$$\begin{aligned} \text{gens}(T) \supseteq & \left\{ \text{gens}(N) \setminus \{f\mathbf{y} \mid f \in \mathcal{F}\} \right\} \cup \{fw\mathbf{z} \mid f \in \mathcal{F}\} \\ & \cup \{f\mathbf{y}z_j \mid f \in \mathcal{F}, 2 \leq j \leq r, \max(f) < z_j\}. \end{aligned}$$

The inclusion  $\text{gens}(T) \supseteq \left\{ \text{gens}(N) \setminus \{f\mathbf{y} \mid f \in \mathcal{F}\} \right\}$  follows from  $\text{gens}(N) \setminus \text{gens}(T) = \{f\mathbf{y} \mid f \in \mathcal{F}\}$ . By (1) and (2), we also have that

$$\{T\} \setminus \{N\} = \{fw\mathbf{z} \mid N_f \text{ is not squarefree lex}\}.$$

Therefore,  $\text{gens}(T) \supseteq \{fw\mathbf{z} \mid f \in \mathcal{F}\}$ . The inclusion

$$\text{gens}(T) \supseteq \{f\mathbf{y}z_j \mid f \in \mathcal{F}, 2 \leq j \leq r, \max(f) < z_j\}$$

holds because if  $\frac{f\mathbf{y}z_j}{a} \in T$  for some variable  $a$  then, since  $T$  is squarefree Borel, we get the contradiction  $f\mathbf{y} \in T$ .

It remains to prove (6). Fix an  $f$  such that  $N_f$  is not squarefree lex. In view of (1), we need to consider only  $(T_f)_r$ . By (2), we conclude that we have to check two properties: We have to show that, if a squarefree monomial  $m$  is obtained from  $f\mathbf{wz}$  by replacing a variable with a lex-greater variable, then  $m$  is in  $T$ . We also have to show that, if a squarefree monomial  $u$  is obtained from  $f\mathbf{y}$  by replacing a variable with a lex-smaller variable, then  $u$  is not in  $T$ .

There are several possibilities for  $m$  and  $u$ . First, we consider four cases for  $m$ .

Suppose  $m = \frac{fc}{a}w\mathbf{z}$ , where  $a$  divides  $f$  and  $c \in k[\mathcal{A}^c]$ . Since  $f\mathbf{y} \in N$  and  $N$  is squarefree Borel, we have that  $\frac{fc}{a}\mathbf{y} \in N$ . Hence,  $\mathbf{y} \in N_{\frac{fc}{a}}$ . Therefore  $w\mathbf{z} \in T_{\frac{fc}{a}}$ , and so  $m \in T$ .

Suppose  $m = \frac{fe}{a}w\mathbf{z}$ , where  $a$  divides  $f$  and  $e \in k[\mathcal{A}]$ . It follows that  $e = y_j$  for some  $j$ . So,  $m = \frac{f}{a}(wy_j\mathbf{z})$ . By Lemma 4.4, we have that  $(N_{\frac{f}{a}})_{r+1} = (T_{\frac{f}{a}})_{r+1}$ , so we have to prove that  $m \in N$ . The ideal  $N$  is  $(\{a\} \cup \mathcal{A} \setminus \{y_j\})$ -compressed by Construction 4.2 since  $a < y_j$ . Hence,  $f\mathbf{y} = \frac{fy_j}{a} \frac{a\mathbf{y}}{y_j} \in N$  implies that  $\frac{fy_j}{a}w\mathbf{z} = m \in N$ .

Suppose  $m = f\frac{w\mathbf{z}y_j}{e}$ , where  $e$  divides  $w\mathbf{z}$ . Since  $T_f$  is squarefree lex in  $k[\mathcal{A}]$ , we conclude that  $m \in T$ .

Suppose  $m = f\frac{w\mathbf{z}c}{e} = (fc)\frac{w\mathbf{z}}{e}$ , where  $e$  divides  $w\mathbf{z}$  and  $c \in k[\mathcal{A}^c]$ . By Lemma 4.4, we have that  $(N_{fc})_{r-1} = (T_{fc})_{r-1}$ , so we have to prove that  $m \in N$ . First, we consider the subcase when either  $e = w$  or  $c < y_r$ . The ideal  $N$  is  $(\{c\} \cup \mathcal{A} \setminus \{e\})$ -compressed by Construction 4.2 since  $c > e$ . Hence,  $f\mathbf{y} \in N$  implies that  $m = f(\frac{w\mathbf{z}c}{e}) \in N$ . Now, let  $e = z_i$  for some  $i$  and  $c > y_r$ . Since  $N$  is squarefree Borel,  $f\mathbf{y} \in N$  implies that  $fc\frac{\mathbf{y}}{y_r} \in N$ . As  $(N_{fc})_{r-1}$  is squarefree lex, we get that  $m \in N$ .

Recall that we also have to show that every squarefree monomial  $u$ , obtained from  $f\mathbf{y}$  by replacing a variable with a lex-smaller variable, is not in  $T$ . Similarly, we consider four cases for  $u$ . We assume the opposite, that is  $u \in T$ , and we will arrive at the contradiction that  $f\mathbf{wz} \in N$ .

Suppose  $u = \frac{fc}{a}\mathbf{y}$ , where  $a$  divides  $f$  and  $c \in k[\mathcal{A}^c]$ . Since  $\mathbf{y} \in T_{\frac{fc}{a}}$ , by Lemma 4.4



we conclude that  $(T_{\frac{fc}{a}})_r = (N_{\frac{fc}{a}})_r$  is squarefree lex, and so  $w\mathbf{z} \in N_{\frac{fc}{a}}$ . As  $N$  is squarefree Borel, it follows that  $\left(\frac{fc}{a}\right)\frac{a}{c}w\mathbf{z} = fw\mathbf{z} \in N$ .

Suppose  $u = \frac{f}{a}e\mathbf{y}$ , where  $a$  divides  $f$  and  $e \in k[\mathcal{A}]$ . By Lemma 4.4 we conclude that  $\left(T_{\frac{f}{a}}\right)_{r+1} = \left(N_{\frac{f}{a}}\right)_{r+1}$  is squarefree lex. Hence,  $u \in N$  implies that  $\frac{f}{a}ew\mathbf{z} \in N$ .

As  $N$  is squarefree Borel, we conclude that  $\left(\frac{fe}{a}\right)\left(\frac{a}{e}\right)w\mathbf{z} = fw\mathbf{z} \in N$ .

Suppose  $u = f\frac{\mathbf{y}e}{y_j}$ , where  $e \in k[\mathcal{A}]$ . Then  $e = z_i$ . Since  $\frac{\mathbf{y}e}{y_j} \in T_f$ , by Lemma 4.4 we conclude that  $(T_f)_r = (N_f)_r$  is squarefree lex, and so  $w\mathbf{z} \in N_f$ .

