Which Borel ideals are Gotzmann?

Costantini, Francisco, Mermin, Schweig

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Abstract: To be written later

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1 Introduction

To be written later

Why is Gotzmann relevant.

What is known about Gotzmann?

What is known about Borel—Gotzmann?

Compare our result to [BE], emphasizing the new approach. Restate their conjectures, and point out that they're all answered in the positive.

2 Background and notation

Let $S = k[x_1, \dots, x_n]$, and set $\mathfrak{m} = (x_1, \dots, x_n)$, the homogeneous maximal ideal.

In the examples, for simplicity of notation, we use variables $a, b, c, d, \dots$ instead of $x_1, x_2, \dots$. The letter d will thus sometimes stand for x_4 and sometimes for a positive integer degree. The meaning will be clear from context.

Definition 2.1. Fix a degree d. A *monomial space* of degree d is a k-subspace of S_d with a basis consisting of monomials. If M_d is a monomial space, we write $|M_d|$ for its dimension, which is equal to the number of monomials in M_d.

If I is a monomial ideal, then for all d we let I_d stand for the degree d component of I; this is a monomial space. If in addition I is equigenerated in degree one, it is natural to abuse notation by omitting the subscript in I_1. In particular, if $V = V_d$ is a monomial space, then $\mathfrak{m}V \subset S_{d+1}$ stands for the (finite-dimensional) k-vector space spanned by the degree $(d+1)$ monomials divisible by some monomial of V, rather than the ideal of S generated by all such monomials (which is infinite-dimensional as a k-space).

Definition 2.2. Let M_d be a monomial space. Then M_d is *Borel* if it satisfies the exchange condition

> Suppose $f \in S$ is such that $fx_j \in M_d$ and $i < j$. Then $fx_i \in M_d$ as well.

Definition 2.3. A monomial space L_d is *lex* if its monomial basis consists of an initial segment in the (graded) lex order.

Theorem 2.4 (Macaulay's theorem)**.** *For any degree-d monomial space M_d, there is a unique degree-d lex space L_d with $|L_d| = |M_d|$. Furthermore, $|\mathfrak{m}L_d| \leq |\mathfrak{m}M_d|$.*

Definition 2.5. If $|L_d| = |M_d|$, we say that L_d is the monomial space *associated to M_d*.

Theorem 2.6 (Macaulay's theorem, ideal version)**.** *Let I be a homogeneous ideal, and for each d let L_d be the monomial space associated to I_d. Put $L = \oplus L_d$. Then L is an ideal. In particular, there is a lex ideal with the same Hilbert function as I.*

Theorem 2.7 (Gotzmann's persistence theorem)**.** *Let G_d be a degree-d monomial space, and let L_d be the associated lex space. If $|\mathfrak{m}L_d| = |\mathfrak{m}G_d|$, then $|\mathfrak{m}^t L_d| = |\mathfrak{m}^t G_d|$ for all t.*

Definition 2.8. A monomial space G_d satisfying the assumptions of Gotzmann's persistence theorem is called *Gotzmann*.

Proposition 2.9 (Hilbert Basis Theorem)**.** *Let M_d be a monomial space. Then for all sufficiently large N, $\mathfrak{m}^N M_d$ is Gotzmann.*

Proof. Let L be the associated lex ideal. Since S is Noetherian, L is finitely generated. Let δ be the maximum degree among the generators of L, and choose N such that $N + d \geq \delta$. □

In view of Proposition 2.9, the question of whether a monomial space M_d is Gotzmann is subsumed by the question of which t make $\mathfrak{m}^t M_d$ Gotzmann. We will see that the second question is in many ways easier to answer.

Definition 2.10 (Bonanzinga-Eliahou, [?BE2, Definition 1.3])**.** Let B_d be equigenerated, and let N be minimal such that $\mathfrak{m}^N B_d$ is Gotzmann. Then we refer to N as the *Gotzmann threshold* of B_d.

3 Macaulay representations and growth of lex ideals

There is a standard decomposition of a lex space, used for induction in many contexts:

Lemma 3.1. *Let $L_d \subset S$ be a lex space. Then there exists an α such that L_d decomposes as a direct sum of k-vector spaces,*

$$L_d = x_1^{d-\alpha}\mathfrak{m}^\alpha \oplus x_1^{d-(\alpha+1)}(L'_{\alpha+1}),$$

where $L'_{\alpha+1}$ is a lex space inside $S^2 = k[x_2, \ldots, x_n]$.

Example 3.2. Let $L_5 \subset k[a, b, c]$ be the lex segment consisting of all degree 5 monomials greater than a^2b^2c in the lex order. Then L_5 has basis $\{a^5, a^4b, a^4c, a^3b^2, a^3bc, a^3c^2, a^2b^3, a^2b^2c\}$.

We may decompose L_5:

$$\begin{aligned} L_5 &= \left(a^5, a^4b, a^4c, a^3b^2, a^3bc, a^3c^2\right) \oplus \left(a^2b^3, a^2b^2c\right) \\ &= a^3\left(a, b, c\right)^2 \oplus a^2\left(b^3, b^2c\right). \end{aligned}$$

Here, (b^3, b^2c) is lex inside $k[b, c]$.

Proof of Lemma 3.1. We may take α to be maximal such that $x_1^{d-\alpha}x_n^{\alpha} \in L_d$. (This means that $L'_{\alpha+1}$ can have empty basis. It would also work to choose α minimal such that $x_1^{d-\alpha}x_n^{\alpha} \notin L_d$, but that would break down for our purposes later.) $\square$

Before inducting, we need some notation for the smaller rings.

Notation 3.3. For all i, let S^i and T^i be the subrings $S^i = k[x_i, \ldots, x_n]$ and $T^i = k[x_1, \ldots, x_i]$ of S. Let $\mathfrak{m}_i = (x_i, \ldots, x_n)$ be the homogeneous maximal ideal of S^i. Abusing notation as in Definition 2.1, we also use $\mathfrak{m}_i$ to represent the vector space spanned by the variables of S^i. This allows us to avoid the notational monstrosity $(\mathfrak{m}_i)_1$.

We obtain the following by a straightforward induction on the number of variables.

Proposition 3.4. *Let L_d be a lex space. Then we have the following:*

1. *There exists a weakly decreasing sequence $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_{n-1}$ and monomials $m_i \in T^i$ such that*

$$L_d = m_1\mathfrak{m}_1^{\alpha_1} \oplus m_2\mathfrak{m}_2^{\alpha_2} \oplus \cdots \oplus m_{n-1}\mathfrak{m}_{n-1}^{\alpha_{n-1}}$$

2. *With notation as above, we have*

$$\mathfrak{m}L_d = m_1\mathfrak{m}_1^{1+\alpha_1} \oplus m_2\mathfrak{m}_2^{1+\alpha_2} \oplus \cdots \oplus m_{n-1}\mathfrak{m}_{n-1}^{1+\alpha_{n-1}}$$

3. *We have*

$$|L_d| = \binom{\alpha_1+n-1}{n-1} + \binom{\alpha_2+n-2}{n-2} + \cdots + \binom{\alpha_{n-1}+1}{1}.$$

4. *We have*

$$|\mathfrak{m}L_d| = \binom{1+\alpha_1+n-1}{n-1} + \binom{1+\alpha_2+n-2}{n-2} + \cdots + \binom{1+\alpha_{n-1}+1}{1}.$$

Proof. Detailed proofs are unavoidably very notational; see [`?Buchanan`, Structure Theorem 4.13] for one such. The key idea is that, putting $\gamma_1 = d-\alpha-1$ and using Lemma 3.1 inductively, we produce nonnegative integers γ_i such that $m_1 = x_1^{1+\gamma_1}$, $m_2 = x_1^{\gamma_1}x_2^{1+\gamma_2}$, and in general $m_i = x_1^{\gamma_1}\ldots x_{i-1}^{\gamma_i}x_i^{1+\gamma_i}$. Setting $\alpha_i = d - \deg(m_i)$, we see that the α_i are weakly decreasing.

