DG-resolutions via Morse theory

Buchanan, Chau, Mermin

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Abstract: To be written later

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1 Introduction

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Morse-closed condition appears in [?GMPR, Theorem 2.26] and [?CCMT, Theorem 4.2]. We're giving it a name.

2 Background and Notation

Let $S = k[x_1, \ldots, x_n]$ be a polynomial ring over a field k.

We endow S with a multigrading indexed by its monomials and written multiplicatively. That is, for every monomial $m \in S$, there is a one-dimensional k-vector space $S_m = k\langle m\rangle$ consisting of the multihomogeneous elements with multidegree m. Viewed as a k-vector space, S decomposes as $S = \bigoplus_{m\in S} S_m$, and for all multidegrees m and m' we have $(S_m)(S_{m'}) \subset S_{mm'}$.

The multigraded ideals of S are precisely the monomial ideals. The twisted module $S(m^{-1})$ stands for a copy of S generated in multidegree m; that is, $[S(m^{-1})]_{m'}$ is isomorphic (as a k-vector space) to $S_{m^{-1}m'}$ (which is one-dimensional if m' divides m and trivial if not). Note that $S = S(1)$.

For a monomial ideal I, a multigraded *free resolution* of S/I is a collection of multigraded modules and maps

$$\mathbb{F}_\bullet : 0 \to F_p \to [d_n]F_{p-1} \to [d_{n-1}] \cdots \to [d_2]F_1 \to [d_1]F_0 \to [d_0]S/I \to 0,$$

where each F_i is a multigraded free module, each d_i is multigraded of multidegree 1, and $\ker(d_i) = \mathrm{im}(d_{i+1})$ for all i. The resolution is *minimal* if each F_i has minimum possible rank, or (equivalently) if $\mathrm{im}(d_i) \subset (x_1, \ldots, x_n)F_{i-1}$ for all i. If $\mathbb{F}_\bullet$ is minimal and we write $F_i = \bigoplus S(m^{-1})^{\beta_{i,m}}$, then we say that $\beta_{i,m}$ is the i^{th} *multigraded Betti number* of I with multidegree m.

Every free resolution is a complex, so, as with any complex, it is possible to view $\mathbb{F}_\bullet$ as a module over $S[\partial]$ (or even over $\dfrac{S[\partial]}{(\partial^2)}$), by setting $\mathbb{F} = \bigoplus_i F_i$, and defining the action of ∂ by $\partial(f) = d_i(f)$ whenever $f \in F_i$ and extending by linearity. Observe that $S/I = \dfrac{\ker(\partial)}{\mathrm{im}(\partial)}$. The direct sum decomposition $\mathbb{F}_\bullet = \bigoplus F_i$ provides a grading in which ∂ is a map of degree -1; this grading is compatible with the multigrading on each of the F_i. If $f \in F_i$ is multihomogeneous of multidegree m, we may view f as an homogeneous element of $\mathbb{F}$ in both senses; we say that f has *multidegree* m and *homological degree* i.

2.1 DG-Algebras

It is natural to ask whether a resolution $\mathbb{F}$, viewed as a module, has additional algebraic structure. In some important cases, the answer is resoundingly positive.

Example 2.1. Perhaps the most important example of a free resolution is the *exterior algebra* See [?Eisenbud, ?Chapter??] or [?Peeva, ?Chapter??] for a thorough discussion. We provide a quick-and-dirty definition of the exterior algebra on the variables of S here; these may be replaced harmlessly with a multigraded regular sequence (or, at more cost, with an arbitrary list of multigraded elements: in this case the construction below yields the *Koszul complex* on those elements, which is not a resolution but still contains important information).

For each variable x_i, define an *exterior variable* e_i of homological degree 1 and multidegree x_i. We impose an associative anticommutative multiplication, called the *wedge product*, on the e_i, so that $e_{j_1} \wedge e_{j_2} \wedge \cdots \wedge e_{j_i}$ has homological degree i and multidegree $x_{j_1} x_{j_2} \ldots x_{j_i}$. The anticommutativity means that $e_i \wedge e_j = -e_j \wedge e_i$ and $e_i \wedge e_i = 0$ for all i and j. (The second requirement here follows straightforwardly from the first unless k has characteristic 2, in which case it is necessary to state explicitly.) Let E_i be the free S-module spanned by all wedge products of homological degree i; then E_i has rank $\binom{n}{i}$ and basis consisting of wedge products of the form $e_{j_1} \wedge \cdots \wedge e_{j_i}$ with $j_1 \lneqq \cdots \lneqq j_i$. For a monomial $\mu = x_{j_1} x_{j_2} \ldots x_{j_i}$ with $j_1 \leq j_2 \ldots j_i$ we define the exterior element e_μ as the wedge product $e_\mu = e_{j_1} \wedge \ldots e_{j_i}$; this is nonzero if and only if μ is squarefree, in which case it has homological degree $\operatorname{hdeg}(e_\mu) = \deg(\mu) = i$ and multidegree μ.

We also define a differential $\partial_i : E_i \to E_{i-1}$ by setting

$$\partial_i(f e_{j_1} \wedge \ldots e_{j_i}) = \sum_{\ell=1}^{i} (-1)^{\ell+1} f x_{j_\ell} e_{j_1} \wedge e_{j_2} \wedge \cdots \wedge \widehat{e_{j_\ell}} \wedge \cdots \wedge e_{j_i}$$

for any $f \in S$ and any $j_1, \ldots, j_i$. (The hat, as usual, means that the factor e_{j_ℓ} is omitted.)

One may verify, with some tedium, that the $S[\partial]$ module $E = \bigoplus E_i$ is a minimal free resolution of $\dfrac{S}{(x_1, \ldots, x_n)}$. But the wedge product gives this module an associative algebra structure that has some additional nice properties:

1. The wedge product respects homological degree: $E_i \wedge E_j \subset E_{i+j}$.
2. The exterior variables are graded-commutative: $e_\mu \wedge e_\nu = (-1)^{\operatorname{hdeg}(e_\mu)\operatorname{hdeg}(e_\nu)} e_\nu \wedge e_\mu$.
3. The regular variables commute with the exterior variables: $(f e_\mu) \wedge (g e_\nu) = f g e_\mu \wedge e_\nu$.
4. The wedge product and the exterior variables satisfy an anticommutative Leibniz rule: $\partial(e_\mu \wedge e_\nu) = \partial(e_\mu) \wedge e_\nu + (-1)^{\operatorname{hdeg}(e_\mu)} e_\mu \wedge \partial(e_\nu)$.
5. The wedge product is nontrivial.