Suppose  $u = f\frac{\mathbf{y}c}{y_j}$ , where  $c \in k[\mathcal{A}^c]$ . By Lemma 4.4 we conclude that  $(T_{fc})_{r-1} = (N_{fc})_{r-1}$ , so  $u \in N$ . By Construction 4.2, the ideal  $N$  is  $(\{c\} \cup \mathcal{A} \setminus y_j)$ -compressed since  $c < y_j$ . Hence,  $u \in N$  implies that  $fw\mathbf{z} \in N$  as  $w\mathbf{z} > \frac{\mathbf{y}c}{y_j}$ .  $\square$

**Construction 4.7.** Each of the colon ideals  $(N : \sigma)$  can be decomposed in the notation of Definition 3.5 as follows:

$$(N : \sigma) = \bigoplus_f (N : \sigma)_f.$$

Each  $(N : \sigma)_f$  is an ideal in  $k[x_i \in \mathcal{A} \mid i \notin \sigma] / (x_i^2 \mid i \notin \sigma)$ . Similarly, we have

$$(T : \sigma) = \bigoplus_f (T : \sigma)_f.$$

**Lemma 4.8.** Let  $f$  be a squarefree monomial in  $k[\mathcal{A}^c]$ .

- (1) For every  $\sigma \subseteq \mathcal{A}$  we have  $(N : \sigma)_f = (N_f : \sigma)$ .
- (2) If  $\sigma, \gamma \subseteq \mathcal{A}$  and  $\sigma \cap \gamma = \emptyset$ , then  $(N : \sigma\gamma)_f = ((N_f : \sigma) : \gamma)$ .
- (3) If  $\tau \subseteq \mathcal{A}^c$ , then  $(N : \tau)_f = N_{f\tau}$ .

*Proof:* First, we prove (1). Let  $m \in k[\mathcal{A} \setminus \sigma]$  be a monomial. We have that

$$m \in (N : \sigma)_f \Leftrightarrow fm \in (N : \sigma) \Leftrightarrow fm\sigma \in N \Leftrightarrow m\sigma \in N_f \Leftrightarrow m \in (N_f : \sigma).$$

Applying (1), we prove (2) as follows:

$$((N_f : \sigma) : \gamma) = ((N : \sigma)_f : \gamma) = ((N : \sigma) : \gamma)_f = (N : \sigma\gamma)_f.$$

(3) For a monomial ideal  $U$ , we denote by  $\{U\}$  the set of (squarefree) monomials in  $U$ . We have that

$$\begin{aligned} \{(N : \tau)_f\} &= \{\text{monomials } m \in k[A] \mid mf \in (N : \tau)\} \\ &= \{\text{monomials } m \in k[A] \mid mf\tau \in N\} \\ &= \{N_{f\tau}\}. \end{aligned}$$

□

**Lemma 4.9.** Let  $\mathcal{N} = (w\frac{\mathbf{y}}{y_r}, \dots, wy_r\frac{\mathbf{z}}{z_2}, \mathbf{y})$  be an ideal in  $k[\mathcal{A}]$ , (where “ $\dots$ ” means that we take all the squarefree monomials that are lex-between  $w\frac{\mathbf{y}}{y_r}$  and  $wy_r\frac{\mathbf{z}}{z_2}$ ). Then

- (1)  $\mathcal{T} = (w\frac{\mathbf{y}}{y_r}, \dots, w\mathbf{z}, \mathbf{y}z_2, \dots, \mathbf{y}z_r)$  is the squarefree lex ideal with the same Hilbert function, and  $\mathcal{N}_{r+1} = \mathcal{T}_{r+1}$ .
- (2) If  $\mu \subset \mathcal{A}$  is not a subset of  $\text{supp}(\mathbf{y})$  or of  $\text{supp}(w\mathbf{z})$ , then  $(\mathcal{N} : \mu) = (\mathcal{T} : \mu)$ . All other possibilities for  $\mu$ , and the corresponding ideals  $(\mathcal{N} : \mu)$  and  $(\mathcal{T} : \mu)$ , are listed in the two tables below. The first table lists the cases when  $\text{gens}(\mathcal{N} : \mu) \subset \text{gens}(\mathcal{T} : \mu)$ :

$\mu$	$(\mathcal{N} : \mu)$	$(\mathcal{T} : \mu)$
$\emptyset \subset \zeta \subset \text{supp}(\mathbf{z})$	$((w(y_1, \dots, y_r))_{r- \zeta }, \mathbf{y})$	$((w(y_1, \dots, y_r))_{r- \zeta }, w\frac{\mathbf{z}}{\zeta}, \mathbf{y})$
$\text{supp}(w)$	$((y_1, \dots, y_r)_{r-1})$	$((y_1, \dots, y_r)_{r-1}, \mathbf{z})$
$\text{supp}(w)\zeta$ with	$((y_1 \cdots, y_r)_{r-1- \zeta })$	$((y_1 \cdots, y_r)_{r-1- \zeta }, \frac{\mathbf{z}}{\zeta})$
$\emptyset \subset \zeta \subset \text{supp}(\mathbf{z})$		

The second table lists the remaining cases:

$\mu$	$(\mathcal{N} : \mu)$	$(\mathcal{T} : \mu)$
$\emptyset$	$((w(y_1, \dots, y_r))_r, \mathbf{y})$	$((w(y_1, \dots, y_r))_r, w\mathbf{z}, \mathbf{y}z_2, \dots, \mathbf{y}z_r)$
$\emptyset \subset \rho \subset \text{supp}(\mathbf{y})$	$((w)_{r- \rho }, \frac{\mathbf{y}}{\rho})$	$((w)_{r- \rho }, \frac{\mathbf{y}}{\rho}z_2, \dots, \frac{\mathbf{y}}{\rho}z_r)$
$\text{supp}(\mathbf{y})$	$(1)$	$(w, z_2, \dots, z_r)$
$\text{supp}(\mathbf{z})$	$(wy_1, \dots, wy_r, \mathbf{y})$	$(w, \mathbf{y})$
$\text{supp}(w\mathbf{z})$	$(y_1, \dots, y_r)$	$(1)$

(3) Let  $Y$  be a  $(2r-2)$ -compressed ideal in the polynomial ring  $k[\mathcal{A}]/(P \cap k[\mathcal{A}])$  such that  $Y_r = \mathcal{N}_r$ , and let  $Z$  be the lex ideal of  $k[\mathcal{A}]/(P \cap k[\mathcal{A}])$  with the same Hilbert function as  $Y$ . For every subset  $\mu$  of  $\mathcal{A}$ , we have:

$$\text{gens}(\mathcal{N} : \mu) \setminus \text{gens}(\mathcal{T} : \mu) = \text{gens}(Y : \mu) \setminus \text{gens}(Z : \mu)$$

$$\text{gens}(\mathcal{T} : \mu) \setminus \text{gens}(\mathcal{N} : \mu) = \text{gens}(Z : \mu) \setminus \text{gens}(Y : \mu).$$

*Proof:* (1) The ideal  $\mathcal{T}$  is clearly squarefree lex. We need to show that  $\mathcal{T}$  has the same Hilbert function as  $\mathcal{N}$ . Obviously,  $\dim_k((\mathcal{N})_r) = \dim_k((\mathcal{T})_r)$ . Let  $\mathcal{T}' = (w\frac{\mathbf{y}}{y_r}, \dots, w\mathbf{z})$ .

It is straightforward to verify that  $\mathcal{N}_{r+1} \supset \mathcal{T}'_{r+1}$  and  $\{\mathcal{N}_{r+1}\} \setminus \{\mathcal{T}'_{r+1}\} = \{\mathbf{y}z_2, \dots, \mathbf{y}z_r\}$ . Therefore,  $\mathcal{N}_{r+1} = \mathcal{T}'_{r+1}$ .

(2) Recall that  $\mu \subseteq \mathcal{A}$ . A simple computation shows that  $(\mathcal{N} : wy_i) = (\mathcal{T} : wy_i)$  and  $(\mathcal{N} : y_iz_j) = (\mathcal{T} : y_iz_j)$  for all  $i, j$ . If  $\mu$  is not divisible by  $\mathbf{y}$  or  $w\mathbf{z}$ , it follows that  $\mu$  contains either  $\text{supp}(wy_i)$  or  $\text{supp}(y_iz_j)$ , for some  $i, j$ , so  $(\mathcal{N} : \mu) = (\mathcal{T} : \mu)$ . For all other  $\mu$ , straightforward computation yields the ideals  $(\mathcal{N} : \mu)$  and  $(\mathcal{T} : \mu)$  listed in the tables.

