Compressed ideals

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Abstract: Compressed ideals have been used by Macaulay and others to study Hilbert functions and lex ideals in a polynomial ring. We formalize the theory of compression, classify the compressed ideals, and provide new proofs of theorems of Macaulay, Kruskal-Katona, and Bigatti, Hulett, and Pardue.

1 Introduction

Lex ideals are important in the study of a polynomial ring $R = k[x_1, \cdots, x_n]$ because they can be used to classify the Hilbert functions of ideals in R. An important tool in the study of lex ideals, dating back to Macaulay [Ma], has been compression, which allows one to move carefully towards the lex ideal while controlling the Hilbert function. Compressed ideals are combinatorially very well-behaved, which allows us to compare their invariants to those of lex ideals in ways which are impossible for monomial or even Borel ideals. The goal of this paper is to codify the theory of compression and show how it may be used to recover some classical results on Hilbert functions and Betti numbers.

Throughout the paper, R is the polynomial ring $k[x_1, \cdots, x_n]$ and S is the quotient of R by the squares of the variables, $S = R/(x_1^2, \cdots, x_n^2)$.

In section 2, we introduce notation that will be used throughout the paper, and develop the basic theory of compression.

In section 3, we study compressed ideals, culminating in the classification of compressed ideals of R and S, respectively, in Theorems 3.12 and 3.13. Theorem 3.10 reduces many questions about lex ideals to questions about lex ideals of $k[a, b, c]$ and an inductive step. This will be illustrated in sections 4 and 5.

In section 4, we use compressed ideals to give new proofs of the theorems of Macaulay [Ma] and Kruskal-Katona [Kr, Ka] that every Hilbert function in R (and, respectively, S), is attained by a lex ideal.

In section 5, we show that Betti numbers are nondecreasing under compression. As an application, we recover the theorem of Bigatti, Hulett, and Pardue [Bi, Hu, Pa] that lex ideals have maximal graded Betti numbers in R.

In the short section 6, we make some comments about possible applications to the Hilbert scheme.

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2 Preliminaries

Let k be an infinite field, and set $R = k[x_1, \cdots, x_n]$, and $S = R/(x_1^2, \cdots, x_n^2)$.

Notation 2.1. The Hilbert function of a graded module M over a graded k-algebra A (or, more generally, any graded k-vector space) associates to every degree d the k-dimension of the vector space $M_d = \{f \in M : \deg f = d\}$,

$$\mathrm{Hilb}_{A,M}(d) = \dim_k(M_d).$$

For simplicity, we use the notation $|M_d|$ or $|M|_d$ for $\mathrm{Hilb}_{A,M}(d)$.

Definition 2.2. We say that a monomial ideal L of R (or of S) is *lex* if it satisfies the following condition:

> Suppose that $u \in L$ is a monomial of degree d, and v is a degree d monomial lex-before u. Then we have $v \in L$.

Lex ideals are important because of the following theorem, due to Macaulay [Ma]:

Theorem 2.3 (Macaulay). *Let I be a homogeneous ideal of R. Then there exists a lex ideal L such that L has the same Hilbert function as I.*

Macaulay's theorem also holds over S, where it is known as Kruskal and Katona's theorem [Kr,Ka].

Notation 2.4. Throughout this section, fix a subset $\mathcal{A}$ of $\{x_1, \cdots, x_n\}$. We denote by $k[\mathcal{A}]$ the polynomial ring in the variables of $\mathcal{A}$ and by $k[\mathcal{A}^c]$ the polynomial ring in the variables of $\mathcal{A}^c$. By abuse, if we work in S rather than in R, $k[\mathcal{A}]$ and $k[\mathcal{A}^c]$ will refer to the quotients of those polynomial rings by the squares of their respective variables.

Notation 2.5. Let N be a monomial ideal of R (or of S). Then we may decompose N as the direct sum $N = \bigoplus_{f \in k[\mathcal{A}^c]} fN_f$, where f ranges over the monomials of $k[\mathcal{A}^c]$ and each N_f is, by Macaulay's theorem, a monomial ideal of $k[\mathcal{A}]$.