Then for (2), we may multiply the decomposition of Lemma 3.1 by $\mathfrak{m}$, yielding

$$\begin{aligned}
\mathfrak{m}L_d &= \mathfrak{m}x_1^{\gamma_1+1}\mathfrak{m}^{d-\gamma_1-1} + \mathfrak{m}(x_1^{\gamma_1}L'_{d-\gamma_1}) \\
&= x_1^{\gamma_1+1}\mathfrak{m}^{d-\gamma_1} + (\langle x_1\rangle + \mathfrak{m}_2)(x_1^{\gamma_1}L'_{d-\gamma_1}) \\
&= x_1^{\gamma_1+1}\mathfrak{m}^{d-\gamma_1} + (x_1^{\gamma_1+1}L'_{d-\gamma_1}) + \mathfrak{m}_2(x_1^{\gamma_1}L'_{d-\gamma_1})
\end{aligned}$$

The second summand is contained in the first, so this simplifies to

$$\mathfrak{m}L_d = x_1^{\gamma_1+1}\mathfrak{m}^{d-\gamma_1} + \mathfrak{m}_2(x_1^{\gamma_1}L'_{d-\gamma_1}).$$

The sum is direct, since the two summands have trivial intersection. By induction, setting $\mu_i = \frac{m_i}{x_1^{\gamma_1}}$, we have $L'_{d-\gamma_1} = \mu_2\mathfrak{m}_2^{\alpha_2} + \cdots + \mu_{n-1}\mathfrak{m}_{n-1}^{\alpha_{n-1}}$; the required decomposition follows.

(3) and (4) follow from (1) and (2) by taking the dimensions of each summand. □

Remark. Setting $c_i = \alpha_{n-i} + i$, statements (3) and (4) of Proposition 3.4 transform into the following statement:

Let I_d be a monomial space. There exists a decreasing sequence $c_{n-1} > c_{n-2} > \cdots > c_1$ such that

$$|I_d| = \binom{c_{n-1}}{n-1} + \binom{c_{n-2}}{n-2} + \cdots + \binom{c_1}{1}. \tag{3.1}$$

If we require that $c_j = -\infty$ whenever $c_j < j$, this expression is unique. Furthermore, under this convention we have

$$|\mathfrak{m}I_d| \geq \binom{1+c_{n-1}}{n-1} + \binom{1+c_{n-2}}{n-2} + \cdots + \binom{1+c_1}{1} \tag{3.2}$$

and, by induction,

$$|\mathfrak{m}^t I_d| \geq \binom{t+c_{n-1}}{n-1} + \binom{t+c_{n-2}}{n-2} + \cdots + \binom{t+c_1}{1},$$

with equality if I_d is lex (or, more precisely, if and only if I_d is Gotzmann).

A similar statement describing the Hilbert function growth of quotients is likely to be more familiar:

If I is a homogeneous ideal equigenerated in degree d, then there is a unique decreasing sequence $b_d > \cdots > b_1$ such that

$$|S/I|_d = \binom{b_d}{d} + \binom{b_{d-1}}{d-1} + \cdots + \binom{b_1}{1}; \tag{3.3}$$

we have

$$|S/I|_{d+1} \leq \binom{b_d+1}{d+1} + \binom{b_{d-1}+1}{d} + \cdots + \binom{b_1+1}{2} \tag{3.4}$$

and, by induction,

$$|S/I|_{d+t} \leq \binom{b_d+t}{d+t} + \binom{b_{d-1}+t}{d-1+t} + \cdots + \binom{b_1+t}{1+t},$$

with equality if and only if I is Gotzmann.

The expressions on the right-hand-sides in (3.1) and (3.3) are called the $(n-1)^{\text{th}}$ *Macaulay representation* of $|I|_d$ and the d^{th} *Macaulay representation* of $|S/I|_d$, respectively. In both cases, the numerators are called the *Macaulay coefficients*. The "increment-the-numerators" operation used to obtain (3.2) from (3.1) is not quite the same as the "increment-the-numerators-and-denominators" operation used to obtain (3.4) from (3.3). There are substantial analogies between the two theories; see [`?Buchanan`] for details and for a more constructive approach to finding the Macaulay coefficients.

The Hilbert function growth of a lex ideal is governed by its Macaulay coefficients. For the purposes of this paper, we find it convenient to work with the a_i instead of the c_{n-i}; we gather the information into a bivariate generating function below.

Definition 3.5. Let L_d be a lex space, and write $|L_d| = \sum_{i=1}^{r} \binom{\alpha_i+n-i}{n-i}$ (here r is maximal such that α_r is positive). The *Macaulay polynomial* of L_d is

$$\mathrm{MP}(L_d) = \sum t^i s^{\alpha_i}$$

4 The Bigatti coefficients and growth of Borel ideals

Comment to the effect that every lex ideal is borel but not conversely, and we want to understand how borel ideals grow too.

Notation 4.1. Let m be a monomial. Denote by $\max(m)$ the index of the last variable appearing in m, $\max(m) = \max\{i : x_i \text{ divides } m\}$.

Notation 4.2. Let B_d be a Borel space. For each i, let $w_i = w_i(B_d)$ count the number of generators whose last variable is x_i,

$$w_i(B_d) = |m \in B_d : \max(m) = i|.$$

Since every $\mu \in \mathfrak{m}B_d$ may be written as $\max(\mu)m$ with $m \in B_d$ and $\max(m) \leq \max(\mu)$, the set of w_i control the Hilbert function growth of B_d.

Proposition 4.3. *Let B_d be a Borel space. Then:*

1. $|B_d| = \sum_{i=1}^{n} w_i(B_d)$.
2. $|\mathfrak{m}B_d| = \sum_{i=1}^{n} (n-i+1) w_i(B_d)$.
3. $|\mathfrak{m}^t B_d| = \sum_{i=1}^{n} \binom{n-i+t}{t} w_i(B_d)$.
4. *For all j, we have* $w_j(\mathfrak{m}B_d) = \sum_{i=1}^{j} w_i(B_d)$.

Proof. Statement (1) is immediate from the definition.