(3) For a monomial  $m$  in  $k[\mathcal{A}]$ , we will use the notation  $m : \mu$  for  $\frac{m}{\gcd(m, \mu)}$ .

Observe first that  $Y_j = Z_j$  for all  $j \neq r$  by Lemma 4.4(3). Hence,  $\{Z\} \setminus \{Y\} = w\mathbf{z}$  and  $\{Y\} \setminus \{Z\} = \mathbf{y}$ .

Suppose that  $m : \mu$  is a minimal monomial generator for  $(Y : \mu)$  but not  $(Z : \mu)$ . We assume that  $m : \mu \notin (\mathcal{N} : \mu)$  and will derive a contradiction. Note that  $m \notin \mathcal{N}$ . Then  $\deg(m) \neq r$ , because  $m \in Y$  and  $Y_r = \mathcal{N}_r$ . Since  $m \in Y$  and  $Y_{\deg(m)} = Z_{\deg(m)}$  we conclude that  $m \in Z$ , and so  $m : \mu \in (Z : \mu)$ . Since  $m : \mu$  is not a minimal monomial generator for  $(Z : \mu)$ , there must be a monomial  $u \in Z \setminus Y$  such that  $u : \mu$

properly divides  $m : \mu$ . The only monomial of  $Z \setminus Y$  is  $w\mathbf{z}$ , so  $u = w\mathbf{z}$ . Then  $\mu(m : \mu)$  is a proper multiple of  $w\mathbf{z}$ , and so must be in  $\mathcal{N}$  by (1). On the other hand, since  $m : \mu \notin (\mathcal{N} : \mu)$ , it follows that  $\mu(m : \mu) \notin \mathcal{N}$ , a contradiction. Thus we must have  $m : \mu \in (\mathcal{N} : \mu)$ . Note that  $Y \supset \mathcal{N}$ . We conclude that  $m : \mu$  is a minimal monomial generator of  $(\mathcal{N} : \mu)$ . We have proved that

$$\text{gens}(Y : \mu) \setminus \text{gens}(Z : \mu) \subseteq \text{gens}(\mathcal{N} : \mu).$$

Now, suppose further that  $m : \mu$  is a minimal monomial generator of  $(\mathcal{T} : \mu)$ . We will derive a contradiction. Note that  $m$  may be chosen to be a minimal monomial generator of  $\mathcal{N}$ , because  $m : \mu$  is a minimal monomial generator of  $(\mathcal{N} : \mu)$ .

Since  $m : \mu$  is not a minimal monomial generator of  $(Z : \mu)$ , there must be a monomial  $u \in Z \setminus \mathcal{T}$  such that  $u : \mu$  properly divides  $m : \mu$ . As  $Y_j = Z_j$  and  $\mathcal{N}_j = \mathcal{T}_j$  for  $j \neq r$ , and  $Y_r = \mathcal{N}_r$ , and  $Z_r = \mathcal{T}_r$ , we get  $\{Z\} \setminus \{\mathcal{T}\} = \{Y\} \setminus \{\mathcal{N}\}$ . Hence  $u \in Y$ , so that  $m : \mu$  is not a minimal monomial generator for  $(Y : \mu)$ , a contradiction. This shows that

$$\text{gens}(Y : \mu) \setminus \text{gens}(Z : \mu) \subseteq \text{gens}(\mathcal{N} : \mu) \setminus \text{gens}(\mathcal{T} : \mu).$$

In order to prove the equality in the first formula in (3), we need to show that the opposite inclusion holds. To this end, suppose that  $m : \mu$  is a minimal monomial generator of  $(\mathcal{N} : \mu)$  but not  $(\mathcal{T} : \mu)$ . Then, by (2), either  $m = \mathbf{y}$  and  $\mu$  divides  $\mathbf{y}$ , or  $\mu = \mathbf{z}$  or  $\mu = w\mathbf{z}$ .

Suppose first that  $m = \mathbf{y}$ . As  $Y \supseteq \mathcal{N}$ , we have that  $\mathbf{y} : \mu \in (Y : \mu)$ . If  $\mathbf{y} : \mu$  were not a minimal monomial generator of  $(Y : \mu)$ , we would have a monomial  $u \in Y \setminus \mathcal{N} = Z \setminus \mathcal{T}$ , such that  $u : \mu$  properly divides  $\mathbf{y} : \mu$ . But then  $u$  must be a proper divisor of  $\mathbf{y}$ , and  $u \in Z$  implies the contradiction  $\mathbf{y} \in Z$ . Hence  $m : \mu \in \text{gens}(Y)$ . If  $m : \mu \in \text{gens}(Z)$ , we would have  $\mathbf{y} \in Z$ , a contradiction. Thus in this case we have  $m : \mu \in \text{gens}(Y : \mu) \setminus \text{gens}(Z : \mu)$ .

Suppose now that  $\mu = \mathbf{z}$  or  $\mu = w\mathbf{z}$ . Straightforward computation using (2) shows that one of the following two cases holds:

$\mu$	$m : \mu$	$w\mathbf{z} : \mu$
$\mathbf{z}$	$wy_j$	$w$
$w\mathbf{z}$	$y_j$	$1$

where  $1 \leq j \leq n$ . In particular,  $w\mathbf{z} : \mu$  is a proper divisor of  $m : \mu$  in either case. As  $Z_r = \mathcal{T}_r$ , we get  $w\mathbf{z} : \mu \in (Z : \mu)$ , so  $m : \mu$  is not a minimal monomial generator for  $(Z : \mu)$ . If  $m : \mu$  were not a minimal monomial generator for  $Y : \mu$ , there would be a monomial  $u \in Y \setminus \mathcal{N}$  such that  $u : \mu$  is a proper divisor of  $m : \mu$ . The table above implies that one of the following two cases holds:

$\mu$	$u : \mu$	$\mu(u : \mu) \in Y$
$\mathbf{z}$	$w$ , or $y_j$ , or $1$	$w\mathbf{z}$ , or $y_j\mathbf{z}$ , or $\mathbf{z}$
$w\mathbf{z}$	$1$	$w\mathbf{z}$

where  $1 \leq j \leq n$ . As  $Y_r = \mathcal{N}_r$ , none of the monomials in the third column are in  $Y$ . This is a contradiction. Thus  $m : \mu$  is a minimal monomial generator for  $(Y : \mu)$ . Therefore,  $m : \mu \in \text{gens}(Y : \mu) \setminus \text{gens}(Z : \mu)$  in this case.

We have shown that the first formula in (3) holds:

$$\text{gens}(\mathcal{N} : \mu) \setminus \text{gens}(\mathcal{T} : \mu) = \text{gens}(Y : \mu) \setminus \text{gens}(Z : \mu).$$

A very similar argument yields the second formula:

$$\text{gens}(\mathcal{T} : \mu) \setminus \text{gens}(\mathcal{N} : \mu) = \text{gens}(Z : \mu) \setminus \text{gens}(Y : \mu).$$

□

**Notation 4.10.** We write an EK-triple in the form  $(\tau\mu, gq, \alpha)$  so that  $\tau \in k[\mathcal{A}^c]$ ,  $\mu \in k[\mathcal{A}]$ ,  $g \in k[\mathcal{A}^c]$ ,  $q \in k[\mathcal{A}]$ . By  $\tau\mu$  we mean the union of  $\tau$  and  $\mu$ .