The notion of a differential graded algebra is an attempt to generalize the nice properties of the exterior algebra. We recall some definitions:

Definition 2.2. A pair (M, ∂) is called a *DG-module* if M is a graded $S[\partial]$ module $M = \bigoplus M_d$ such that

- $\partial^2 M = 0$.
- ∂ has homological degree -1: For all d, $\partial M_d \subset M_{d-1}$.

Definition 2.3. A triple $(M, \partial, \star)$ is a *DG-algebra* if (M, ∂) is a DG-module, $M_0 = S$, and $\star : M \times M \to M$ is an operator satisfying the additional properties:

- Associativity: $(a \star b) \star c = a \star (b \star c)$ for all $a, b, c \in M$.
- Graded anticommutativity: If $a \in M_d$ and $b \in M_e$, then $a \star b = (-1)^{de} b \star a$. Also, if $d \neq 0$, $a \star a = 0$.
- Leibniz rule: If $a \in M_d$ and $b \in M_e$, then $\partial(a \star b) = \partial(a) \star b + (-1)^d a \star \partial(b)$.
- Scalar commutativity: If $a \in M_0 = S$ and $b \in M$, then $a \star b = b \star a = ab$.

I have possibly modified the definition to force $a \star a = 0$ whenever it's a homogeneous non-scalar. This seems like the right generalization to me, but it's not present in at least some of the references and I don't think I can prove it with the others. Will it be a problem? Likewise, I've forced $M_0 = S$.

I also really want to add two homogeneity requirements that $M_d \star M_e \subset M_{d+e}$ and $\operatorname{mdeg}(a \star b) = \operatorname{mdeg}(a) \operatorname{mdeg}(b)$ whenever a, b multihomogeneous. Can/should we do that?

Remark. Without the scalar commutativity requirement, we could make any DG-module into a DG algebra by using the trivial multiplication $a \star b = 0$ for all a, b. (The Leibniz rule prevents us from using the slightly less trivial rule $a \star b = 0$ whenever a and b both have positive homological degree.) The requirement that $\star$ commute with scalar multiplication is equivalent to requiring that M is an S-algebra, so a very natural way to ensure the DG-algebra structure is non-trivial.

Definition 2.4. A DG-algebra is called a *DG-resolution* if in addition it satisfies $\dfrac{\ker \partial}{\operatorname{im} \partial} = \left(\dfrac{\ker \partial}{\operatorname{im} \partial}\right)_0 \cong \dfrac{S}{\operatorname{im} \partial_1}$. Equivalently, a *DG-resolution of S/I* is a resolution of S/I together with a multiplication $\star$ that satisfies the conditions of a DG-algebra.

A natural question is which monomial ideals I have DG-resolutions. It turns out that the Taylor resolution, a (usually highly nonminimal) resolution, is always a DG-resolution. Need to work in citations to [?Taylor] and something more recent, hopefully [?Peeva], as well as [?Gemeda]

Construction 2.5. Let $I = (m_1, \ldots, m_s)$ be a monomial ideal. The Taylor resolution of $I$is the free module with basis given by formal symbols of the form $[\sigma] = [m_{j_1}, \ldots, m_{j_i}]$ for subsets $\sigma \subset \{1, \ldots, s\}$ with $\sigma = \{j_1, \ldots, j_i\}$ and $j_1 < \cdots < j_i$. The symbol $[\sigma]$ has homological degree $i = |\sigma|$ and multidegree $\operatorname{mdeg}(\sigma) = \operatorname{lcm}(m_{j_1}, \ldots, m_{j_i})$.

The differential is given by $\partial([\sigma]) = \sum_{t=1}^{i} (-1)^{t-1} \dfrac{\operatorname{mdeg}(\sigma)}{\operatorname{mdeg}(\sigma \setminus \{j_t\})}$.

Proposition 2.6 (Gemeda)**.** *The Taylor resolution of a monomial ideal is a DG-resolution.*

Proof. We define $\star$ by the rule $[\sigma]\star[\tau] = 0$ if $\sigma\cap\tau \neq \varnothing$ and $[\sigma]\star[\tau] = (-1)^{\text{sign}(\sigma,\tau)}\dfrac{\text{mdeg}(\sigma)\,\text{mdeg}(\tau)}{\text{mdeg}(\sigma\cup\tau)}[\sigma\cup\tau]$ if $\sigma\cap\tau = \varnothing$. (Here, $\text{sign}(\sigma,\tau)$ represents (parity of) the number of transpositions needed to turn the sequence $(\sigma;\tau)$ into an increasing sequence. (The sequence $(\sigma;\tau)$ is obtained by writing the elements of σ in increasing order, then writing the elements of τ in increasing order.) $\square$

It is natural to wonder whether other resolutions, and in particular minimal resolutions, are DG-resolutions. Unfortunately, the answer turns out to be negative in general. One of the first examples is due to Katthän ([`?Katthan`]).

Example 2.7 (Katthän)**.** Let $I = (a^2, ab, b^2c^2, cd, d^2) \subset k[a,b,c,d]$. The minimal resolution of I is not a DG-resolution.

One of the standard ways to build algebras is by taking the quotient of a larger algebra by an ideal. The Taylor resolution in some sense universal; for example, the minimal resolution of I is always a direct summand of the Taylor resolution (see [`?Peeva`, `?chapter??`]). Thus the minimal resolution is a quotient of the Taylor resolution when we view Taylor as a DG-module. If the minimal resolution were a quotient when we view Taylor as an algebra, it should be an DG-resolution as well. To formalize this idea, we define a DG-ideal.