Example 2.6. Take $N = (a^2, ab, ac, b^2, bc) \subset k[a,b,c,d]$, and $\mathcal{A} = \{a, b, d\}$. Then $N = 1(a^2, ab, b^2) \oplus c(a,b) \oplus c^2(a,b) \oplus c^3(a,b) \oplus \cdots$.

Definition 2.7. If every N_f is a lex ideal of $k[\mathcal{A}]$, we say that N is $\mathcal{A}$-*compressed.* In general, let T_f be the lex ideal with the same Hilbert function as N_f, and set $T = \bigoplus_{f \in k[\mathcal{A}^c]} fT_f$. We say that T is the $\mathcal{A}$-*compression* of N.

Example 2.8. Taking N and $\mathcal{A}$ as in example 2.6 above, N is not $\mathcal{A}$-compressed since $N_1 = (a^2, ab, b^2)$ is not lex in $k[a, b, d]$. The $\mathcal{A}$-compression of N is $T = 1(a^2, ab, ad) \oplus c(a, b) \oplus c^2(a, b) \oplus \cdots = (a^2, ab, ac, ad, bc)$.

Theorem 2.9. *Let N be any monomial ideal of R (or S), and let T be its $\mathcal{A}$-compression. Then T is an ideal of R (respectively, S).*

Proof. It suffices to show that $x_i(fT_f) \subset T$, for all i and f. Observe that $T_f \subset T_g$ if and only if $|T_f|_d \leq |T_g|_d$ for all d, since the T_f are lex ideals of $k[\mathcal{A}]$.

If $x_i \in \mathcal{A}$, then $x_i T_f \subset T_f$ because T_f is an ideal of $k[\mathcal{A}]$, so we have $x_i f T_f \subset f T_f \subset T$.

If $x_i \notin \mathcal{A}$, we have $x_i f N_f \subset x_i f N_{x_i f}$ since N is an ideal. Thus $N_f \subset N_{x_i f}$ and $|N_f|_d \leq |N_{x_i f}|_d$ for all d. Hence $|T_f|_d \leq |T_{x_i f}|_d$ for all d, and so $T_f \subset T_{x_i f}$. Thus $x_i f T_f \subset x_i f T_{x_i f} \subset T$. □

3 Structure of compressed ideals

We make a number of observations about $\mathcal{A}$-compressed ideals.

Proposition 3.1. *Suppose that $N \subset R$ is $\mathcal{A}$-compressed. Set $B = R[y]$. Then NB is $\mathcal{A}$-compressed as an ideal of B.*

Remark 3.2. Proposition 3.1 holds regardless of the position of y in the lexicographic order.

Proposition 3.3. *If N is lex, then N is $\mathcal{A}$-compressed.*

Proposition 3.4. *If N is $\mathcal{A}$-compressed and $\mathcal{A} \supset \mathcal{B}$, then N is $\mathcal{B}$-compressed.*

Proposition 3.5. *N is $\{x_i\}$-compressed for any x_i.*

Definition 3.6. Let r be a positive integer. If N is $\mathcal{A}$-compressed for every r-element set $\mathcal{A}$, we say that N is r-*compressed.* If N is $\mathcal{A}$-compressed for every proper subset $\mathcal{A}$ of $\{x_1, \cdots, x_n\}$, we simply say that N is *compressed.*

Definition 3.7. A monomial ideal N is *0-Borel-fixed* or *strongly stable* if it satisfies the following condition:

> Let $m \in N$ be a monomial and suppose that x_j divides m and $i < j$. Then $\frac{x_i}{x_j} m \in N$ as well.

Proposition 3.8. *N is 2-compressed if and only if N is strongly stable.*

Remark 3.9. Up to this point, everything has held in somewhat more generality. The ring R could have been replaced by, for example, a quotient of the form $k[x_1, \cdots, x_n]/(x_1^{e_1}, \cdots, x_n^{e_n})$ with $e_1 \leq \cdots \leq e_n \leq \infty$ without meaningful modification to any of the statements or proofs. (Macaulay's theorem in such a ring is known as Clements-Lindström's theorem [CL].) Beginning with Theorem 3.10, however, it will be essential that we work over the correct ring.