For  $\mu \subset \mathcal{A}$ , we set  $\mathbf{n}$  to be the homogeneous maximal ideal of the ring  $k[\mathcal{A} \setminus \mu]/P$ .

The next lemma provides a list of the possible EK-triples for  $N$ :

**Lemma 4.11.**

(1) *There are three types of EK-triples  $(\tau\mu, gq, \alpha)$  for  $N$ :*

*Type 1:  $(\tau\mu, gq, \alpha)$  is an EK-triple for  $T$ .*

*Type 2:  $(\tau\mu, gq, \alpha)$  is not an EK-triple for  $T$  and  $q$  is a minimal monomial generator of both  $(N : \tau\mu)_g$  and  $(T : \tau\mu)_g$ .*

*Type 3:  $(\tau\mu, gq, \alpha)$  is not an EK-triple for  $T$  and  $q$  is a minimal monomial generator of  $(N : \tau\mu)_g$  but not of  $(T : \tau\mu)_g$ .*

- (2) The EK-triples of Type 2 satisfy  $q\mu = wz$  and  $\max(g) > \max(q)$ . In particular,  $\mu \neq 1$ , because  $x_n = z_r$  divides  $\mu$ .
- (3) Let  $\mathcal{N}$  and  $\mathcal{T}$  be as in Lemma 4.9. If  $(\tau\mu, gq, \alpha)$  is an EK-triple of Type 3 for  $N$ , then  $q$  is a minimal monomial generator for  $(\mathcal{N} : \mu)$  but not for  $(\mathcal{T} : \mu)$ .
- (4) Suppose that  $(\tau\mu, gq, \alpha)$  is an EK-triple of Type 3 for  $N$ . All possibilities for  $\mu$ , and the corresponding ideals  $(\mathcal{N} : \mu)$  and  $(\mathcal{T} : \mu)$ , are listed in the second table in Lemma 4.9(2).

*Proof:* (1) It suffices to show that  $q$  is a minimal monomial generator of the ideal  $(N : \tau\mu)_g$ . Since  $gq \in (N : \tau\mu)$ , we have that  $q \in (N : \tau\mu)_g$ . If we had  $\frac{q}{a} \in (N : \tau\mu)_g$ , it would follow that  $\frac{gq}{a} \in (N : \tau\mu) = (N : \tau\mu)$ , so that  $gq$  would not be a minimal monomial generator of  $(N : \tau\mu)$ . Thus,  $q$  is a minimal monomial generator of  $(N : \tau\mu)_g$ .

(2) Since  $gq \in (T : \tau\mu)$  is not a minimal monomial generator, there exists a variable  $c$  dividing  $gq$  such that  $\frac{gq}{c} \in (T : \tau\mu)$ . Since  $T$  is Borel, we may take  $c = x_{\max(gq)}$ . If  $c$  divides  $q$ , we have  $\frac{q}{c} \in (T : \tau\mu)_g$ , so that  $q$  is not a minimal monomial generator of  $(T : \tau\mu)_g$ . Therefore  $c$  divides  $g$ . Since  $c = x_{\max(gq)}$ , we have  $\max(g) > \max(q)$ . As  $gq$  is a minimal monomial generator of  $(N : \tau\mu)$ , we have  $\frac{g}{c}q \notin (N : \tau\mu)$ , so  $\frac{g}{c}q\tau\mu \in T \setminus N$ . By Lemma 4.6(2) it follows that  $q\mu = wz$ .

(3) By Lemma 4.8, we have that

$$(N : \tau\mu)_g = ((N : \tau)_g : \mu) \quad \text{and} \quad (T : \tau\mu)_g = ((T : \tau)_g : \mu).$$

We are going to apply Lemma 4.9(3) to the ideals  $Y = (N : \tau)_g$  and  $Z = (T : \tau)_g$ . By Lemma 4.8(3), we have that  $Y = N_{g\tau}$  and  $Z = T_{g\tau}$ . By Lemma 4.6 we get that  $Y_r = \mathcal{N}_r$ . Clearly,  $Z$  is the squarefree lex ideal with the same Hilbert function as  $Y$ . Note that  $Y$  is  $(2r - 2)$ -compressed since  $N$  is. Therefore,  $Y$  and  $Z$  satisfy the conditions of Lemma 4.9(3) and we can apply it.

We have that

$$q \in \text{gens}(Y : \mu) \setminus \text{gens}(Z : \mu)$$

since we consider EK-triples of Type 3. Lemma 4.9(3) yields

$$q \in \text{gens}(\mathcal{N} : \mu) \setminus \text{gens}(\mathcal{T} : \mu).$$

(4) follows from (3) and Lemma 4.9(2).  $\square$

Next, we construct a map from the set of EK-triples for  $N$  to the set of EK-triples for  $T$ . We will use this map to prove the Main Lemma 3.9.

**Construction 4.12.** We will define a map  $\phi$  from the set of EK-triples for  $N$  to the EK-triples for  $T$ . First, we introduce notation.

If  $\alpha = v \prod_{i \in \mathcal{I}} y_i \prod_{j \in \mathcal{J}} z_j$ , where  $v$  is a squarefree monomial in  $k[\mathcal{A}^c]$  and we use the convention  $w = z_1$ , then let  $\hat{\alpha}$  be the monomial

$$\hat{\alpha} = v \prod_{i \in \mathcal{I}} z_i \prod_{j \in \mathcal{J}} y_j.$$

Let  $t_1 > \cdots > t_s = z_r = x_n$  be all the variables of  $S$  not in  $\tau \mathbf{y}$ , ordered lexicographically. For a monomial  $m$ , such that  $m\tau$  is squarefree, set

$$t_m = \begin{cases} x_{\max(m)} & \text{if } x_{\max(m)} \notin \mathbf{y} \\ \text{the lex-last variable among the } t\text{-variables} \\ \text{that is lex-before } x_{\max(m)} & \text{if } x_{\max(m)} \in \mathbf{y}, \end{cases}$$

and furthermore, for a monomial  $m$  and an integer  $j$ , set

$$t_{m+j} = t_{p+j} \text{ where the integer } p \text{ is defined by } t_p = t_m.$$

In cases 2 and 3 below, we will set the integer  $d$  such that  $\max(g) = \max(t_d)$ . In case 2,  $e$  will be the integer such that  $t_d$  is between  $z_{r-e}$  and  $z_{r-e+1}$ . Thus,  $r - e = \#\{z_j : \max(z_j) < \max(t_d)\}$  (recall the convention  $z_1 = w$ ). In case 3 we will set  $i$  such that  $t_d$  is between  $y_{i-1}$  and  $y_i$ .

Denote by  $\tilde{\alpha}$  the monomial

$$\tilde{\alpha} = \alpha \prod_{\substack{y_j \text{ divides } \alpha \\ j \neq 1}} \frac{t_{gq+j-1}}{y_j} \prod_{y_1 \text{ divides } \alpha} \frac{w}{y_1}.$$

and denote by  $\bar{\alpha}$  the monomial

$$\bar{\alpha} = \alpha \left( \prod_{\substack{y_j \text{ divides } \alpha, \\ \max(z_j) < \max(g)}} \frac{z_j}{y_j} \right) \left( \prod_{\substack{y_j \text{ divides } \alpha, \\ \max(z_j) > \max(g)}} \frac{t_{d+j-(r-e)}}{y_j} \right).$$

The map  $\phi$  is defined as follows: If  $\Gamma$  is an EK-triple for  $N$  of the form given in the third column in the table below, then  $\phi(\Gamma)$  is given in the fourth column.