Definition 2.8. Let $(A,\partial,\star)$ be a DG-algebra. We say that $B\subset A$ is a *DG-ideal of* A if:

- B is an S-submodule of A.
- B is closed under ∂. That is, if $b\in B$, then $\partial(b)\in B$.
- B is absorptive under $\star$. That is, if $a\in A$ and $b\in B$, then $a\star b\in B$.

Proposition 2.9. *Suppose A is a DG-algebra and B is a DG-ideal of A. Then A/B is a DG-algebra.*

Proof. Yeah, need to actually write this out. Blah. $\square$

The primary tool that commutative algebraists have used to view resolutions (minimal or otherwise) as quotients of the Taylor resolution is discrete Morse theory. We review the basics in the next subsection.

2.2 Discrete Morse Theory

Discrete Morse theory is a way of building downward from the Taylor resolution toward the minimal resolution of a monomial ideal.

Let $I = (m_1,\ldots,m_s)$. The standard basis for the Taylor resolution of I is given by formal symbols $[\sigma]$ corresponding to subsets $\sigma\subset\{1,\ldots,s\}$. These subsets naturally form a Boolean lattice under inclusion, and we see that $\partial([\tau])$

has a nonzero coefficient on $[\rho]$ if and only τ covers ρ in this lattice. The Taylor resolution is non-minimal if and only if one of the nonzero coefficients is a scalar, if and only if there exist subsets τ covering ρ with $\text{mdeg}(\tau) = \text{mdeg}(\rho)$. The fundamental intuition of discrete Morse theory is that we ought to be able to delete such pairs (τ, ρ) from the Boolean lattice, and the resulting poset should describe a new, smaller resolution.

We make this precise as follows.

Lemma 2.10. *Let $I = (m_1, \ldots, m_s)$ be a monomial ideal, with Taylor resolution $\mathbb{T}$. Suppose that $\tau = \rho \cup \{j\}$ are subsets of $\{1, \ldots, s\}$, and $\text{mdeg}(\tau) = \text{mdeg}(\rho)$. Then the trivial complex $\mathbb{F} : 0 \to S[\tau] \to S \cdot \partial([\tau]) \to 0$ is a direct summand of $\mathbb{T}$. In particular, the quotient $\mathbb{G} = \dfrac{\mathbb{T}}{\mathbb{F}}$ is a resolution of I.*

Proof. □

Remark. With notation as in Lemma 2.10, the smaller resolution $\mathbb{G}$ has a non-minimal generating set consisting of formal symbols $[\sigma]$ for every subset σ of $\{1, \ldots, s\}$, and the formulas for the differential are unchanged from the Taylor differential. Since we are taking a quotient by $\mathbb{F}$, we have $[\tau] = 0$ and $\partial[\tau] = 0$; that is, $[\rho]$ is equal to an S-linear combination of the other subsets of τ. In particular, $\mathbb{G}$ has basis consisting of formal symbols $[\sigma]$ for every subset $\sigma \neq \rho, \tau$, and the formula for every $\partial([\sigma \cup \ell])$ in the new basis becomes more complicated.

Unfortunately, the Taylor resolution is in general not close to minimal, so making a single cancellation-and-quotient, as in Lemma 2.10, is usually insufficient to find the minimal resolution of I. We could apply Lemma 2.10 (or a sufficiently close analog) to $\mathbb{G}$ inductively until we reach the minimal resolution, but doing so is computationally prohibitive and offers very little intuition. Instead, we'd like to cancel many pairs of basis elements simultaneously. The state of the art here is the *Morse matching.*

Definition 2.11. Let $I = (m_1, \ldots, m_s)$ be a monomial ideal, and let $\mathbb{T}$ be the Taylor resolution of I. Let $\mathcal{P}$ be the poset of all subsets of $\{1, \ldots, s\}$, ordered by inclusion. Define a directed graph $G = (V, E)$ where V is the vertices of $\mathcal{P}$ and E is the Hasse diagram of $\mathcal{P}$, consisting of directed edges from $\rho \cup \{j\}$ to ρ for all $j \notin \rho$. We say that a set of edges $A \subset E$ is a *Morse matching* if it satisfies the three properties:

- A is a matching: Distinct edges in A are disjoint.
- A is acyclic: If we reverse the arrows on every edge in A, the resulting collection of edges $(E \smallsetminus A) \cup A^*$ does not contain a directed cycle.
- A is homogeneous: If A contains the edge from τ to ρ, then $\text{mdeg}(\tau) = \text{mdeg}(\rho)$.

Construction 2.12. Let $I = (m_1, \ldots, m_s)$ be a monomial ideal, and let $\mathbb{T}$ be the Taylor resolution of I. Suppose that A is a Morse matching. For every edge

$e \in A$, pointing from τ_e to ρ_e, the trivial complex $\mathbb{F}_e : 0 \to S[\tau_e] \to S \cdot \partial[\tau_e] \to 0$ is a direct summand of $\mathbb{T}$. Set $\mathbb{F} = \bigoplus_{e \in A} \mathbb{F}_e$. Then the quotient $\mathbb{G} = \mathbb{T}/\mathbb{F}$ is a resolution of I. Furthermore, the collection of Taylor symbols $\overline{[\sigma]}$ that do not appear in any edge of A forms a basis for $\mathbb{G}$.

Proof. For an actual proof, see [?IDon'tKnowTheStandardTextHere]. Briefly, the homogeneity guarantees that the coefficient a_{τ_e,ρ_e} on $[\sigma_e]$ in $\partial[\tau_e]$ is a scalar, so that $\mathbb{F}_e$ is a summand of $\mathbb{T}$. The acyclicity and disjointness mean that, in the quotient, we may use the relations $\overline{\partial[\tau_e]} = 0$ to rewrite $\overline{\rho_e}$ uniquely in terms of the surviving $\overline{[\sigma]}$, so that these do form a basis.

□

Definition 2.13. Let I be a monomial ideal, and A a Morse matching on its generators. The resolution $\mathbb{G}$ arising from A in Construction 2.12 is called the *Morse resolution* induced by A.