For (2), observe that if $\mu \in \mathfrak{m}B_d$, then $\dfrac{\mu}{x_j} \in B_d$ for some x_j dividing μ. But, since B_d is Borel, we have $\dfrac{\mu}{x_{\max(\mu)}} = \dfrac{\mu}{x_j}\dfrac{x_j}{x_{\max(\mu)}} \in B_d$ as well. Thus, taking

$\ell = \max(\mu)$, μ may be written uniquely in the form $\mu = mx_\ell$ with $m \in B_d$ and $\max(m) \le \ell$. Conversely, every such expression is a monomial in $\mathfrak{m}B_d$.

For (3), a similar analysis allows us to decompose every element of $\mathfrak{m}^t B_d$ uniquely into a "beginning" and an "end", $\mu = \mathrm{beg}(\mu)\mathrm{end}(\mu)$, where $\mathrm{beg}(\mu) \in B_d$ and $\mathrm{end}(\mu) \in \mathfrak{m}_{\max(\mathrm{beg}(\mu))}$, resulting in a Stanley decomposition

$$\mathfrak{m}^t B_d = \bigoplus_{i=1}^{n} \left(\bigoplus_{m \in B_d, \max(m)=i} k[x_{\max(m)}, \ldots, x_n] \right).$$

(The beginning and end of monomials are useful in studying the Eliahou-Kervaire resolution of a Borel ideal; see [?PS].)

For (4), observe that the "multiplication by x_j" map is a bijection between the monomials $m \in B_d$ with $\max(m) \le j$ and the monomials $\mu \in \mathfrak{m}B_d$ with $\max(\mu) = j$. $\square$

Do we ever use the notation S^i? The ring in this Stanley decomposition is a place for it.

Definition 4.4. Let B_d be a Borel space. We refer to the $w_i(B_d)$ as the *Bigatti coefficients* of B_d, and gather them into a generating function.

The *Bigatti polynomial* of B_d is $\mathrm{BP}(B_d) = \sum_{i=1}^n w_i(B_d)t^i$.

5 Translating between the Macaulay and Bigatti polynomials

Every lex space is also Borel, so we'd like to be able to find the Bigatti polynomial of a lex space from its Macaulay polynomial. We accomplish this by translating the terms of the Macaulay polynomial back to the decomposition of Proposition 3.4.

Theorem 5.1. *Let L_d be a lex space with Macaulay polynomial $\mathrm{MP}_{L_d}(t,s)$. Then the Bigatti polynomial $BP_{L_d}(t)$ is obtained by truncating $\mathrm{MP}_{L_d}(t, \frac{1}{1-t})$ after the t^n term.*

Proof. From the Macaulay polynomial $\mathrm{MP}_{L_d}(t,s) = \sum_i t^i s^{\alpha_i}$ we can read off the Macaulay decomposition

$$|L_d| = \sum_i \binom{\alpha_i - n - i}{n - i}.$$

Each summand $\binom{\alpha_i+n-i}{n-i}$ corresponds to the summand $m_i \mathfrak{m}_i^{\alpha_i}$ in the decomposition of Proposition 3.4 (1). Since $m_i \in T^i$ and $\mathfrak{m}_i^{\alpha_i} \subset k[x_i, \ldots, x_n]$, we have $\max(m_i f) = \max(f)$ for all monomials $f \in \mathfrak{m}_i^{\alpha_i}$. Thus for all $j \in \{i, \ldots, n\}$ we have $w_j(m_i \mathfrak{m}_i^{\alpha_i}) = w_j(\mathfrak{m}_i^{\alpha_i}) = \left|(x_i, \ldots, x_j)^{\alpha_i - 1} x_j\right| = \binom{j-i+\alpha_i-1}{j-i}$.

In particular, the term $t^i s^{\alpha_i}$ in MP_{L_d} contributes

$$\sum_{j=i}^{j=n} \binom{j-i+\alpha_i-1}{j-i} t^j = \sum_{\ell=0}^{\ell=n-i} \binom{\ell+\alpha_i-1}{\ell} t^{\ell+i} = t^i \sum_{\ell=0}^{\ell=n-i} \binom{\ell+\alpha_i-1}{\ell} t^\ell$$

to BP_{L_d}. This expression is the truncation, after the t^n term, of $t^i \left(\dfrac{1}{1-t}\right)^{\alpha_i}$ (Taylor's version). We conclude that $\mathrm{BP}_{L_d} = \mathrm{MP}_{L_d}(t, \frac{1}{1-t})$, truncated after the t^n term, as desired.

$\square$

In principle this allows us to identify precisely which Borel ideals are Gotzmann. The following result is a key ingredient in proving a generalized version of Green's Hyperplane Restriction Theorem ([?MP2, Theorem 2.15]): **BUT SHOULD FIND IT IN PEEVA'S BOOK OR IN ONE OF GASHAROV'S PAPERS**

Theorem 5.2. *Let B_d be a Borel space. B_d is Gotzmann if and only if there is a lex space L_d such that $w_i(B_d) = w_i(L_d)$ for all i.*

Proof. Let $\mathcal{L}_d$ be the lex monomial space with $|\mathcal{L}_d| = |B_d|$, and for all i put $\tau_i(B_d) = w_1(B_d) + \cdots + w_i(B_d)$ (and similarly for $\tau_i(\mathcal{L}_d)$). By [?Peeva, Comparison Theorem 44.4] we have $\tau_i(\mathcal{L}_d) \le \tau_i(B_d)$ for all i.

By Proposition 4.3(1 and 4), we have $|\mathfrak{m}B_d| = \sum \tau_i(B_d)$, and similarly $|\mathfrak{m}\mathcal{L}_d| = \sum \tau_i(\mathcal{L}_d)$. We conclude that $|\mathfrak{m}\mathcal{L}_d| \le |\mathfrak{m}B_d|$. B_d is Gotzmann if and only if equality holds, if and only if $\tau_i(\mathcal{L}_d) = \tau_i(B_d)$ for all i, if and only if $w_i(L_d) = w_i(B_d)$ for all i. $\square$

Since any weakly decreasing sequence of α_i yields a lex space, we get the following: **FIND A SENTENCE WHICH DOESN'T RELY ON THE PROOF OF THE LEMMA** The criterion suggested by Theorem 5.2 is impractical, as one needs to check the equalities $w_i(B_d) = w_i(L_d)$ for all candidates L. A more effective criterion is the following.

Proposition 5.3. *Fix n, and let $B_d \subset k[x_1, \ldots, x_n]_d$ be a Borel space. Then B_d is Gotzmann if and only if there exist α_i satisfying $d \ge \alpha_1 \ge \alpha_2 \ge \cdots \ge \alpha_{n-1}$ such that*

1. *All α_i are nonnegative or equal to $-\infty$.*
2. $\mathrm{BP}(B_d)$ *is equal to the truncation, after the t^n term, of $\sum_{\alpha_i \ne -\infty} \frac{t^i}{(1-t)^{\alpha_i}}$.*

Furthermore, if B_d is Gotzmann, then these α_i are unique.

Proof. If B_d is Gotzmann, then there exists a lex space $L_d \subset S_d$ with $\mathrm{BP}_{L_d} = \mathrm{BP}_{B_d}$. Write $\mathrm{MP}_{L_d} = \sum_{i=1}^{r} t^i s^{\alpha_i}$, and set $\alpha_i = -\infty$ for all $i > r$. Then all α_i are nonnegative (if $i \le r$) or equal to $-\infty$ by construction. Since L_d is the unique lex space in S_d with $|L_d| = |B_d|$, such α_i are unique.