Case	Type of $\Gamma$	EK-triple $\Gamma$	$\phi(\Gamma)$
1)	Type 1	$\Gamma = (\tau\mu, gq, \alpha)$	$\Gamma = (\tau\mu, gq, \alpha)$
2)	Type 2	$(\tau\mu, g\frac{wz}{\mu}, \alpha),$ $\emptyset \subset \mu \subset \text{supp}(wz), \max(g) > \max(y_r)$	$(\tau\hat{\mu}, \frac{g}{x_{\max(g)}}t_{d+e}\frac{y}{\hat{\mu}}, \hat{\alpha}),$
3)	Type 2	$(\tau z, gw, \alpha), \max(y_1) < \max(g) < \max(y_r)$ or $(\tau wz, g, \alpha)$	$(\tau\frac{y}{y_1}, \frac{g}{x_{\max(g)}}t_{d+i-1}y_1, \tilde{\alpha})$ or $(\tau y, \frac{g}{x_{\max(g)}}t_{d+i-1}, \tilde{\alpha})$
4)	Type 2	$(\tau z, gw, \alpha),$ $\max(w) < \max(g) < \max(y_1)$	$(\tau\frac{y}{y_1y_2}, \frac{g}{x_{\max(g)}}y_1y_2z_2, \alpha x_{\max(g)})$
5)	Type 3	$(\tau, gy, \alpha)$	$(\tau, gwz, \hat{\alpha})$
6)	Type 3	$(\tau\rho, g\frac{y}{\rho}, \alpha), \emptyset \subset \rho \subset \text{supp}(y), \rho \neq \frac{y}{y_1}$	$(\tau\hat{\rho}, g\frac{wz}{\hat{\rho}}, \hat{\alpha})$
7)	Type 3	$(\tau\frac{y}{y_1}, gy_1, \alpha)$	$(\tau\frac{z}{z_r}, gwz_r, y_r\hat{\alpha})$
8)	Type 3	$(\tau y, g, \alpha)$	$(\tau\frac{wz}{z_r}, gz_r, y_r\hat{\alpha})$
9)	Type 3	$(\tau z, gwy_j, \alpha)$	$(\tau\frac{y}{y_j}, gy_jt_{y_j+j}, \tilde{\alpha})$
10)	Type 3	$(\tau wz, gy_j, \alpha), j \neq 1$	$(\tau y, gt_{y_j+j}, \tilde{\alpha})$
11)	Type 3	$(\tau wz, gy_1, \alpha)$	$(\tau\frac{y}{y_1}, gy_1t_{y_1+1}, w\tilde{\alpha})$

**Lemma 4.13.** *The third column in the table in Construction 4.12 lists all the possibilities for EK-triples for  $N$ .*

*Proof:* Apply Lemma 4.11. For EK-triples of Type 3, straightforward computation yields  $q$  since we know all possibilities for  $(\mathcal{N} : \mu)$  and  $(\mathcal{T} : \mu)$  (see the second table in Lemma 4.9(3)) and in view of Lemma 4.11(3).  $\square$

**Lemma 4.14.**

- (1) *The map  $\phi$  is well-defined (i.e.,  $\phi(\Gamma)$  is an EK-triple for  $T$ , for all  $\Gamma$ ).*
- (2) *The map  $\phi$  preserves bidegree.*
- (3) *The map  $\phi$  is an injection.*

*Proof:* (2) Straightforward verification shows that  $\phi$  preserves bidegrees.

(3) Let  $\Gamma = (\tau\mu, gq, \alpha)$ ,  $\Gamma' = (\tau'\mu', g'q', \alpha')$ ,  $\phi(\Gamma) = (\theta, \ell, \beta)$ ,  $\phi(\Gamma') = (\theta', \ell', \beta')$ . Write  $\theta = \tau\lambda$ ,  $\theta' = \tau'\lambda'$ ,  $\ell = uv$  and  $\ell' = u'v'$  with  $\lambda, \lambda', u, u' \in k[\mathcal{A}]$ ,  $v, v' \in k[\mathcal{A}^c]$ .



Suppose that  $\Gamma \neq \Gamma'$ .

If  $\tau \neq \tau'$ , then  $\phi(\Gamma) \neq \phi(\Gamma')$  and we are done. For the rest of the proof, we assume that  $\tau = \tau'$ . In this case, note that we have the same choice for the variables  $t_j$  in Construction 4.12 for  $\Gamma$  and  $\Gamma'$ . Also, note that if  $\alpha \neq \alpha'$ , then  $\hat{\alpha} \neq \hat{\alpha}'$ ,  $\bar{\alpha} \neq \bar{\alpha}'$ , and  $\tilde{\alpha} \neq \tilde{\alpha}'$  by Construction 4.12.

If  $\Gamma$  and  $\Gamma'$  fall under the same Case in Construction 4.12, it is immediate that  $\phi(\Gamma) \neq \phi(\Gamma')$ , except in Cases 2 and 3. In Cases 2 and 3, we have to consider the situation when  $\frac{g}{x_{\max(g)}} = \frac{g'}{\max(g')}$ , and  $\tau = \tau'$ ,  $\mu = \mu'$ ,  $\alpha = \alpha'$ . Let  $d', e', i'$  be defined analogously to  $d, e, i$ . Without loss of generality we can assume that  $\max(g) > \max(g')$ . Hence,  $\max(t_d) > \max(t_{d'})$ , and so  $d > d'$ . Therefore, we have the inequalities  $i \geq i'$  and

$$e' - e = \#\{z_j : z_j \text{ between } t_d \text{ and } t_{d'}\} \leq 1 + \#\{t_j : t_j \text{ between } t_d \text{ and } t_{d'}\} = d - d'.$$

It follows that  $d + e > d' + e'$  and  $d + i - 1 > d' + i' - 1$ . Therefore,  $\ell \neq \ell'$ .

Thus, we may assume that  $\Gamma$  and  $\Gamma'$  belong to different Cases.

Suppose first that  $\Gamma'$  falls under Case 1. We will show that  $\phi(\Gamma)$  is not an EK-triple for  $N$ , for  $\Gamma$  in each of the Cases 2 through 11. In Cases 2, 3, and 4, we have that  $\ell\theta$  is properly divisible by the monomial  $\tau \frac{g}{x_{\max(g)}} \mathbf{y}$ , which is in  $N$  by Lemma 4.16; hence  $\ell\theta$  is not a minimal monomial generator of  $N$ . In Cases 9, 10, and 11, we have that  $\ell\theta$  is properly divisible by the monomial  $\tau g \mathbf{y}$ , which is in  $N$  because  $(N : \tau)_g \neq (T : \tau)_g$  implies  $\mathbf{y} \in (N : \tau)_g$ ; hence  $\ell\theta$  is not a minimal monomial generator of  $N$ . In Cases 5 through 8,  $\lambda = \mu$  and  $u = q$  have concrete values, and the second table in Lemma 4.9(3) shows that  $q$  is not a minimal monomial generator for  $(\mathcal{N} : \mu)$ ; hence Lemma 4.11(3) implies that  $\phi(\Gamma)$  is not an EK-triple for  $N$ .

For the rest of the proof, we assume that that neither  $\Gamma$  nor  $\Gamma'$  is in Case 1.

In many cases, it is clear that  $\lambda \neq \lambda'$ . These cases are listed in the following table.