For any given ideal, it can be difficult to find a Morse matching such that the corresponding Morse resolution is minimal. (In fact, in many cases this is impossible: for example, the Morse resolution is always characteristic-independent, but there exist monomial ideals whose minimal resolution depends on the characteristic of k.) Finding recipes for large Morse matchings is an ongoing direction of active research; the current champions seem to be generalized Barile-Macchia resolutions ([?ChauKara]) and pruning resolutions ([????]).

Our project, locating minimal DG-resolutions, requires us to focus on ideals and matchings that do yield a minimal Morse resolution. In the language of Construction 2.12, we are looking for ideals and Morse matchings for which the trivial complex $\mathbb{F}$ is a DG-ideal. It is immediate that $\mathbb{F}$ is closed under ∂, so we need to establish only that it is absorptive under $\star$. A sufficient condition, which we call *Morse-closed*, is introduced in the next section.

2.3 Graphs and Edge Ideals

Just need to get the basics here. Recording what's necessary as it arises. Define (simple) graph, edge ideal, induced subgraph. Is that it? Need to define cycles and paths too.

3 Morse-closed Matchings

For the rest of the paper, let $I = (m_1, \ldots, m_s)$ be a monomial ideal with Taylor resolution $\mathbb{T}$.

Notation 3.1. Let I be a monomial ideal with Morse matching A, and write each edge in A as $e = (\rho_e \cup \{j_e\}, \rho_e)$. Set $A^+ = \{\rho_e \cup \{j_e\} : e \in A\}$, the set of tails of directed edges in A, and $A^- = \{\rho_e : e \in A\}$ the set of heads.

Definition 3.2. Let $A = \{e = (\tau_e, \rho_e)\}$ be a Morse matching for I. We say that A is *Morse-closed* if A^+ is closed under taking supersets. That is, for every

$e \in A$, write $e = (\rho \cup \{j\}, \rho)$. A is Morse-closed if, for every $i \neq j$, there exists an ℓ such that $(\rho \cup \{i, j\}, \rho \cup \{i, j\} \setminus \{\ell\}) \in A$.

We say that I is *Morse-closed* if there exists a Morse-closed Morse matching for I that induces a minimal Morse resolution.

Proposition 3.3. *Suppose that I is Morse-closed. Then I has a minimal DG-resolution.*

Proof. By assumption, there exists a Morse-closed Morse matching A which induces a minimal Morse resolution. We will show that this Morse resolution is a DG-resolution.

It suffices to verify that, in Construction 2.12, the trivial complex $\mathbb{F}$ is a DG-ideal. By construction, $\mathbb{F}$ is closed under addition, additive inverses, the action of S, and the action of ∂. Thus it suffices to show that $\mathbb{F}$ is absorptive under $\star$.

First, fix a subset $\tau \subset \{1, \dots, s\}$ such that $\tau \in A^+$, and an arbitrary subset σ. If $\tau \cap \sigma \neq \varnothing$, then $[\tau] \star [\sigma] = 0 \in \mathbb{F}$. Otherwise, $\tau \cup \sigma \in A^+$ since A is Morse-closed, so $[\tau \cup \sigma] \in \mathbb{F}$ and $[\tau] \star [\sigma] = \dfrac{\mathrm{mdeg}[\tau]\,\mathrm{mdeg}[\sigma]}{\mathrm{mdeg}[\tau \cup \sigma]}[\tau \cup \sigma] \in \mathbb{F}$ as required.

Now, let f be an arbitrary element of $\mathbb{F}$, and write $f = \sum_e (a_e[\tau_e] + b_e \partial[\tau_e])$. Then, for any σ, $f \star [\sigma] = \sum_e (a_e[\tau_e] * [\sigma_e] + b_e(\partial([\tau] \star [\sigma]) \pm [\tau] \star \partial[\sigma])$; each summand is in $\mathbb{F}$, so $f \star [\sigma]$ must be in $\mathbb{F}$ as well. □

Proof is probably not worth the space it takes up. Also, it's Theorem 2.24 of [?GellerMartinPotts-Rubin].

Remark. For a Morse matching A to induce a DG-resolution, the Morse-closed condition is sufficient but not necessary. For example, consider the matching $A = \{(12345, 1234), (1235, 125), (1245, 124), (1345, 134), (2345, 234), (123, 12)\}$ on the ideal $I = (cdehij, abefgj, bdgi, acfh, fghij)$. A is not Morse-closed, since 123 is a tail but 1234 is not. However, $[1234] = \partial[12345] + [1235] - [1245] + [1345] - [2345] \in \mathbb{F}$, so the Morse resolution induced by A is a DG-resolution. And if I did it right, I is the Phan ideal of the lattice where the matched sets are equal and everything else is distinct. Anyway, is there a smaller/easier/more symmetric example? Trung has an example somewhere.

We can identify a few small classes of Morse-closed ideals.

Theorem 3.4. *Every ideal with at most four generators is Morse-closed.*

Proof. By CITATION NEEDED, every ideal with four generators has a Morse matching A which induces the minimal resolution. Observe that A^+ cannot contain a two-generator set: If it did, A would contain an edge $(\{f, g\}, f)$ satisfying $\mathrm{lcm}(f, g) = f$ and f would not be a minimal generator.

Write $I = (f_1, f_2, f_3, f_4)$. If I is not Morse-closed, we conclude that, without loss of generality, A^+ contains $\{f_1, f_2, f_3\}$ but not $\{f_1, f_2, f_3, f_4\}$. It follows that (without loss) $\mathrm{lcm}(f_1, f_2) = \mathrm{lcm}(f_1, f_2, f_3)$, so $\mathrm{lcm}(f_1, f_2, f_4) = \mathrm{lcm}(f_1, f_2, f_3, f_4)$. Consequently, $[f_1, f_2, f_3, f_4] \in \mathbb{F}$; since $\{f_1, f_2, f_3, f_4\} \notin A^-$, we conclude that it is in A^+ after all. □

Definition 3.5. Let $I = (m_1, \dots, m_s)$ be a monomial ideal, and write $m_i = \prod x_j^{e_{i,j}}$. We say that m_i is a *dominant* generator for I if there exists some variable x_j such that $e_{i,j} > e_{i',j}$ for all $i' \neq i$.