Conversely, suppose such α_i exist. If $\alpha_1 = d$, observe that $\mathrm{BP}(S_d) = \frac{t}{(1-t)^d}$, so $B_d = S_d$ is lex and we have $\alpha_i = -\infty$ for all $i > 1$. Otherwise, set $r = \max\{i : \alpha_i \neq -\infty\}$ and put

$$L_d = x_1^{d-\alpha_1}\mathfrak{m}_1^{\alpha_1} \oplus x_1^{d-\alpha_1-1}x_2^{\alpha_1-\alpha_2+1}\mathfrak{m}_2^{\alpha_2} \oplus \dots$$
$$\oplus x_1^{d-\alpha_1-1}x_2^{\alpha_1-\alpha_2}\dots x_{r-1}^{\alpha_{r-2}-\alpha_{r-1}}x_r^{\alpha_{r-1}-\alpha_r+1}\mathfrak{m}_r^{\alpha_r}.$$

Observe that L_d has basis consisting of all monomials greater than or equal to $x_1^{d-\alpha_1-1}x_2^{\alpha_1-\alpha_2}\dots x_{r-1}^{\alpha_{r-2}-\alpha_{r-1}}x_r^{\alpha_{r-1}-\alpha_r+1}x_n^{\alpha_r}$ in the lex order, so in particular L_d is lex. Since $\mathrm{MP}_{L_d} = \sum_{i=1}^{r} t^i s^{\alpha_i}$, we have $\mathrm{BP}_{L_d} = \mathrm{BP}_{B_d}$ by Theorem 5.1, so B_d is Gotzmann as desired.

CAN WE FIND A WAY TO CLEAN THIS UP WITH A CITATION TO BUCHANAN? □

In practice, it is far from obvious whether or not such α_i exist. Luckily, the greedy algorithm works.

Algorithm 5.4. *This algorithm takes as input a polynomial $BP \in t\mathbb{Z}[t]$ with degree at most n and linear term $1t$, and outputs a sequence $\boldsymbol{\alpha}(BP) = \{\alpha_1, \dots, \alpha_{n-1}\} \in \mathbb{Z} \cup \{-\infty\}$ such that BP is equal to the truncation, after the t^n term, of $\sum \frac{t^i}{(1-t)^{\alpha_i}}$.*

Initialization Step: *Set all α_i equal to $-\infty$ and set the working polynomial* WP *equal to* BP.

Sanity Check: *Observe that the leading term of* WP *is t^p for some p and that α_i are still equal to $-\infty$ for all $i \geq p$. (If not, FAIL.) Then attempt the conclusion step.*

Conclusion Step: *If* WP *is equal to (the truncation of) $\frac{t^p}{(1-t)^q}$ for some (not necessarily positive) q, set α_p equal to q and exit. Otherwise, move to the recursive step.*

Recursive Step: WP *has the form* $\mathrm{WP}(t) = t^p + qt^{p+1} +$ *(higher order terms). Set $\alpha_p = q - 1$ and replace* WP *with $WP(t) - \frac{t^p}{(1-t)^{q-1}}$. Truncate after the t^n term if necessary, then return to the Sanity Check.*

GO THROUGH THIS AND USE THE NOTATION $\boldsymbol{\alpha}(\mathrm{BP}_{B_d})$ THROUGHOUT

Theorem 5.5. *Algorithm 5.4 detects whether B_d is Gotzmann. That is, if $\boldsymbol{\alpha}(\mathrm{BP}_{B_d})$ is a weakly decreasing sequence of nonnegative integers (or $-\infty$), then B_d is Gotzmann. If not, then B_d is not Gotzmann.*

Proof. First, we show that, if the input is BP_{B_d} for a Borel space $B_d \subset k[x_1, \dots, x_n]$, Algorithm 5.4 terminates without FAILing, after at most $n-2$ iterations of the recursive step. In fact, we claim that, whenever we visit the Sanity Check, the required p is one more than the number of times the Recursive Step has already been executed. The proof is by induction on this number. After the Initialization Step, we have $\mathrm{WP} = \mathrm{BP}_{B_d} = 1t +$ (higher order terms) since $w_1(B) = 1$ for all nontrivial Borel spaces B. Inductively, after $p-1$ iterations of the Recursive Step, we have $\mathrm{WP} = t^p + qt^{p+1} + (\text{higher order terms})$. If we iterate the Recursive Step a p^{th} time, we subtract $\frac{t^p}{(1-t)^{q-1}} = t^p + (q-1)t^{p+1} + (\text{higher order terms})$, yielding a new Working Polynomial of $t^{p+1} + (\text{higher order terms})$. Now observe that, after $n-2$ iterations of the Recursive Step, we have $p = n-1$ and, since we always truncate after the t^n term, $\mathrm{WP} = t^{n-1} + qt^n$, which is the truncation of $\dfrac{t^p}{(1-t)^{-q}}$. Thus the Conclusion Step sets $a_{n-1} = -q$ and exits without an $(n-1)^{\text{th}}$ iteration of the Recursive Step.

Second, observe that the output of Algorithm 5.4 is a sequence of integers (not necessarily nonnegative or decreasing) α_i such that the input polynomial is equal to $\displaystyle\sum_{i=1}^{r} \frac{t^i}{(1-t)^{\alpha_i}}$. If the input is in fact BP_{B_d} for a Borel space, observe that $w_2(B_d) \leq d$ (since the only monomials with $\max(m) = 2$ are $m = x_1^{\alpha} x_2^{d-\alpha}$ with $\alpha \in \{0, \ldots, d-1\}$). Since α_1 is equal either to $w_2(B_d)$ (if the Conclusion Step holds immediately) or to $w_2(B_d) - 1$ (if not), we also have $d \geq \alpha_1$. Thus, if the sequence of α_i is weakly decreasing, we may conclude by Proposition 5.3 that B_d is Gotzmann.