Theorem 3.10. *Let N be a monomial ideal of R. N is 3-compressed if and only if N is lex.*

Theorem 3.10 is a corollary of the following sharper result:

Proposition 3.11. *Suppose that $N \subset R$ is 2-compressed and also $\mathcal{A}$-compressed for every set $\mathcal{A}$ of the form $\{x_i, x_{i+1}, x_n\}$. Then N is lex.*

Proof. Let $u \in N$ be a monomial of degree d, and let v be another monomial of degree d, lex-before u. We will show that $v \in N$.

Write $u = \prod x_i^{e_i}$ and $v = \prod x_i^{f_i}$. Let i be minimal such that $e_i \neq f_i$; we have $e_i < f_i$. Put $w = \prod_{j=1}^{i} x_i^{e_i}$, $u' = \frac{u}{w}$, and $v' = \frac{v}{x_i w}$. Set $D = \deg u'$, and observe that $u \in k[x_{i+1}, \cdots, x_n]$, $v \in k[x_i, \cdots, x_n]$.

Since $u = wu' \in N$ and N is strongly stable, we have $wx_{i+1}^D \in N$.

Since N is $\{x_i, x_{i+1}x_n\}$-compressed, we have $wx_ix_n^{D-1} \in N$.

Since N is strongly stable, we have $wx_iv' = v \in N$. □

Propositions 3.5 and 3.8 and Theorem 3.10 combine to give us the following structure theorem for compressed ideals of R:

Theorem 3.12. *We classify the compressed ideals of R as follows:*

- *If $n < 3$, every monomial ideal is compressed.*
- *If $n = 3$, the compressed ideals are precisely the strongly stable ideals.*
- *If $n > 3$, the compressed ideals are precisely the lex ideals.*

We can also describe the compressed ideals of S, as follows:

Theorem 3.13. *Let N be a compressed ideal of S. Then:*

- *If n is odd, N is lex.*
- *If $n = 2r$ is even, the vector space N_d is lex for every $d \neq r$.*
- *If $n = 2r$ and $u \in N$, $v \notin N$ are degree r monomials with v lex-before u, then $u = x_2x_3 \cdots x_{r+1}$ and $v = x_1x_{r+2}x_{r+3} \cdots x_{2r}$.*

In particular, if N is not lex, then N_d is generated by $\{(x_1)_d\} \smallsetminus \{v\}$ and u, where u and v are as above and $\{(x_1)_d\}$ denotes the monomials in $(x_1)_d$. That is, if N is not lex, N_d is generated by the lex segment terminating at u, with a single gap at v. Note that u is the successor of v in the lex order.

Proof. Suppose that $u \in N$ and $v \notin N$ both have degree r, and v is lex-before u. Write $u = \prod x_i^{e_i}$ and $v = \prod x_i^{f_i}$. Set $\mathcal{A} = \{x_i : e_i \neq f_i\}$.

N cannot be $\mathcal{A}$-compressed, so we must have $\mathcal{A} = \{x_1, \cdots, x_n\}$. On the other hand, $\mathcal{A} \subset \operatorname{supp}(u) \cup \operatorname{supp}(v)$. Thus $\mathcal{A} = \operatorname{supp}(u) \cup \operatorname{supp}(v)$ and $n = 2r$.

We have $\operatorname{supp}(u) \cap \operatorname{supp}(v) = \varnothing$, and x_1 divides v, since v is lex-before u. If $n = 2$ we are done, otherwise suppose that x_i divides v, for some $2 \leq i \leq r+1$. Then there exists x_j dividing u with $j > r+1$. Since N is $\{x_i, x_j\}$-compressed, we have $u\frac{x_i}{x_j} \in N$. Then, since N is $(\{x_1, \cdots, x_n\} \setminus \{x_i, x_j\})$-compressed, we have $v \in N$. Thus every $x_i, 2 \leq i \leq r+1$, must divide u, so $u = x_2 x_3 \cdots x_{r+1}$ and $v = x_1 x_{r+2} x_{r+3} \cdots x_{2r}$. □

4 Macaulay's Theorem

Macaulay's theorem classifies the Hilbert functions over R in terms of lex ideals:

Theorem 4.1 (Macaulay). *Every Hilbert function over R is attained by a lex ideal. That is, let I be any homogeneous ideal of R. Then there exists a lex ideal L such that* $\operatorname{Hilb}_{R,L} = \operatorname{Hilb}_{R,I}$.