We say that I is a *dominant* ideal if every generator is dominant, and 1-*semi-dominant* if all but one generator is dominant. More generally, I is t-semi-dominant if all but t generators are dominant.

The language of dominance was developed by Alesandroni, who in [?AlesandroniDominance] proved the following:

Theorem 3.6 (Alesandroni)**.** *Let I be a monomial ideal. Then:*

1. *If I is dominant, then I is minimally resolved by the Taylor resolution.*
2. *If I is* 1*-semi-dominant, then I is minimally resolved by the Scarf complex, which is the sub-complex of the Taylor resolution consisting of all $[\sigma]$ such that* $\operatorname{mdeg}(\sigma)$ *is unique.*
3. *If I is* 2*-semi-dominant, then every maximal Morse matching induces the minimal resolution.*
4. *If I is* 3*-semi-dominant, then there exists a minimal Morse resolution for I.*

Proposition 3.7. *If I is dominant or* 1*-semi-dominant, then I is Morse-closed.*

Proof. If I is dominant, then the Taylor resolution is minimal, so we may take A to be the empty matching.

If I is one-semi-dominant, write $I = (f_1, \dots, f_s, g)$ so that all f_i are dominant monomials. Let $A = \{(\sigma \cup \{g\}, \sigma) : \operatorname{lcm}(\sigma, g) = \operatorname{lcm}(\sigma)\}$. By the dominance assumption, two sets of generators σ and τ have the same least common multiple only if $\sigma \cap \{f_1, \dots, f_s\} = \tau \cap \{f_1, \dots, f_s\}$. Thus A consists of all pairs of sets with the same multidegree, so $\frac{\mathbb{T}}{\mathbb{F}}$ is isomorphic to the Scarf complex, and in particular is minimal. It remains to verify that A^+ is closed under taking supersets. To that end, suppose that $\sigma \cup \{g\} \in A^+$ and let $f_i \notin \sigma$. Then we have $\operatorname{lcm}(\sigma, f_i, g) = \operatorname{lcm}(\operatorname{lcm}(\sigma, g), f_i) = \operatorname{lcm}(\sigma, f_i)$, so that $\sigma \cup \{f_i\} \cup \{g\} \in A^+$ as desired. □

Remark. Example 2.7, due to Katthän, is a 2-semidominant Scarf ideal, $I = (a^2, ab, b^2c^2, cd, d^2)$, which is not DG and hence not Morse-closed. Thus Proposition **??** cannot be extended in either direction.

It is similarly difficult to identify clean necessary conditions for ideals to be Morse-closed. The following technical conditions will be useful later in the paper.

Lemma 3.8. *Let I be a Morse-closed ideal with Morse-closed matching A, and assume $\sigma \in A^+$. Then σ contains a non-dominant generator.*

Proof. Suppose not. Then all subsets of σ have distinct lcm, and in particular all proper subsets of σ have distinct lcm from σ, contradicting the homogeneity of A. □

Lemma 3.9. *Let I be a Morse-closed ideal with Morse-closed matching A. If I is not dominant, there exists a non-dominant generator f such that every $\sigma \in A^+$ contains f.*

Proof. Since I is not dominant, its projective dimension is strictly less than the number of generators. In particular, if $\mathcal{G}(I)$ is the set of all monomial generators, then A contains a directed edge $(\mathcal{G}(I), \mathcal{G}(I) \smallsetminus \{f\})$ for some generator f, which cannot be dominant. We claim that every $\sigma \in A^+$ f. Suppose not: then there exists $\sigma \in A^+$ with $\sigma \subseteq \mathcal{G}(I) \smallsetminus \{f\}$. But $\mathcal{G}(I) \smallsetminus \{f\} \notin A^+$, contradicting the assumption that A^+ is closed under taking supersets. □

I think it is worth discussing all the things that it looks like this element is saying but isn't. For example, the ideal is not 1-semidominant, as shown by P_4. It is also not the case that the ideal generated by $G(I) \smallsetminus \{f\}$ is dominant, as shown by C_4. I assume it's not the case that every ideal with a Morse element is Morse-closed, but an example is going to have to be pretty big, and I don't have one quickly. Nope! $P_6 = (ab, bc, cd, de, ef)$ has Morse element cd. In fact, it'll probably smooth out the next section to put that as an example here.

4 Classification of Morse-closed edge ideals

In this section, we classify the graphs whose edge ideals are Morse-closed. The strategy is to identify a large enough collection of forbidden subgraphs. For such an argument to work, we require that the Morse-closed property persists when passing to an induced subgraph. We prove a stronger statement below.

Proposition 4.1. *Let $I = (f_1, f_s)$ be a Morse-closed monomial ideal and m a multidegree. Let $I_{\leq m} = (f_i : f_i$ divides $m)$ be the restriction of I to multidegrees less than m. Then $I_{\leq m}$ is also Morse-closed.*

Proof. Let A be a Morse-closed Morse matching on I, and let $A_{\leq m}$ be the collection of directed edges in A whose vertices have multidegree dividing m. (Since A is a matching, both vertices of every edge have the same multidegree, so this is unambiguous.)

Let $\mathbb{T}$ be the Taylor resolution of I, and $\mathbb{F}$ be the trivial complex arising from A, so that $\frac{\mathbb{T}}{\mathbb{F}}$ is the minimal resolution of I. Setting $\mathbb{T}_{\leq m}$ equal to the subcomplex of $\mathbb{T}$ generated by symbols of multidegree dividing m, and similarly for $\mathbb{F}_{\leq m}$, we see that, for every multidegree μ dividing m, the complexes of k-vector spaces $(\frac{\mathbb{T}_{\leq m}}{\mathbb{F}_{\leq m}})_\mu$ and $(\frac{\mathbb{T}}{\mathbb{F}})_\mu$ are equal. It follows that $\frac{\mathbb{T}_{\leq \mu}}{\mathbb{F}_{\leq \mu}}$ minimally resolves $I_{\leq m}$.