Conversely, if B_d is Gotzmann, then there exists a lex ideal L_d with $\mathrm{BP}_{L_d} = \mathrm{BP}_{B_d}$. We claim that Algorithm 5.4 finds the Macaulay coefficients for L_d. Indeed, write $\mathrm{MP}_{L_d} = \sum_{i=1}^{i=r} t^i s^{\alpha_i}$, so $\mathrm{BP}_{B_d} = \mathrm{BP}_{L_d}$ is the truncation of $\displaystyle\sum_{i=1}^{i=r} t^i \left(\frac{1}{1-t}\right)^{\alpha_i}$. Rearranging terms, we have

$$\begin{aligned}
\mathrm{BP}_{L_d} &= \sum_{i=1}^{r} \left(\binom{\alpha_i}{\alpha_i} t^i + \binom{\alpha_i + 1}{\alpha_i} t^{i+1} + \cdots + \binom{\alpha_i + n - i}{\alpha_i} t^n \right) \\
&= \sum_{j=1}^{j=n} \left(\sum_{i=1}^{i=\min(j,r)} \binom{\alpha_i + j - i}{\alpha_i} \right) t^j
\end{aligned}$$

CHECK THIS STUFF FOR OFF-BY-ONES

Inductively, after executing the Recursive Step $(p-1) < (r-2)$ times, we get $\mathrm{WP} = \binom{\alpha_p}{\alpha_p} t^p + \left[\binom{\alpha_p+1}{\alpha_p} + \binom{\alpha_{p+1}}{\alpha_{p+1}}\right] t^{p+1} + \left[\binom{\alpha_p+2}{\alpha_p} + \binom{\alpha_{p+1}+1}{\alpha_{p+1}} + \binom{\alpha_{p+2}}{\alpha_{p+2}}\right] t^{p+2} +$ (higher order terms), so the Conclusion Step fails and the p^{th} iteration of the Recursive Step identifies α_p correctly. After executing the Recursive Step $(p-1) = (r-2)$ times, we are left with $\mathrm{WP} = \binom{\alpha_p}{\alpha_p} t^p + \left[\binom{\alpha_p+1}{\alpha_p} + \binom{\alpha_{p+1}}{\alpha_{p+1}}\right] t^{p+1} + \left[\binom{\alpha_p+2}{\alpha_p} + \binom{\alpha_{p+1}+1}{\alpha_{p+1}}\right] t^{p+2} +$ (higher order terms), so the Conclusion Step fails and the Recursive Step identifies α_{r-1} correctly. Finally, after executing the Recursive Step $(r-1)$ times, we have $\mathrm{WP} = \sum_{i=r}^{n} \binom{\alpha_r + i - r}{\alpha_r} t^i$, which is the truncation of $\dfrac{t^r}{(1-t)^{\alpha_r}}$, so the Conclusion Step successfully identifies α_r and exits.

$\square$

Thus we can use Algorithm 5.4 to detect whether a Borel ideal B is Gotzmann. We also show in Corollary 5.9 that, whenever B is not Gotzmann, Algorithm 5.4 determines its Gotzmann threshold.

Proposition 5.6. *Let B be a Borel ideal. Then $\mathrm{BP}_{\mathfrak{m}B}(t)$ is equal to the truncation, after the t^n term, of $\frac{1}{1-t}\,\mathrm{BP}_B(t)$.*

Proof. It suffices to show that, for all j, $w_j(\mathfrak{m}B) = \sum_{i \le j} w_i(B)$. This statement is Proposition 4.3(4). $\square$

Proposition 5.7. *Multiplying a Borel ideal B by $\mathfrak{m}$ increments the Macaulay coefficients output by Algorithm 5.4. That is, suppose that $\boldsymbol{\alpha}(\mathrm{BP}_B) = \{\alpha_1, \ldots, \alpha_{n-1}\}$. Then $\boldsymbol{\alpha}(\mathrm{BP}_{\mathfrak{m}B}) = \{\alpha_1 + 1, \ldots, \alpha_{n-1} + 1\}$*

Proof. We have $\sum_{\alpha_i \neq -\infty} t^i \left(\frac{1}{1-t}\right)^{\alpha_i} = \mathrm{BP}_B$. Thus $\sum_{\alpha_i \neq -\infty} t^i \left(\frac{1}{1-t}\right)^{\alpha_i+1} = \frac{1}{1-t}\,\mathrm{BP}_B = \mathrm{BP}_{\mathfrak{m}B}$, as required. $\square$

Corollary 5.8. *Let B_d be Borel. The output of Algorithm 5.4, given an input of BP_{B_d}, is a weakly decreasing sequence $\{\alpha_1 \ge \alpha_2 \ge \cdots \ge \alpha_{n-1}\}$. Furthermore, $\alpha_1 \le d$.*

Proof. By the Hilbert Basis Theorem, we may choose N sufficiently large so that $\mathfrak{m}^N B_d$ is Gotzmann. Then, applying Algorithm 5.4 to $\mathfrak{m}^N B_d$, the output is $\{\alpha_1 + N, \alpha_2 + N, \ldots, \alpha_{n-1} + N\}$. By Theorem 5.5, we have $d + N \ge \alpha_1 + N \ge \alpha_2 + N \ge \cdots \ge \alpha_{n-1} + N$. $\square$

Corollary 5.9. *Algorithm 5.4 detects the Gotzmann threshold. To be precise, let B_d be given, and suppose that the output of Algorithm 5.4, given input BP_{B_d}, is $\{\alpha_1, \ldots, \alpha_{n-1}\}$. Let r be maximal such that $\alpha_r \neq -\infty$. If $\alpha_r \ge 0$, then B_d is Gotzmann. If $\alpha_r \le 0$, then the Gotzmann threshold of B_d is equal to $-\alpha_r$.*

Proof. If $N \ge -\alpha_r$, then we have $d + N \ge \alpha_1 + N \ge \cdots \ge \alpha_r + N \ge 0$, so $\mathfrak{m}^N B_d$ is Gotzmann. If not, then $\alpha_r + N < 0$. $\square$

Remark. The Gotzmann threshold depends on the choice of ring, as well as the list of generators for B_d. In general, the index r in Corollary 5.9 will be equal to $n-1$; adding an extra variable x_{n+1} to the ring will change r to n, and α_n will be considerably more negative than α_{n-1}.

6 Examples

GONNA HAVE TO COMPLETELY REWRITE THIS SECTION IN LIGHT OF THE NEW SECTION 7

Example 6.1. In this example, we use Algorithm 5.4 to show that the principal Borel ideal B generated by abc is not Gotzmann in $k[a, b, c, d]$.

This ideal is generated by $(a^3, a^2b, a^2c, ab^2, abc)$, so we have $w_1 = 1$, $w_2 = w_3 = 2$, and $w_4 = 0$.

We have $\mathrm{BP}(t) = t + 2t^2 + 2t^3$.

Since the coefficient on the t^2 term is $2 = 1 + 1$, we set $\alpha_1 = 1$ and subtract the truncation of $\frac{t}{(1-t)^1} = t + t^2 + t^3 + t^4$, yielding a new working polynomial of $t^2 + t^3 - t^4$.

Since the coefficient on the t^3 term is $1 = 0 + 1$, we set $\alpha_2 = 0$ and subtract the truncation of $\frac{t^2}{(1-t)^0} = t^2$, yielding a new working polynomial of $t^3 - t^4$.

Since $t^3 - t^4$ is equal to $\frac{t^3}{(1-t)^{-1}}$, we set $\alpha_3 = -1$.

Thus the Macaulay coefficients for the hypothetical 3-generated lex ideal corresponding to B are 1, 0, and -1. Since the last of these is negative, such an ideal does not exist. Thus B is not Gotzmann.

(The actual lex ideal corresponding to B contains a^2d instead of abc in degree 3, and it has an extra generator, abc^2, in degree four.)

Example 6.2. In this example, we use Algorithm **??** to show that the principal Borel ideal B generated by $abcd$ is Gotzmann in $k[a, b, c, d]$.

We may compute $w_1 = 1$, $w_2 = 3$, and $w_3 = w_4 = 5$.