Macaulay's original proof is probably the first example of a compression argument (using $\mathcal{A} = \{x_2, \cdots, x_n\}$), but is sufficiently opaque that he felt it necessary to warn his readers away, saying:

> This proof of the theorem ... is given only to place it on record. It is too long and complicated to provide any but the most tedious reading.

A number of other proofs have appeared since, most notably that of Green [Gr]. In this section, we present one more.

Our proof is by induction on n, and so will make free use of Theorem 2.9, which was proved using Macaulay's theorem on the smaller ring $k[\mathcal{A}]$. We begin with some remarks on compression:

Definition 4.2. Let $\{m_i\}$ and $\{n_i\}$ be two sets of monomials, ordered so that m_i is before m_j in the graded lex order (respectively, n_i before n_j) whenever $i < j$. We say that $\{m_i\}$ is *lexicographically greater than* $\{n_i\}$ if the following condition is satisfied:

> Let j be minimal such that $m_j \neq n_j$. Then m_j is before n_j in the graded lex order.

If I and J are two monomial ideals, and $\{I\}, \{J\}$ are the sets of monomials appearing in each, we say that I is *lexicographically greater than* J if $\{I\}$ is lexicographically greater than $\{J\}$.

Lemma 4.3. *Let N be a monomial ideal, $\mathcal{A} \subsetneq \{x_1, \cdots, x_n\}$ any set of variables, and T the $\mathcal{A}$-compression of N. Then T has the same Hilbert function as N, and T is lexicographically greater than N.*

Proof. Every T_f has the same Hilbert function as N_f, and is lexicographically greater than N_f. □

Lemma 4.4. *Let N be any homogeneous ideal, and let $\mathfrak{A}$ be any collection of proper subsets of $\{x_1, \cdots, x_n\}$. Then there exists an ideal T, having the same Hilbert function as N, which is $\mathcal{A}$-compressed for all $\mathcal{A} \in \mathfrak{A}$.*

Proof. Set $T_0 = N$ and proceed inductively as follows: If $\mathcal{A} \in \mathfrak{A}$ is such that T_i is not $\mathcal{A}$-compressed, let T_{i+1} be the $\mathcal{A}$-compression of T_i. Then all T_i have the same Hilbert function, and T_{i+1} is lexicographically greater than T_i for all i. Since "lexicographically greater than" is a well-ordering on the (finite) sets of monomials in R, this process must stabilize in degree less than or equal to d, say at $T_{s(d)}$, for all d. Let T be the ideal whose degree-d components are the $T_{s(d)}$. □

Corollary 4.5. *Let N be any homogeneous ideal. Then there exists a compressed ideal T having the same Hilbert function as N.*

We are now ready for the proof of Macaulay's theorem:

Proof of Macaulay's Theorem 4.1. We may assume by Gröbner basis theory that N is monomial. If $n = 1$ or 2, the theorem is now obvious.

Otherwise, by corollary 4.5, we may assume that N is compressed. If $n \geq 4$, Theorem 3.12 shows that N is lex.

This leaves the case that $R = k[a, b, c]$ and N is strongly stable. It suffices to show for every degree d that, if L_d is the vector space spanned by the lex-first $|N_d|$ monomials of R, we have $|(a, b, c)L_d| \leq |(a, b, c)N_d|$. Since N_d is strongly stable, we have

$$|(a, b, c)N_d| = |N_d| + |N_d \cap k[a, b]| + |N_d \cap k[a]|$$

and likewise

$$|(a, b, c)L_d| = |L_d| + |L_d \cap k[a, b]| + |L_d \cap k[a]|.$$

$|N_d| = |L_d|$ by construction, and $N_d \cap k[a] = L_d \cap k[a] = (a^d)$, so it suffices to show that $|N_d \cap k[a, b]| \geq |L_d \cap k[a, b]|$.