By construction, $A^+_{\leq m}$ is closed under taking supersets (within the generators of $I_{\leq m}$).

We conclude that $A_{\leq m}$ is a Morse-closed Morse matching for $I_{\leq m}$. □

A natural question is: What happens, above and below, if we replace Morse-closed with DG? Trung points out that "above" is done by Geller-Martin-Potts-Rubin, and this leads to a ridiculously difficult characterization of DG trees; general graphs appear hopeless. We should mention that.

Definition 4.2. Let G be a simple graph, and $I = I(G)$ its edge ideal. We say that G is *Morse-closed* if I is Morse-closed.

Corollary 4.3. *Let G be a simple graph. If G is Morse-closed, then every induced subgraph of G is Morse-closed. Equivalently, if any induced subgraph of G is not Morse-closed, then G is not Morse-closed.*

We begin by translating Lemma 3.9 into the context of edge ideals. Let G be a simple graph, and $I = I(G)$ be its edge ideal. We abuse notation by identifying an edge $e = (x_i, x_j)$ with the corresponding monomial generator $x_i x_j$. Observe that e is dominant if and only if e is a leaf.

Lemma 4.4. *Let G be a Morse-closed simple graph with edge ideal I_G. Then all induced copies of C_3, C_4, and P_4 share a common edge, which is not a leaf of G.*

Proof. Let H be an induced copy of C_3, C_4, or P_4. Let α be the multidegree of H and σ_H be the collection of edges of H. Observe that σ_H is the maximal subset of $G(I)$ having multidegree α. On the other hand, H is not dominant (since C_3 and P_4 have projective dimension 2 and C_4 has projective dimension 3). Consequently, $\sigma_H \in A^+$.

By Lemma 3.9, we conclude that every σ_H shares a generator f of I_G which is not dominant. The corresponding edge e is thus not a leaf of G and is contained in every H, as desired. □

Lemma 4.4 allows us to classify the Morse-closed graphs on four vertices.

Corollary 4.5. *Let G be a graph on four vertices. Then G is Morse-closed if and only if G is not K_4.*

Proof. By Theorem 3.4, every graph with four or fewer edges is Morse-closed. Up to isomorphism, the only graphs on 4 vertices with more than four edges are K_4 and the complement of a single edge $H = (ab, bc, cd, da, ac)$. K_4 is not Morse-closed, since it has four induced copies of C_3 which do not share a common edge. To see that H is Morse-closed, let A^+ be the set of all subsets of $\mathcal{G}(H)$ with cardinality at least three and containing ac, except $\{ab, ac, cd\}$ and $\{da, ac, bc\}$. Then A, consisting of all pairs $(\sigma, \sigma \setminus \{ac\})$ with $\sigma \in A^+$, is a Morse-closed Morse matching. □

I think it's worth including this, even though it's not really part of the path to the classification.

We now proceed to the classification of Morse-closed simple graphs. We begin with connected graphs, and organize the classification by the type of cycles.

4.1 Trees

Proposition 4.6. *The path graph P_n is Morse-closed if and only if $n \leq 5$.*

Proof. By Corollary 4.3, it suffices to show that P_5 is Morse-closed and P_6 is not. Since P_5 contains only four edges, it is Morse-closed by Theorem 3.4.

To see that P_6 is not Morse-closed, write $I_{P_6} = (ab, bc, cd, de, ef)$, and assume that there exists a Morse-closed matching A. Because I_{P_6} has no Betti numbers of multidegree $abcd$, but two Taylor symbols ($\{ab, bc, cd\}$ and $\{ab, cd\}$) of multidegree $abcd$, the matching A must contain an edge connecting these two subsets. In particular, A^+ must contain $\{ab, bc, cd\}$. Similarly, A^+ contains $\{cd, de, ef\}$. It follows that A^+ must contain the larger subsets $\{ab, bc, cd, ef\}$ and $\{ab, cd, de, ef\}$, which each have multidegree $abcdef$. Thus, A must contain edges connecting each of these two subsets to a smaller subset of multidegree $abcdef$. But there is only one subset of cardinality three and multidegree $abcdef$, namely $\{ab, cd, ef\}$. □

Since all paths in a tree are induced, it follows that Morse-closed trees must have diameter at most four.

Proposition 4.7. *The E_5 graph, consisting of five edges ab, bc, cd, de, and cz, is not Morse-closed.*

Proof. The induced subgraphs of multidegree $abcz$ and $cdez$ are both isomorphic to P_4, but intersect only in the leaf cz. By Lemma 4.4, we conclude that E_5 is not Morse-closed. □

Theorem 4.8. *Suppose G is a tree. Then G is Morse-closed if and only if it is one of the following:*

1. *A star: G has a vertex b and an arbitrary collection of vertices $\{a_i\}$, and edges (a_i, b) for each i.*

2. *A double broom graph of diameter three: G has vertices b, c, and arbitrary collections $\{a_i\}$ and $\{e_j\}$, with edges (b, c), and (a_i, b) and (c, e_j) for all i and j.*

3. *A double broom graph of diameter four: G has vertices b, c, d, and arbitrary collections $\{a_i\}$ and $\{f_j\}$, with edges (b, c), (c, d), and (a_i, b) and (d, e_j) for all i and j.*

Proof. First, suppose G is Morse-closed. If G has diameter at least five, then it contains an induced P_6, contradicting Proposition 4.6. If G has diameter four, it has an induced copy of P_5; call this $H = \{ab, bc, cd, de\}$. If c has degree greater than 2 in G, then there is another edge cz, and the induced subgraph on vertices a, b, c, d, e, z is a copy of E_5, contradicting Proposition 4.7. We conclude that c has degree two, so G is a double broom graph in this case. Finally, every tree of diameter three is a double broom graph, and every tree of diameter at most two is a star. □

Aw, nuts. I just realized I've been using "leaf" to mean "whisker". Will it be necessary to go through and change that everywhere?

4.2 Non-chordal graphs

It turns out that the only connected non-chordal Morse-closed graph is the four-cycle.