We have $\mathrm{BP}(t) = t + 3t^2 + 5t^3 + 5t^4$.

Since the coefficient on the t^2 term is $3 = 2 + 1$, we set $\alpha_1 = 2$ and subtract the truncation of $\frac{t}{(1-t)^2} = t + 2t^2 + 3t^3 + 4t^4$, yielding a new working polynomial of $t^2 + 2t^3 + t^4$.

Since the coefficient on the t^3 term is $2 = 1 + 1$, we set $\alpha_2 = 1$ and subtract the truncation of $\frac{t^2}{(1-t)^1} = t^2 + t^3 + t^4$, yielding a new working polynomial of t^3.

Since t^3 is equal to $\frac{t^3}{(1-t)^0}$, we set $\alpha_3 = 0$.

Thus the Macaulay coefficients for the hypothetical 3-generated lex ideal corresponding to B are 2, 1, and 0. Since none of these is negative, such an ideal does exist. Thus B is Gotzmann.

(It is not a coincidence that the Macaulay coefficients in Example 6.2 are all augmented by 1 from those in Example 6.1. In general, multiplying a Borel ideal by $\mathfrak{m}^r$ will augment all Macaulay coefficients by r, or, equivalently, multiply the Macaulay polynomial by s^r, as the next example illustrates.)

Example 6.3. In this example, we determine how large r has to be to make the principal Borel ideal B generated by $a^4b^6c^9d^r$ Gotzmann.

Using the techniques of [?FMS], we compute $w_1 = \binom{r-1}{0}$, $w_2 = 15\binom{r-1}{0} + 1\binom{r}{1}$, $w_3 = 99\binom{r-1}{0} + 15\binom{r}{1} + 1\binom{r+1}{2}$, and $w_4 = 99\binom{r}{1} + 15\binom{r+1}{2} + 1\binom{r+2}{3}$.

The Bigatti polynomial is

$$\mathrm{BP}(t) = \binom{r-1}{0}t + \left(15\binom{r-1}{0} + 1\binom{r}{1}\right)t^2 + \left(99\binom{r-1}{0} + 15\binom{r}{1} + 1\binom{r+1}{2}\right)t^3 + \left(99\binom{r}{1} + 15\binom{r+1}{2} +$$

which simplifies in Mathematica to

$$\mathrm{BP}(t) = t + (15+r)t^2 + \left(99 + \frac{31}{2}r + \frac{1}{2}r^2\right)t^3 + \left(\frac{641}{6}r + 8r^2 + \frac{1}{6}r^3\right)t^4.$$

Setting $\alpha_1 = 14 + r$, we subtract the truncation of $\frac{t}{(1-t)^{14+r}}$, which is $t + (14+r)t^2 + \left(105 + \frac{29}{2}r + \frac{1}{2}r^2\right)t^3 + \left(560 + \frac{337}{3}r + \frac{15}{2}r^2 + \frac{1}{6}r^3\right)t^4$. When the dust clears, the new working polynomial is

$$t^2 + (-6+r)t^3 + \left(-560 - \frac{11}{2}r + \frac{1}{2}r^2\right)t^4.$$

We now set $\alpha_2 = -7 + r$, and subtract the truncation of $\frac{t^2}{(1-t)^{-7+r}}$, which is $t^2 + (-7+r)t^3 + \left(42 - \frac{13}{2}r + \frac{1}{2}r^2\right)t^4$.

The new working polynomial is

$$t^3 + (-602 + r)t^4.$$

We set $\alpha_3 = 602 + r$.

For B to be Gotzmann, all three of these coefficients must be nonnegative. That is, we require $\alpha_1 = 14 + r$, $\alpha_2 = -7 + r$, and $\alpha_3 = -602 + r$ to all be nonnegative. We conclude that B is Gotzmann if and only if $r \geq 602$.

DO WE EVEN WANT THIS EXAMPLE ANY MORE? OR IF SO, DO WE WANT TO MOVE IT TO THE NEXT SECTION?

7 Principal Borel ideals

The primary obstacle to using Algorithm 5.4 is the determination of the $w_i(B_d)$ and the Bigatti polynomial BP_{B_d}. In general, given a minimal monomial generating set for B_d, this information can be read off from the generators; unfortunately, such generating sets tend to be prohibitively large. The class of *principal Borel ideals* is easier to work with.

Definition 7.1. Let $m \in S_d$ be a monomial. Then Borel(m), the *principal Borel space* generated by m is the smallest Borel space containing m. (If we write $m = x_{i_1} \dots x_{i_d}$ with $i_1 \leq \cdots \leq i_d$, then Borel(m) is spanned by the monomials $x_{j_1} \dots x_{j_d}$ such that $j_\ell \leq i_\ell$ for all ℓ.)

In [?BE1, ?BE2], Bonanzinga and Eliahou make a detailed study of the "gaps" in principal Borel ideals (loosely speaking, the monomials that are missing from Borel(m) but lexicographically greater than m), and prove a number of results relating the Gotzmann threshold to the exponent vector of the Borel generator. Specifically, they compute in [?BE1, Theorem 7.7] an explicit formula for the Gotzmann threshold for an arbitrary principal Borel ideal in $k[x_1, \dots, x_4]$, and in [?BE2, Theorem 4.15] describe an efficient procedure for computing the Gotzmann threshold of an arbitrary monomial.

They also make several conjectures, which we prove below.

Given a principal Borel ideal, the Bigatti polynomial can be determined efficiently from the exponent vector of the principal generator. The following is a translation of [?FMS, §5] to our setting.

Theorem 7.2 ([?FMS], 5.2–5.4)**.** *Suppose* $B_d = \text{Borel}(m)$*, and write* $m = x_1^{e_1} x_2^{e_2} \dots x_n^{e_n}$*. The* Catalan diagram of shape m *is a left-justified staircase diagram having* e_i *rows of length* i *for all* i*. If one fills in the boxes as with Catalan's Triangle (place a* 1 *in the top left box, then fill each remaining box with the sum of the numbers in the boxes immediately above it and immediately to its left), then* $w_i(B_d)$ *is equal to the entry in the* i^{th} *box of the bottom row.*

1				
1	1			
1	2	2		
1	3	5	5	
1	4	9	14	14

1				
1	1			
1	2			
1	3	3		
1	4	7	7	7

Figure 1: The Catalan diagrams $C(x_1x_2x_3x_4x_5)$ and $C(x_1x_2^2x_3x_5)$, respectively.

In principle, we can use these Catalan diagrams to compute $\text{BP}_B(t)$, and use it as input to Algorithm 5.4, to determine precisely which principal Borel ideals are Gotzmann. In practice, the computations become prohibitively difficult to do by hand as the number of variables gets large. However, we can use the structure of the Catalan diagram to bypass Algorithm 5.4 entirely.

Construction 7.3. Let m be a monomial of degree d, and extend the Catalan diagram of m upward and downward indefinitely, creating an infinite array Ω_m with entries $\omega_{i,j}$, such that $\omega_{i,d} = w_{i,d}(\text{Borel}(m))$ for all i; $\omega_{1,j} = 1$ for all j, and $\omega_{i,j} = \omega_{i-1,j} + \omega_{i,j-1}$ for all $i \geq 1$ and all j. We call Ω_m the *extended Catalan diagram* of m.