Suppose that u, v are degree d monomials with $u \in N$, $v \notin N$, and v lex-before u. Then c divides v, as otherwise the strongly stable condition, applied to u, would require $v = (\frac{a}{b})^i(\frac{b}{c})^j u \in N$. In particular, any $v \in L_d \setminus N_d$ is not in $k[a, b]$, yielding the desired inequality. □

Macaulay's theorem is known to hold in many quotients of R [Kr, Ka, CL, Sh, MP1, MP2, Me]. The first extension was due to Kruskal [Kr] and Katona [Ka], who showed that it holds in $S = R/(x_1^2, \cdots, x_n^2)$:

Theorem 4.6 (Kruskal, Katona). *Macaulay's theorem holds in S. That is, if N is any homogeneous ideal of S, there exists a lex ideal L such that L and N have the same Hilbert function.*

We prove Kruskal-Katona's theorem, in the spirit of our proof of Macaulay's theorem:

Proof. We may assume by Gröbner basis theory that N is monomial. If $n = 1$ or 2, the theorem is now obvious.

Otherwise, we may assume by corollary 4.5 that N is compressed. If N is lex we are done.

If N is not lex, we have by Theorem 3.13 that $n = 2r$ and that N fails to be lex only in degree r, and only by containing $u = x_2 \cdots x_{r+1}$ but not $v = x_1 x_{r+2} \cdots x_n$. Let $\{N\}$ be the set of monomials of N, and let L be the vector space spanned by $\{N\} \setminus \{u\} \cup \{v\}$. Clearly, L has the same Hilbert function as N; we claim that it is an ideal. Indeed N (hence L) contains every multiple of v (as $x_i v = x_n(\frac{x_i}{x_n} v)$) and no divisor of u (as $\frac{u}{x_i} \in N$ would force $\frac{x_n}{x_i} u \in N$). $\square$

5 Betti numbers

Definition 5.1. If I is a homogeneous R-module, a *free resolution* of I is an exact sequence $\mathbb{F} : \cdots \to F_1 \to F_0 \to I \to 0$, with each F_i a free module. $\mathbb{F}$ is *minimal* if each F_i has minimum possible rank. The F_i may be graded $F_i = \oplus R(-d)^{b_{i,d}}$ so that each map $F_{i+1} \to F_i$ is homogeneous of degree zero. If a minimal free resolution is graded in this way, the *graded Betti numbers* of I are the $b_{i,d}$.

Bigatti [Bi], Hulett [Hu], and Pardue [Pa] showed that the graded Betti numbers of lex ideals of R are maximal among those of all homogeneous ideals with a fixed Hilbert function:

Theorem 5.2 (Bigatti, Hulett, Pardue). *Let I be a homogeneous ideal of R, and let L be the lex ideal having the same Hilbert function as I. Then, for all i, d, we have $b_{i,d}(L) \geq b_{i,d}(I)$.*

Analogous results are known in many rings which satisfy Macaulay's theorem [AHH, GHP, MPS], and are widely conjectured in general [FR].

In this section we show that the graded Betti numbers are nondecreasing under $\mathcal{A}$-compression, and provide a new proof of the results of Bigatti, Hulett, and Pardue over R. Our argument is in the spirit of Hartshorne's proof that the Hilbert scheme is connected [Ha].

Lemma 5.3. *Let k have characteristic zero, let $R = k[a, b]$, and let N be a monomial ideal of R. Let L be the lex ideal with the same Hilbert function as N. Then there exists a change of variables f such that the initial ideal of $f(N)$ with respect to the lex order is L, and the initial ideal of $f(N)$ with respect to the inverse lex order is N.*

Proof. Set $f(a) = a$ and $f(b) = a + b$. (Alternatively, a generic linear form for $f(b)$ will work.) $\square$

Lemma 5.4. *Let $R = k[a, b, c]$ (over an arbitrary infinite field k), and let N be strongly stable but not lex. Then there exists an ideal $\tilde{N}$ such that:*

- *N and $\tilde{N}$ have the same Hilbert function.*
- *N is the initial ideal of $\tilde{N}$ with respect to the grevlex order.*
- *The initial ideal of $\tilde{N}$ with respect to the lex order is strictly lexicographically greater than N.*
- *N and $\tilde{N}$ have the same graded Betti numbers.*

In particular, if N' is the lex initial ideal of $\tilde{N}$, we have $b_{i,j}(N') \geq b_{i,j}(\tilde{N}) = b_{i,j}(N)$ for all i, j.