Proposition 4.9. *Suppose that G is a connected simple graph and has a proper induced four-cycle. Then G is not Morse-closed.*

Proof. Let H be the induced four-cycle on vertices a, b, c, d, with edges $\{ab, bc, cd, da\}$. Since H is proper there is another vertex; since G is connected, there is without loss of generality an edge de. Since G is not chordal, there cannot be edges connecting e to a or to c. In particular, the induced subgraph G' on vertices $\{a, b, c, d, e\}$ consists of either the five edges $\{ab, bc, cd, da, de\}$ or the six edges $\{ab, bc, cd, de, ea\}$. In the first case the two induced copies of P_4 (on $abde$ and $bcde$) intersect only in a leaf. In the second case, the three induced copies of C_4 (on $abcd$, $abde$, and $bcde$) do not share a common edge. We conclude by Lemma 4.4 that G' is not Morse-closed, and by Corollary 4.3 that G is not Morse-closed. □

Theorem 4.10. *Suppose that G is a connected non-chordal graph. Then G is Morse-closed if and only if $G \cong C_4$.*

4.3 Chordal graphs

Finally, suppose that G is a connected chordal graph which is not a tree. Then G contains a copy of C_3, whose vertices we name a, b, c. If C is a proper subgraph of G, then there are strong restrictions on what other edges are available.

Lemma 4.11. *The following graphs are not Morse-closed:*

1. *The tadpole graph, consisting of C_3 and a whisker of length two: (ab, bc, ca, ax, xw).*
2. *The net graph, consisting of C_3 and a whisker at each vertex: (ab, bc, ca, ax, by, cz).*
3. *The kite graph, consisting of two triangles with a shared edge and a whisker off one of the non-common vertices: (ab, bc, ca, bd, da, cz).*

Proof. For the tadpole graph, observe that the two induced copies of P_4 (on $abxw$ and $acxw$ and the induced C_3 (on abc) do not share a common edge. By Lemma 4.4, the tadpole is not Morse-closed.

For the net graph, observe that the three induced copies of P_4 (on $abxy$, $bcyz$, and $acxz$) do not share a common edge. By Lemma 4.4, the net is not Morse-closed.

For the kite graph, observe that the two induced copies of P_4 (on $acdz$ and $bcdz$) share only the edge cz, which is not a leaf. By Lemma 4.4, the kite is not Morse-closed. □

Since all triangles must share a single edge by Lemma 4.4, the only chordal graphs with an induced C_3 but without an induced subgraph forbidden by

Lemma 4.11 consist of the edge ab together with an arbitrary collection of whiskers x_i connected only to a, an arbitrary collection of whiskers y_j connected only to b, and an arbitrary nonempty collection of ears c_k connected to both a and b but nothing else. Such graphs are called "Lyubeznik graphs" because they are minimally resolved by a Lyubeznik resolution. CITATION NEEDED

Definition 4.12. A *Lyubeznik graph* of type (r, s, t) is a graph consisting of $r+s+t+2$ edges named a, b, $c_1, \ldots, c_r$, $x_1, \ldots, x_s$, $y_1, \ldots, y_t$ and $2r+s+t+1$ edges, namely ab and ac_i, bc_i, ax_j, by_k for all i, j, k.

Theorem 4.13. *Suppose that G is a connected chordal graph, and is not a tree. Then G is Morse-closed if and only if G is a Lyubeznik graph of type (r, s, t) for $r \geq 0$, $s \geq 0$, $t \geq 1$.*

Proof. If G is Morse-closed, then by Lemma 4.11 and the subsequent discussion G must be Lyubeznik.

Conversely, suppose G is Lyubeznik of type (r, s, t). Then by Theorem 3.8 in Morse-DG paper the matching A consisting of all pairs $(\sigma \cup ab, \sigma)$ such that $ab \notin \sigma$ and ab divides $\operatorname{mdeg} \sigma$ is a Morse matching. This matching is immediately Morse-closed. Their proof relies on understanding how the Lyubeznik resolution works. But I think we can produce a reasonably short version that doesn't. If so, it's worth including. □

4.4 Disconnected graphs

Proposition 4.14. *Let G be a simple graph. Then G is Morse-closed if and only if all but one connected component of G is a star, and the remaining component is Morse-closed.*

Proof. Suppose that G has two components G_1 and G_2 which are not stars. Then there are induced subgraphs $H_1 \subset G_1$ and $H_2 \subset G_2$ which are isomorphic to C_3, C_4, or P_3. These subgraphs are disjoint, so by Lemma 4.4 G is not Morse-closed.

Conversely, suppose that G has one Morse closed connected component H and the remaining components are an arbitrary collection of stars $G_i = \{x_i y_{ij}\}$. Let A be a Morse-closed matching for H. Define a matching A' consisting of all pairs $(\sigma^+ \cup \tau, \sigma^- \cup \tau)$ such that $(\sigma^+, \sigma^-) \in A$ and τ is an arbitrary subset of the edges of the stars. Then A' is a Morse-closed matching for G.

I suppose it might not be obvious that A' is a matching. We need a Lemma up in Section 2 to the effect that if $I = (J, f)$ with f dominant, then I is Morse-closed if and only if J is Morse-closed. This requires a (presumably, known?) result that an arbitrary Morse matching extends as described above; it's immediate that one is closed if and only if both are. □

5 Powers of edge ideals

In this section, we show that almost no nontrivial powers of edge ideals are Morse-closed.