Example 7.4.

Lemma 7.5. *Let* $m = x_1^{e_1} x_2^{e_2} \dots x_n^{e_n}$ *be a monomial, and let* Ω_m *be its extended Catalan diagram. Then* $\omega_{2,e_1} = \omega_{3,e_1+e_2} = \cdots = \omega_{n,e_1+e_2+\cdots+e_{n-1}} = 0$.

The extended Catalan diagrams will allow us to rewrite the Bigatti coefficients of Borel(m) using discrete calculus. To this end, we recall the relevant elementary definitions and results of discrete calculus. **FIND A CITATION**

CONSIDER MOVING TO SECTION 2

Theorem 7.6. *Let* $f : \mathbb{Z} \to \mathbb{R}$ *be a function defined on the integers. Then*

1. *The* discrete derivative *of* f *is* $\Delta f : \mathbb{Z} \to \mathbb{R}$*, defined by* $\Delta f(n) = f(n) - f(n-1)$.

2. *For integers $a < b$, the* discrete definite integral *of f from a to b is* $\mathcal{I}_a^b(f\Delta n) = \displaystyle\sum_{n=a}^{n=b} f(n)$.

3. *The* discrete indefinite integral *of f is the family of functions $\mathcal{I}(f\Delta n)$ whose discrete derivatives are equal to f; for any one such discrete antiderivative F, we may write $\mathcal{I}(f\Delta n) = F + C$.*

4. *Every discrete initial value problem*
$$\begin{cases} \Delta f = g \\ f(a) = b \end{cases}$$
has a unique solution, namely
$$f(m) = \begin{cases} b & \text{if} \quad m = a \\ b + \mathcal{I}_{a+1}^{m}(g\Delta n) & \text{if} \quad m > a \\ b - \mathcal{I}_{m}^{a-1}(g\Delta n) & \text{if} \quad m < a \end{cases}$$

5. *The discrete fundamental theorem of calculus states that $\Delta\mathcal{I}(f) = f$ for all f.*

6. *The discrete power rule states that, for any positive integer i, we have*
$$\mathcal{I}\left(\binom{n+k}{i}\Delta n\right) = \binom{n+k+1}{i+1} + C.$$

7. *The discrete sum rule states that, for any f and g, we have $\mathcal{I}((f+g)\Delta n) = \mathcal{I}(f\Delta n) + \mathcal{I}(g\Delta n)$.*

Proof. Should we "prove" (6)? (4)? □

Proposition 7.7. *Let $m = x_1^{e_1}x_2^{e_2}\ldots x_n^{e_n}$ be a monomial of degree d, and let Ω_m be its extended Catalan diagram. Define functions $W_{i,m} : \mathbb{Z} \to \mathbb{Z}$ by the rule $W_{i,m}(j) = \omega_{i,j}$ for all j. Then we have the following:*

1. *For all i, $W_{i,m}(d) = w_i(\mathrm{Borel}(m))$.*

2. *For all i and all $t \geq 0$, $W_{i,m}(d+t) = w_i(\mathfrak{m}^t \,\mathrm{Borel}(m)) = w_i((\mathrm{Borel}(m))_{d+t})$*

3. *$W_{1,m} = 1$. For all other i, $W_{i,m}$ is the solution to the discrete initial value problem*
$$\begin{cases} \Delta W_{i,m} = W_{i-1,m} \\ W_{i,m}(e_1 + e_2 + \cdots + e_{i-1}) = 0 \end{cases}$$

Proof. □

For the rest of the section, fix a monomial $m = x_1^{e_1}\ldots x_n^{e_n} \in k[x_1,\ldots,x_n]$ of degree $d = e_1 + \cdots + e_n$. Let Ω_m be the extended Catalan diagram of m, and adopt the notation from Construction 7.3. We begin by solving the first three discrete initial value problems.

Proposition 7.8. *With notation as above, adopt the convention that, when $q < k$, the binomial coefficient $\binom{q}{k}$ is equal to $\frac{(q)(q-1)\ldots(q-k+1)}{k!}$. We have the following:*

1. $W_{2,m}(r) = \binom{r - e_1}{1}$.

2. $W_{3,m}(r) = \binom{r - e_1}{2} + \binom{r - e_1 - e_2 - \binom{e_2}{2}}{1}$.

3. $W_{4,m}(r) = \binom{r - e_1 + 1}{3} + \binom{r - e_1 - e_2 - \binom{e_2}{2}}{2} + \binom{r - e_1 - e_2 - e_3 - \binom{e_3 - \binom{e_2}{2}}{2} - \binom{e_2+e_3+1}{3}}{1}$.

Proof. **WE CAN ASSUME $e_1 = 0$ WLOG. DOES IT GAIN US ANYTHING?**

1. Since $W_{1,m}(r) = 1 = \binom{r}{0}$, we have $W_{1,m}(r) = \binom{r+1}{1} + C = r + 1$ by the discrete power rule. Since $W_{2,m}(e_1) = 0$, we have $e_1 + 1 + C = 0$, i.e., $C = -e_1 - 1$. Thus $W_{2,m}(r) = r + 1 - e_1 - 1 = r - e_1 = \binom{r-e_1}{1}$ as desired.

2. Applying the discrete power rule to (1), we get $W_{3,m}(r) = \binom{r-e_1+1}{2} + C$. Since $W_{3,m}(e_1 + e_2) = 0$, we compute $\binom{e_1+e_2-e_1+1}{2} + C = 0$, i.e., $C = -\binom{e_2+1}{2}$. Thus

$$\begin{aligned} W_{3,m}(r) &= \binom{r - e_1 + 1}{2} - \binom{e_2 + 1}{2} \\ &= \left[\binom{r - e_1}{2} + \binom{r - e_1}{1}\right] - \left[\binom{e_2}{2} + \binom{e_2}{1}\right] \\ &= \binom{r - e_1}{2} + r - e_1 - e_2 - \binom{e_2}{2} \\ &= \binom{r - e_1}{2} + \binom{r - e_1 - e_2 - \binom{e_2}{2}}{1} \end{aligned}$$

   as desired.

3. Applying the discrete power rule to (2), we get $W_{4,m}(r) = \binom{r-e_1+1}{3} + \binom{r-e_1-e_2-\binom{e_2}{2}+1}{2} + C$. Since $W_{4,m}(e_1 + e_2 + e_3) = 0$, we compute $C =$

$-\binom{e_3-\binom{e_2}{2}+1}{2}-\binom{e_2+e_3+1}{3}$. Thus

$$\begin{aligned}
W_{4,m}(r) &= \binom{r-e_1+1}{3}+\binom{r-e_1-e_2-\binom{e_2}{2}+1}{2} \\
&\quad -\binom{e_3-\binom{e_2}{2}+1}{2}-\binom{e_2+e_3+1}{3} \\
&= \binom{r-e_1+1}{3}+\left[\binom{r-e_1-e_2-\binom{e_2}{2}}{2}+\binom{r-e_1-e_2-\binom{e_2}{2}}{1}\right] \\
&\quad -\left[\binom{e_3-\binom{e_2}{2}}{2}+\binom{e_3-\binom{e_2}{2}}{1}\right]-\binom{e_2+e_3+1}{3} \\
&= \binom{r-e_1+1}{3}+\binom{r-e_1-e_2-\binom{e_2}{2}}{2} \\
&\quad +r-e_1-e_2-e_3-\binom{e_3-\binom{e_2}{2}}{2}-\binom{e_2+e_3+1}{3} \\
&= \binom{r-e_1+1}{3}+\binom{r-e_1-\binom{e_2+1}{2}}{2} \\
&\quad +\binom{r-e_1-e_2-e_3-\binom{e_3-\binom{e_2}{2}}{2}-\binom{e_2+e_3+1}{3}}{1},
\end{aligned}$$

as desired.