Proof. Let R^{po} and N^{po} be the polarizations of R and N with respect to the variable b. (So if N is minimally generated by $\{a^{e_{i,1}} b^{e_{i,2}} c^{e_{i,3}}\}$, we have N^{po} generated by $\{a^{e_{i,1}} b_1 b_2 \cdots b_{e_{i,2}} c^{e_{i,3}}\}$). Let $f : R^{\mathrm{po}} \to R$ be defined by sending each b_j to a generic linear function of the form $f_j = f_{1,j} a + f_{2,j} b + f_{3,j} c$. Let $\tilde{N}$ be the image of N^{po} under f.

That N and $\tilde{N}$ have the same Hilbert function and graded Betti numbers is immediate from the theory of polarization (e.g., [Pe2]) (in particular, the operation $\tilde{(\cdot)}$ extends to any graded free resolution of N: a straightforward argument in the degree of the b-variables shows that the forms $b_j - b$ form a regular sequence on R/N; then, since the f_j are generic, we may assume that the forms $b_j - f_j$ form a regular sequence as well. Then, by a well-known Tor argument, the Betti numbers of N and $\tilde{N}$ over R are equal to those of N^{po} over R^{po}).

To see that N is the grevlex initial ideal of $\tilde{N}$, observe that, if $m = a^{e_1} b^{e_2} c^{e_3}$ is any monomial of N, then we have $\tilde{m} = c^{e_3}(g_0 a^{e_1+e_2} + \cdots + g_{e_2} a^{e_1} b^{e_2}) + c^{e_3+1}(\text{other terms}) \in \tilde{N}$. Subtracting an appropriate linear combination of $\tilde{n}$, for $n = a^{e_1+i} b^{e_2-i} c^{e_3} \in N$ gives us $m + c^{e_3+1}(\text{other terms}) \in \tilde{N}$, whose initial term is m.

To see that the lex initial ideal of $\tilde{N}$ is lexicographically greater than N, let $u \in N$ be the lex-first degree d monomial such that there exists $v \notin N$ lex-before u; choose v to be the first such. Since the f are generic, there exists a linear combination $\sum \tilde{m}$, for $m \in N$ the monomials lex-before or equal to u, with leading term v: since N is strongly stable, we have $u = a^{e_1} b^{e_2}$, and so $v = a^{e_1+1} b^i c^{e_2-i-1}$; we could have chosen $f_j = \alpha_j a + b$ for $j \leq i+1$ and $f_j = \alpha_j a + b + \gamma_j c$ for $j > i+1$. ☐

Remark 5.5. We believe, based on some extremely limited computer experiments, that if k has characteristic zero, the functions $f_1 = a+b$ and $f_i = a+b+c$, for $i > 1$, produce $\tilde{N}$ satisfying the desired conditions.

Remark 5.6. We believe that the functions f_i may be chosen so that the lex initial ideal of $\tilde{N}$ is the lexicographic ideal L. Unfortunately, we have no proof at this time.

Remark 5.7. A similar argument in two variables shows that lemma 5.3 holds in arbitrary characteristic.

Remark 5.8. The operation $\tilde{(\cdot)}$ defined in the proof of lemma 5.4 does not depend on the dimension of R. The proof continues to hold if $\{a, b, c\}$ is a subset of $\{x_1, \cdots, x_n\}$.

Theorem 5.9. *Let N be any monomial ideal of R, and let $\mathcal{A}$ be any subset of variables. If T is the $\mathcal{A}$-compression of N, then $b_{i,j}(T) \geq b_{i,j}(N)$ for all i, j.*

Proof. By induction on the cardinality of $\mathcal{A}$, we may assume that N is $\mathcal{B}$-compressed for all proper subsets $\mathcal{B}$ of $\mathcal{A}$. If $|\mathcal{A}| \geq 4$, we have $N = T$, by Theorem 3.12. If $|\mathcal{A}| = 2$ or 3, the proof of lemma 5.3 or 5.4, respectively, gives us a monomial ideal N' satisfying:

- The $\mathcal{A}$-compression of N' is T.
- N' is lexicographically greater than N
- $b_{i,j}(N') \geq b_{i,j}(N)$.