Case of $\Gamma$	Case of $\Gamma'$	Difference between $\lambda$ and $\lambda'$
2	5	$\mu \neq \emptyset$ , so $\lambda = \hat{\mu} \neq \emptyset$ . But $\lambda' = \emptyset$ .
2	6,7,8	$\mu \subset \text{supp}(w\mathbf{z})$ by Lemma 4.11(2), so $\emptyset \neq \lambda = \hat{\mu} \subset \text{supp}(\mathbf{y})$ . But $\emptyset \neq \lambda' \subset \text{supp}(w\mathbf{z})$ .
3	4	If $\lambda = \lambda' = \frac{\mathbf{y}}{y_1 y_2}$ then $\mu = \frac{\mathbf{z}}{z_2}$ , but $\mu$ is $\mathbf{z}$ or $w\mathbf{z}$ .
3	5	$\mu = \mathbf{z}$ or $\mu = w\mathbf{z}$ , so $\lambda = \hat{\mu}$ is $\frac{\mathbf{y}}{y_1}$ or $\mathbf{y}$ , but $\lambda' = \emptyset$ .
3	6,7,8	$\mu = \mathbf{z}$ or $\mu = w\mathbf{z}$ , so $\lambda = \hat{\mu}$ is $\frac{\mathbf{y}}{y_1}$ or $\mathbf{y}$ , but $\emptyset \neq \lambda' \subset \text{supp}(w\mathbf{z})$ .
4	6	$\lambda = \frac{\mathbf{y}}{y_1 y_2}$ , but $\emptyset \neq \lambda' = \hat{\rho} \subset \text{supp}(w\mathbf{z})$ .
4	8,9,10,11	$\deg(\lambda) = r - 2$ , but $\deg(\lambda') \geq r - 1$ .
5	6	$\lambda = \emptyset$ and $\lambda' \neq \emptyset$ .
5	8,9,10,11	$\lambda = \emptyset$ and $\lambda' \neq \emptyset$ .
6	9,10,11	$\emptyset \subset \rho \subset \text{supp}(\mathbf{y})$ , so $\emptyset \neq \lambda = \hat{\rho} \subset \text{supp}(w\mathbf{z})$ . But $\emptyset \neq \lambda' \subseteq \text{supp}(\mathbf{y})$ .
7	8,9,10,11	$\lambda = \frac{\mathbf{z}}{z_r}$ , but $\lambda'$ has a different value.
8	9,10,11	$\lambda = \frac{w\mathbf{z}}{z_r}$ , but $\lambda'$ has a different value.
9	10	$\lambda = \frac{\mathbf{y}}{y_j}$ , but $\lambda' = \mathbf{y}$ .
10	11	$\lambda = \mathbf{y}$ , but $\lambda' = \frac{\mathbf{y}}{y_1}$ .

If  $\lambda = \lambda'$ , then  $\Gamma$  and  $\Gamma'$  must belong to one of the pairs of Cases listed in the following table. We assume that  $\lambda = \lambda'$  and give the differences between  $\Gamma$  and  $\Gamma'$  in the last column of the table.

Case of $\Gamma$	Case of $\Gamma'$	Difference
2	3	$\lambda = \lambda' = \frac{\mathbf{y}}{y_1}$ , so $u = t_{d+e}y_1$ and $u' = t_{d'+i'-1}y_1$ . By Lemma 4.17 (1,2), $u \neq u'$ .
2	4	If $\lambda = \lambda' = \frac{\mathbf{y}}{y_1y_2}$ , then $\mu = \frac{\mathbf{z}}{z_2}$ so $\max(g) > \max(z_2)$ by Lemma 4.10(2), but $\max(g') < \max(z_2)$ .
2	9,10,11	$\max(t_d) = \max(g) > \max(y_r)$ , so $\max(\ell) = \max(t_{d+e}) > \max(t_{y_r+r})$ by Lemma 4.17(1). But $\max(\ell') = \max((\max(g'), \max(t_{y_j+j})) \leq \max(t_{y_r+r})$ , because $\max(t_{y_j+j}) \leq \max(t_{y_r+j}) \leq \max(t_{y_r+r})$ and $\max(g') < \max(y_r) < \max(t_{y_r+r})$ by Lemmas 4.15 and 4.17(4).
3	9,10,11	$\max(\ell) = \max(t_{d+i-1})$ . In case 11, set $j' = 1$ . Then $\max(g') < \max(y_{j'})$ . So $\max(\ell') = \max(t_{y_{j'}+j'})$ by Lemma 4.15. By Lemma 4.17(3), $\max(\ell)$ and $\max(\ell')$ are different.
4	5,7	$u = y_1y_2z_2$ , but $wz_r$ divides $u'$ .
5	7	$y_r \notin \beta$ , but $y_r \in \beta'$
6	7,8	$y_r \in \text{supp}(\beta')$ . But $z_r = x_n \notin \alpha$ so $y_r \notin \text{supp}(\beta)$ .
9	11	$w \in \text{supp}(\beta')$ . If $w \in \text{supp}(\beta)$ , then $y_1 \in \alpha$ , so $j \neq 1$ , and then $y_1 \notin \ell$ but $y_1 \in \ell'$ .

It remains to prove (1). In each of the Cases in Construction 4.12, we will show that  $\phi(\Gamma)$  is an EK-triple for  $T$ . Set  $\phi(\Gamma) = (\theta, \ell, \beta)$ . In all cases it is immediate that  $\theta\ell\beta$  is squarefree and that  $\max(\beta) < \max(\ell)$ . Thus we need only verify that  $\ell$  is a minimal monomial generator for  $(T : \theta)$ .

Case 1 is clear.

Case 2: We have that  $\tau\mu \frac{gq}{x_{\max(g)}} = \tau \frac{g}{x_{\max(g)}} w\mathbf{z} \in T \setminus N$ . Hence  $\tau \frac{g}{x_{\max(g)}} \mathbf{y} \in N$ , so  $\tau \frac{g}{x_{\max(g)}} \mathbf{y}z_r \in N$ . Thus  $\tau \frac{g}{x_{\max(g)}} \mathbf{y}z_r \in T$ . Since  $T$  is squarefree Borel, we have  $\tau \frac{g}{x_{\max(g)}} \mathbf{y}t_{d+e} \in T$ , and hence  $\frac{g}{x_{\max(g)}} t_{d+e} \frac{\mathbf{y}}{\hat{\mu}} \in (T : \tau\hat{\mu})$ . Suppose that this is not a minimal monomial generator. Then  $\tau\hat{\mu} \frac{g\hat{q}}{x_{\max(g)}c} t_{d+e} \in T$ . Since  $T$  is squarefree Borel

and  $t_{d+e}$  is lex-after  $x_{\max(g)}y_n$ , it follows that  $\tau\hat{\mu}\frac{g}{x_{\max(g)}}\hat{q} = \tau\frac{g}{x_{\max(g)}}\mathbf{y} \in T$ . Hence

$\tau\frac{g}{x_{\max(g)}}w\mathbf{z} \in N$ . So we get the contradiction  $\frac{g}{x_{\max(g)}q} \in (N : \tau\mu)$ .

Case 3: Note that  $\max(q) < \max(g) < \max(y_r) < \max(z_2)$  implies that either  $q = w$  or  $q = 1$ . If  $\max(g) > y_1$  or  $q = 1$ , we have that  $t_{d+i-1}$  is lex-after  $x_{\max(g\hat{q})}$  and the proof of Case 2 holds, mutatis mutandis. If not, we are in Case 4.