□

Notation 7.9. With notation as above, define a family of functions recursively by

- $y_2(r)=0$
- $z_2(r)=r-e_1-1$
- $y_3(r)=\binom{z_2(r)+1}{2}$
- $z_i(r)=r-e_1-\cdots-e_{i-1}-y_i(e_1+e_2+\cdots+e_{i-1})-1$
- $y_i(r)=\binom{z_2(r)+i-2}{i-1}+\binom{z_3(r)+i-3}{i-2}+\cdots+\binom{z_{i-1}(r)+1}{2}$

Theorem 7.10. *With notation as above, we have $W_{i,m}(r)=1+y_i(r)+z_i(r)$ for all $i\geq 2$.*

Proof. The cases $i=2,3,4$ are handled in Proposition 7.8. We proceed by induction on i.

We may assume, inductively, that $W_{i-1,m}(r)=1+y_{i-1}(r)+z_{i-1}(r)$. To verify that $W_{i,m}(r)=1+y_i(r)+z_i(r)$, it suffices to show that $1+y_i+z_i$ satisfies the appropriate discrete initial value problem. Indeed, we have

$$\begin{aligned}\Delta(1+y_i+z_i) &= \Delta(1)+\Delta y_i+\Delta z_i\\ &= 0+\sum_{j=2}^{i-1}\Delta\binom{z_j+i-j}{i-j+1}+\Delta\binom{z_i}{1}\end{aligned}$$

Since each z_j has the form $z_j(r) = r - $ (a constant), we may apply the discrete power rule to each summand, yielding

$$\begin{aligned}\Delta(1+y_i+z_i) &= \sum_{j=2}^{i-1}\binom{z_j+i-j-1}{i-j}+\binom{z_i}{0}\\ &= \left(\binom{z_{i-1}}{1}+\sum_{j=2}^{i-2}\binom{z_j+(i-1)-j}{(i-1)-j+1}\right)+1\\ &= (z_{i-1}+y_{i-1})+1\\ &= W_{i-1}.\end{aligned}$$

Furthermore, we compute $[1+y_i+z_i](e_1+\cdots+e_{i-1})=0$ as required. ☐

Theorem 7.11. *Let $m = x_1^{e_1}x_2^{e_2}\dots x_n^{e_n} \in k[x_1,\dots,x_n]$ be a monomial of degree d, and set $I = \mathrm{Borel}(m)$. For all i, define y_i and z_i as in Notation 7.9. Then the following are equivalent:*

1. *I is Gotzmann.*
2. *For all $i \geq 2$, we have $z_i(d) \geq 0$.*
3. *$z_n(d) \geq 0$.*
4. *$e_n \geq y_n(e_1+\cdots+e_{n-1})-1$.*

Check statement and proof carefully against known bounds and examples. Off-by-one errors seem likely.

Proof. For the equivalence of (1) and (2), adopt the convention that $\binom{z_{n+1}(d)}{0} = 1$. We compute

$$\begin{aligned}\mathrm{BP}_I(t) &= \sum_{i=1}^{n} W_i(d)t^i\\ &= \sum_{i=1}^{n}(y_i(d)+z_i(d)+1)t^i\\ &= \sum_{i=1}^{n}\left[t^i\sum_{j=1}^{i}\binom{z_{j+1}(d)+i-j-1}{i-j}\right]\\ &= \sum_{j=1}^{n}\left[\sum_{i=j}^{n}\binom{z_{j+1}(d)+i-j-1}{i-j}t^i\right]\end{aligned}$$

Setting $\ell = i - j$, this becomes

$$= \sum_{j=1}^{n} \sum_{\ell=0}^{n-j} \binom{z_{j+1}(d) + \ell - 1}{\ell} t^{j+\ell}$$
$$= \sum_{j=1}^{n} \left[t^j \sum_{\ell=0}^{n-j} \binom{z_{j+1} + \ell - 1}{\ell} t^{\ell} \right].$$

That is, $\mathrm{BP}_I(t)$ is the truncation at t^n of

$$\sum_{j=1}^{n} t^j \left[\sum_{\ell=0}^{\infty} \binom{z_{j+1} + \ell - 1}{\ell} t^{\ell} \right] = \sum_{j=1}^{n} t^j \left(\frac{1}{1-t} \right)^{z_{j+1}(d)}.$$

Setting $a_i = z_{i+1}$ for all i, it follows from Proposition 5.3 that I is Gotzmann if and only if $z_2(d) \geq z_3(d) \geq \cdots \geq z_{n-1}(d) \geq 0$.

We show that, for all d, $z_2(d) \geq z_3(d) \geq \cdots \geq z_{n-1}(d)$.

$$z_{i-1}(d) - z_i(d) = (W_{i-1,m}(d) - y_{i-1}(d) - 1) + (W_{i,m}(d) - y_i(d) - 1)$$

ugh I don't want to get into a double induction here. Surely there's a better (non-circular) argument based on thresholds increasing. Could bring the algorithm back in, perhaps?

These inequalities also show that (3) implies (2).

For the equivalence of (3) and (4), observe that $z_n(d) = e_n - y_n(e_1 + \cdots + e_{n-1}) - 1$.

□

Corollary 7.12. *Let $m = x_1^{e_1} x_2^{e_2} \ldots x_n^{e_n} \in k[x_1, \ldots, x_n]$. Put $B = \mathrm{Borel}(m)$. Then:*

1. *If $n = 1$, then B is Gotzmann.*
2. *If $n = 2$, then B is Gotzmann.*
3. *If $n = 3$, then B is Gotzmann if and only if $e_3 \geq \binom{e_2+0}{2}$.*
4. *If $n = 4$, then B is Gotzmann if and only if*
$$e_4 \geq \binom{e_3 - \binom{e_2+0}{2} + 0}{2} - \binom{e_2 + e_3 + 1}{3}.$$
5. *If $n = 5$, then B is Gotzmann if and only if*
$$e_5 \geq \binom{e_4 - \binom{e_3 - \binom{e_2+0}{2} + 0}{2} - \binom{e_2+e_3+1}{3}}{2} - \binom{e_3 + e_4 - \binom{e_2+0}{2} + 1}{3} - \binom{e_2 + e_3 + e_4 + 2}{4}$$

The space necessary to write the bound roughly doubles whenever n is incremented.

8 The Bonanzinga-Eliahou conjectures