Since the monomial ideals with a fixed Hilbert function are well-ordered by "lexicographically greater than", we are done by induction. □

In fact, combining the proof of Theorem 5.9 and the proof in [Pe1] we obtain the following stronger result:

Theorem 5.10. *Under the assumptions of Theorem 5.9, the Betti numbers $b_{i,j}(N)$ of N can be obtained from the $b_{i,j}(T)$ by a sequence of consecutive cancellations.*

We obtain as a corollary the result of Bigatti [Bi], Hulett [Hu], and Pardue [Pa].

Theorem 5.11 (Bigatti, Hulett, Pardue). *Let I be any homogeneous ideal of R, and let L be the lex ideal with the same Hilbert function as I. Then $b_{i,j}(L) \geq b_{i,j}(I)$ for all i, j.*

Proof. Let N be the initial ideal of I in any order, so $b_{i,j}(N) \geq b_{i,j}(I)$. Now apply Theorem 5.9 to the ideal N, with $\mathcal{A} = \{x_1, \cdots, x_n\}$. □

Remark 5.12 (On multidegrees). Fix $\mathcal{A}$, and endow R with a multigraded structure as follows: If $m = fg$ is a degree d monomial with $f \in k[\mathcal{A}^c]$ and $g \in k[\mathcal{A}]$, set the *coarse multidegree* of m to be $\text{cmdeg}(m) = (f, d)$. Then we may define coarsely homogeneous ideals, coarse Hilbert functions, etc., by analogy to the usual definitions in a graded ring. With this notation, an $\mathcal{A}$-compressed ideal is coarsely lex, and our results may be restated in more familiar terms:

- Theorem 2.9 states that every coarse Hilbert function is attained by a coarsely lex ideal.

- Theorem 5.9 states that, if N is coarsely homogeneous, and T is coarsely lex with the same coarse Hilbert function, then the coarse Betti numbers of T are greater than or equal to those of N.
- Lemmas 5.3 and 5.4, together with Theorem 3.12, show that the coarse Hilbert scheme is connected.

6 Remarks on the Hilbert scheme

We close with some remarks about possible applications to the Hilbert scheme, which parametrizes all ideals with a fixed Hilbert function.

It is known that the Hilbert scheme is connected. Hartshorne proves this [Ha] by showing that there is a path to the lex ideal from any point on the Hilbert scheme. Reeves [Re] and Pardue [Pa] have shown that there exists a path of length at most $d+2$, where $d \leq n$ is the degree of the Hilbert polynomial.

In section 5, we have shown that one can walk to lex by walking to a sequence of compressions. These moves are much simpler than those defined in [Re], which involved Borel fans. It is natural to ask how many of our "compression steps" are necessary to reach the lex ideal from any monomial ideal. There might be a nice bound in terms of $(n-2)$ and the radius of the Hilbert scheme of $k[a, b, c]$ (since by proposition 3.11 it suffices to be simultaneously Borel and compressed with respect to the $n-2$ sets $\{x_i, x_{i+1}, x_n\}$.)

Furthermore, it should be possible to perform multiple such compressions at once, as the coordinate changes involved in the compressions with respect to, say, $\{x_i, x_{i+1}, x_n\}$ and $\{x_{i+2}, x_{i+3}, x_n\}$, might not interact harmfully.

In exploring these questions, we would like it to be the case that if N is $\mathcal{B}$-compressed and T is its $\mathcal{A}$-compression, then T is $\mathcal{B}$-compressed as well. Unfortunately, this is not true in general; in fact it can be impossible to find $\mathcal{A}$ for which this holds:

Example 6.1. Let $N = (a^2, ab, ac, b^2, bc) \subset k[a, b, c, d]$. Then N is compressed with respect to every proper subset of $\{a, b, c, d\}$ except $\{a, b, d\}$. Its $\{a, b, d\}$-compression is $T = (a^2, ab, ac, ad, bc)$, which is not $\{b, c\}$-compressed.

In [MP2] we show that compression with respect to the set $\{x_1, \cdots, x_{n-1}\}$ is well-behaved in the sense that it takes strongly stable ideals to strongly stable ideals. More research in this direction might prove productive.

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