Case 4: We have  $\tau\frac{g}{x_{\max(g)}}w\mathbf{z} \in T \setminus N$ , so  $\tau\frac{g}{x_{\max(g)}}\mathbf{y} \in N \setminus T$ . Thus  $\tau\frac{g}{x_{\max(g)}}\mathbf{y}z_2 \in N$  and also  $T$ . This yields  $\frac{g}{x_{\max(g)}}y_1y_2z_2 \in (T : \tau\frac{\mathbf{y}}{y_1y_2})$ . If this were not a minimal monomial generator, we would have  $\frac{g}{x_{\max(g)}}y_1y_2 \in (T : \tau\frac{\mathbf{y}}{y_1y_2})$ , so  $\tau\frac{g}{x_{\max(g)}}\mathbf{y} \in T$ .

In all of the remaining cases,  $\Gamma$  is of Type 3, so it is immediate from the table in Lemma 4.9 (2) that  $\ell \in (T : \theta)$ .

Case 5: If  $gw\mathbf{z}$  were not a minimal monomial generator for  $(T : \tau)$ , we would have  $\frac{g}{c}w\mathbf{z} \in (T : \tau)$  which implies  $\frac{g}{c}\mathbf{y} \in (N : \tau)$ , contradicting the assumption that  $g\mathbf{y}$  is a minimal monomial generator of  $(N : \tau)$ .

Case 6: If  $g\frac{w\mathbf{z}}{\hat{\rho}}$  were not a minimal monomial generator for  $(T : \tau\hat{\rho})$ , we would have  $\frac{g}{c}w\mathbf{z} \in (T : \tau)$ .

Case 7: If  $gwz_r$  were not a minimal monomial generator for  $(T : \tau\frac{\mathbf{z}}{z_r})$ , we would have  $\frac{g}{c}w\mathbf{z} \in (T : \tau)$ .

Case 8: If  $gz_r$  were not a minimal monomial generator for  $(T : \tau\frac{w\mathbf{z}}{z_r})$ , we would have  $\frac{g}{c}w\mathbf{z} \in (T : \tau)$ .

Case 9: If  $gy_it_{y_i+i}$  were not a minimal monomial generator for  $(T : \tau\frac{\mathbf{y}}{y_j})$ , we would have  $\frac{g}{c}\mathbf{y}t_{y_j+j} \in (T : \tau)$ . Since  $T$  is squarefree Borel and  $x_{\max(g)}$  is lex-before  $y_j$  and hence lex-before  $t_{y_j+j}$  by Lemma 4.15, it would follow that  $g\mathbf{y} \in (T : \tau)$ .

Case 10: If  $gt_{y_j+j}$  were not a minimal monomial generator for  $(T : \tau\mathbf{y})$ , we would

have  $\frac{g}{c}t_{y_j+j} \in (T : \tau)$ . Since  $T$  is squarefree Borel and  $x_{\max(g)}$  is lex-before  $y_j$  and hence lex-before  $t_{y_j+j}$  by Lemma 4.15, it would follow that  $g\mathbf{y} \in (T : \tau)$ .

Case 11: If  $gy_1t_{y_1+1}$  were not a minimal monomial generator for  $(T : \tau\mathbf{y})$ , we would have  $\frac{g}{c}y_1t_{y_1+1} \in (T : \tau)$ . Since  $T$  is squarefree Borel and  $x_{\max(g)}$  is lex-before  $y_i$  and hence lex-before  $t_{y_i+i}$  by Lemma 4.15, it would follow that  $g\mathbf{y} \in (T : \tau)$ .  $\square$

In the proof of the above Lemma 4.14 we used the following supplementary lemmas:

**Lemma 4.15.** *Let  $\tau \in k[\mathcal{A}^c]$  and  $g$  be a squarefree monomial in  $k[\mathcal{A}^c]$  such that  $g\tau\mathbf{y} \in N$ . Suppose that either  $gy_j$  is a minimal monomial generator of  $(N : \tau w\mathbf{z})$  or that  $gw y_j$  is a minimal monomial generator of  $(N : \tau\mathbf{z})$ . Then  $\max(g) < \max(y_j)$ .*

*Proof:* Suppose the opposite. Let  $c = x_{\max(g)} < y_j$ , and  $\frac{g}{c}c\tau\mathbf{y} \in N$ . By Construction 4.2, it follows that the ideal  $N$  is  $(\{c\} \cup \mathcal{A} \setminus \{y_j\})$ -compressed. Therefore,  $\frac{g}{c}y_j(\tau w\mathbf{z}) \in N$ . Hence, we have that  $\frac{g}{c}y_j \in (N : \tau w\mathbf{z})$  and  $\frac{g}{c}y_j w \in (N : \tau\mathbf{z})$ . This is a contradiction.  $\square$

**Lemma 4.16.** *Let  $(\tau\mu, gq, \alpha)$  be an EK-triple of Type 2 for  $N$ . Then  $\tau \frac{g}{x_{\max(g)}}\mathbf{y} \in N$ .*

*Proof:* Applying Lemma 4.11(2), we have that  $\mu q = w\mathbf{z}$ . Thus  $gq \in T_{\tau\mu}$ , so, since  $(\tau\mu, gq, \alpha)$  is an EK-triple of Type 2, it must be the case that  $\frac{gq}{x_{\max(gq)}} = \frac{g}{x_{\max(g)}}q \in T_{\tau\mu} \setminus N_{\tau\mu}$ . Hence,  $\frac{g}{x_{\max(g)}}\tau q\mu = \frac{g}{x_{\max(g)}}\tau w\mathbf{z} \in T \setminus N$ , so  $\frac{g}{x_{\max(g)}}\tau\mathbf{y} \in N \setminus T$ .  $\square$

**Lemma 4.17.** *Let everything be as in the proof of Lemma 4.14(3). Then:*

- (1) *If  $\max(g) > \max(y_r)$ , then  $\max(t_{d+e}) > \max(t_{y_r+r})$ .*
- (2) *If  $\max(g) < \max(y_r)$ , then  $\max(t_{d+i-1}) < \max(t_{y_r+r})$ .*
- (3)  *$t_{d+i-1} \neq t_{y_j+j}$  for any  $j \geq 1$ .*
- (4)  *$\max(y_j) < \max(t_{y_j+j})$  for any  $j \geq 1$ .*

*Proof:* (1) We have  $\max(t_d) > \max(t_{y_r+r-e})$ , as  $r - e = \#\{z_j : \max(z_j) < \max(t_d)\}$ .

(2) We have  $\max(t_d) < \max(y_r)$  and  $i - 1 < r$ .

(3) If  $\max(t_d) \leq \max(t_{y_j})$ , then  $i < j$ . If  $\max(t_d) > \max(t_{y_j})$ , then  $i \geq j$ .

(4)  $t_{y_j}$  is the lex-last  $t$ -variable that is lex-before  $y_j$ . Hence,  $t_{y_{j+1}}$  is lex-after  $y_j$ . The variable  $t_{y_{j+j}}$  comes lex-later still. Thus,  $y_j > t_{y_{j+j}}$ .  $\square$

We are ready for the proof of the Main Lemma 3.9.

*Proof of the Main Lemma 3.9:* Let  $T$  be the ideal constructed in Construction 4.5. By Lemma 4.6,  $T$  is a squarefree Borel ideal lexicographically greater than  $N$ , and it has the same Hilbert function as  $N$ .

By Theorem 3.3, the graded Betti numbers of  $S/(N + P)$  and of  $S/(T + P)$  can be counted using EK-triples. By Lemma 4.14, there exists an injection  $\phi$  from the set of EK-triples for  $N$  to the EK-triples for  $T$  which preserves bidegree. Therefore, there are at least as many EK-triples for  $T$  as for  $N$  in every bidegree. It follows that for all  $p, s$ , the graded Betti numbers satisfy

$$b_{p,s}(S/(T + P)) \geq b_{p,s}(S/(N + P)).$$

$\square$

## 5. Ideals plus squares

Let  $F$  be a graded ideal containing  $P = (x_1^2, \dots, x_n^2)$ ; we say that  $F$  is an *ideal-plus-squares*. If  $F = I + P$  for some ideal  $I$  which is squarefree Borel or squarefree lex, we say that  $F$  is *Borel-plus-squares* or *lex-plus-squares* respectively. By Kruskal-Katona's Theorem [Kr,Ka], there exists a squarefree lex ideal  $L$  such that  $F$  and the lex-plus-squares ideal  $L + P$  have the same Hilbert function.

**Theorem 5.1.** *Suppose that  $\text{char}(k) = 0$ . Let  $F$  be a graded ideal containing  $P = (x_1^2, \dots, x_n^2)$ . Let  $L$  be the squarefree lex ideal such that  $F$  and the lex-plus-squares ideal  $L + P$  have the same Hilbert function. The graded Betti numbers of  $L + P$  are greater than or equal to those of  $F$ .*

*Proof:* The proof has 5 steps. In each of the first four steps, we replace the original (non-lex) ideal by an ideal with the same Hilbert function and greater graded Betti numbers.

*Step 1:* Let  $F'$  be the initial ideal of  $F$  (with respect to any fixed monomial order). It has the following properties:

- $F' \supseteq P$ .

- $F'$  is a monomial ideal with the same Hilbert function as  $F$ .
- The graded Betti numbers of  $F'$  are greater than or equal to those of  $F$ .

We will prove the theorem by showing that the graded Betti numbers of the lex-plus-squares ideal  $L + P$  are greater than or equal to those of  $F'$ .

*Step 2:* Now, we change the ground field  $k$  to an infinite field  $\tilde{k}$  of characteristic 2. We denote by  $\tilde{F} \subset \tilde{k}[x_1, \dots, x_n]$  the monomial ideal generated by the monomials in  $F'$ . It has the following properties:

- $\tilde{F} \supseteq P$ .
- $\tilde{F}$  is a monomial ideal with the same Hilbert function as  $F'$ .
- The graded Betti numbers of  $\tilde{F}$  are greater than or equal to those of  $F'$ .

We will prove the theorem by showing that the graded Betti numbers of the lex-plus-squares ideal  $L + P$  are greater than or equal to those of  $\tilde{F}$ .

*Step 3:* Now, let  $\tilde{M}$  be a generic initial ideal of  $\tilde{F}$  (with respect to any monomial order). It has the following properties:

- $\tilde{M} \supseteq P$  because the characteristic of  $\tilde{k}$  is 2.
- $\tilde{M}$  is a Borel-plus-squares ideal with the same Hilbert function as  $\tilde{F}$ .
- The graded Betti numbers of  $\tilde{M}$  are greater than or equal to those of  $\tilde{F}$ .

We will prove the theorem by showing that the graded Betti numbers of the lex-plus-squares ideal  $L + P$  are greater than or equal to those of  $\tilde{M}$ .

*Step 4:* The Eliahou-Kervaire resolution [EK] shows that the graded Betti numbers of a squarefree Borel ideal do not depend on the characteristic. By Theorem 2.1(4) and Lemma 3.1(1), it follows that the graded Betti numbers of a Borel-plus-squares ideal do not depend on the characteristic. So now, we return to the ground field  $k$ . We denote by  $M \subset k[x_1, \dots, x_n]$  the monomial ideal generated by the monomials in  $\tilde{M}$ . It has the following properties:

- $M \supseteq P$ .
- $M$  is a Borel-plus-squares ideal with the same Hilbert function as  $\tilde{M}$ .
- The graded Betti numbers of  $M$  are equal to those of  $\tilde{M}$ .

We will prove the theorem by showing that the graded Betti numbers of the lex-plus-squares ideal  $L + P$  are greater than or equal to those of  $M$ .

*Step 5:* Let  $N$  be the squarefree Borel ideal such that  $M = N + P$ . Since  $N + P$  and  $L + P$  have the same Hilbert function, we can apply Theorem 3.4. It yields that the graded Betti numbers of the lex-plus-squares ideal  $L + P$  are greater than or equal to those of  $M = N + P$ .  $\square$

## 6. Ideals plus powers

Let  $\mathbf{a} = \{a_1 \leq a_2 \leq \dots \leq a_n\}$  be a sequence of integers or  $\infty$ . Set  $U = (x_1^{a_1}, \dots, x_n^{a_n})$ , where  $x_i^\infty = 0$ . We say that a monomial  $m \in S$  is an  $\mathbf{a}$ -*monomial* if the image of  $m$  in  $S/U$  is non-zero. Following [GHP], an ideal in  $S$  is called an  $\mathbf{a}$ -*ideal* if it is generated by  $\mathbf{a}$ -monomials.

Set  $\sigma^{\mathbf{a}} = \prod_{i \in \sigma} x_i^{a_i}$ , and for a monomial  $\mathbf{a}$ -ideal  $I$  set  $F_\sigma = S/(I : \sigma^{\mathbf{a}})(-2\sigma^{\mathbf{a}})$ . Note

that  $F_\sigma = 0$  if, for any  $i \in \sigma$ ,  $a_i = \infty$ . Note also that  $(I : \sigma^{\mathbf{a}}) = (I : \prod_{i \in \sigma} x_i^{a_i - 1})$  is

the ideal formed by “erasing” all the variables in  $\sigma$  from a generating set for  $I$ . The argument in the proof of Theorem 2.1 yields:

**Theorem 6.1.** *Let  $I$  be a monomial  $\mathbf{a}$ -ideal.*

(1) *We have the long exact sequence*

(6.2)

$$\begin{aligned} 0 \rightarrow \bigoplus_{|\sigma|=n} F_\sigma \xrightarrow{\varphi_n} \dots \rightarrow \bigoplus_{|\sigma|=i} F_\sigma \xrightarrow{\varphi_i} \bigoplus_{|\sigma|=i-1} F_\sigma \rightarrow \\ \dots \rightarrow \bigoplus_{|\sigma|=1} F_\sigma \xrightarrow{\varphi_1} \bigoplus_{|\sigma|=0} F_\sigma = S/I \rightarrow S/(I+U) \rightarrow 0 \end{aligned}$$

with maps  $\varphi_i$  the Koszul maps for the sequence  $x_1^{a_1}, \dots, x_n^{a_n}$ .

(2) *Each of the ideals  $(I : \sigma^{\mathbf{a}})$  in (1) is an  $\mathbf{a}$ -ideal monomial ideal.*

(3)  *$S/(I+U)$  is minimally resolved by the iterated mapping cones from (6.2).*

**Remark 6.3.** The other results in the previous sections cannot be generalized to this situation. The first problem is that if  $I$  and  $J$  are  $\mathbf{a}$ -ideals, then it is not true that  $I$  and  $J$  have the same Hilbert function if and only if  $I+U$  and  $J+U$  have the same Hilbert function. The following example from [GHP] illustrates this: the ideals  $I = (x^2, y^2)$  and  $J = (x^2, xy)$  have different Hilbert functions, but the ideals  $I + (x^3, y^3)$  and  $J + (x^3, y^3)$  have the same Hilbert function.



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