Lemma 5.1. *Let I be a monomial ideal and m a monomial. Then mI is Morse-closed if and only if I is Morse-closed.*

Proof. The LCM lattices of I and mI are isomorphic (via multiplication by m). It follows that A is a Morse matching for I if and only if mA is a Morse matching for mI. Furthermore, $(mA)^+ = m(A^+)$, so A is Morse-closed if and only if mA is Morse-closed. ☐

We should move this to section 2.
Commented out lemma which is special case of Lemma 3.9

Lemma 5.2. *Let G be P_4, C_3, or C_4. Then I_G^2 is not Morse-closed.*

Proof. Write $P_4 = (ab, bc, cd)$, so that $I = I_{P_4}^2 = (a^2b^2, ab^2c, abcd, b^2c^2, bc^2d, c^2d^2)$. Observe that

$$\begin{aligned} I_{\leq a^2b^2cd} &= (a^2b^2, ab^2c, abcd) \\ I_{\leq abc^2d^2} &= (abcd, bc^2d, c^2d^2) \\ I_{\leq a^2b^2c^2} &= (a^2b^2, ab^2c, b^2c^2). \end{aligned}$$

Each of these minors? induced subideals? is not dominant (with non-dominant generators ab^2c, bc^2d, and ab^2c, respectively), so, if A is any Morse matching for I, A^+ must contain all three of $\{a^2b^2, ab^2c, abcd\}$, $\{abcd, bc^2d, c^2d^2\}$, $\{a^2b^2, ab^2c, b^2c^2\}$. Since these three sets have empty intersection, it follows by Lemma 3.9 that A cannot be Morse-closed. In particular, I is not Morse-closed.

Next, write $C_3 = (ab, bc, ac)$, so that $J = I_{C_3}^2 = (a^2b^2, ab^2c, a^2bc, b^2c^2, abc^2, a^2c^2$. Observe that

$$\begin{aligned} J_{\leq a^2b^2c} &= (a^2b^2, ab^2c, a^2bc) \\ J_{\leq a^2bc^2} &= (a^2bc, abc^2, a^2c^2) \\ J_{\leq ab^2c^2} &= (ab^2c, b^2c^2, abc^2) \end{aligned}$$

are all non-dominant (with all three generators non-dominant in each case) and have empty intersection. It follows from Lemma 3.9 as above that J is not Morse-closed.

Finally, write $C_4 = (ab, bc, cd, ad)$ and set $K = I_{C_4}^2$. Observe that

$$\begin{aligned} K_{\leq a^2b^2c^2} &= (a^2b^2, ab^2c, b^2c^2) \\ K_{\leq b^2c^2d^2} &= (b^2c^2, bc^2d, c^2d^2) \\ K_{\leq a^2c^2d^2} &= (c^2d^2, acd^2, a^2d^2) \end{aligned}$$

are all non-dominant (with non-dominant generators ab^2c, bc^2d, and acd^2, respectively) and have empty intersection. It follows from Lemma 3.9 that J is not Morse-closed. □

Lemma 5.3. *Let G be P_4, C_3, or C_4. Then I_G^n is not Morse-closed for any $n \geq 2$.*

Proof. By [?CHM24-first, Proposition 2.18], there exist a monomial f and a multidegree m such that $(I_G^{n+1})_{\leq m} = fI_G^n$. The result then follows inductively from Proposition 4.1 and Lemma 5.2. □

We conclude that the only graphs G whose edge ideals have Morse-closed powers are (disjoint unions of) stars. But for our purposes the star graph $K_{1,q}$ is essentially a complete intersections of height q.

Lemma 5.4. *Let I_q be the edge ideal of the q-pointed star graph $K_{1,q}$, and let P_q be the monomial prime $(x_1, \dots, x_q)$. For any n, I_q^n is Morse-closed if and only if P_q^n is Morse-closed.*

Proof. Write $K_q = (yx_1, \dots, yx_q)$. Then $I_{K_{1,q}} = yP_q$, and $I_{K_{1,q}}^n = y^n P_q^n$. The result follows from Lemma 5.1. □

These two ought to go in section 3 as well.

Proposition 5.5. *Let $P = (x_1, \dots, x_h)$ be a prime of height h with $h \geq 3$. Then P^n is Morse-closed if and only if $n = 1$.*

Proof. Suppose $n \geq 2$, and observe that the minors?

$$P^n_{\leq x_i^n x_j^2} = (x_i^n, x_i^{n-1}x_j, x_i^{n-2}x_j^2)$$

are non-dominant and have trivial intersection. Since these multidegrees are not dominant, any Morse matching A must have $\{x_i^n, x_i^{n-1}x_j, x_i^{n-2}x_j^2\} \in A^+$. Since the intersection is trivial, it follows from Lemma 3.9 that A cannot be Morse-closed. In particular, P^n is not Morse-closed.

On the other hand, P^1 is dominant so minimally resolved by the Taylor resolution. The empty matching is thus Morse-closed. □

Proposition 5.6. *Let $P = (a, b)$. Then P^n is Morse-closed if and only if $n \leq 4$.*

Proof. If $n \leq 3$, then P^n has at most four generators, so P^n is Morse-closed by Theorem 3.4.

If $n \geq 5$, then observe that, for $2 \leq r \leq n$, the minors?

$$P^n_{\leq a^r b^{n-r+2}} = \{a^r b^{n-r}, a^{r-1}b^{n-r+1}, a^{r-2}b^{n-r+2}\}$$

are non-dominant and have empty intersection. Since these multidegrees are not dominant, any Morse matching A must have $\{a^r b^{n-r}, a^{r-1}b^{rn-r+1}, a^{r-2}b^{n-r+2}\} \in A^+$ for all such r. Since the intersection is trivial, it follows from Lemma 3.9 that A cannot be Morse-closed. In particular, P^n is not Morse-closed.

In $n = 4$, the above lines of reasoning both fail, because P^4 has five generators, but the minors? $P^n_{\leq a^r b^{n-r+2}}$ intersect in a^2b^2. However, one may verify that the matching A consisting of the directed edges

$$\begin{aligned} \{a^4, a^3b, a^2b^2, ab^3, b^4\} &\to \{a^4, a^3b, ab^3, b^4\}, & \{a^4, a^3b, a^2b^2\} &\to \{a^4, a^2b^2\}, \\ \{a^4, a^3b, a^2b^2, ab^3\} &\to \{a^4, a^3b, ab^3\}, & \{a^4, a^2b^2, ab^3\} &\to \{a^4, ab^3\}, \\ \{a^4, a^3b, a^2b^2, b^4\} &\to \{a^4, a^3b, b^4\}, & \{a^4, a^2b^2, b^4\} &\to \{a^4, b^4\}, \\ \{a^4, a^2b^2, ab^3, b^4\} &\to \{a^4, ab^3, b^4\}, & \{a^3b, a^2b^2, ab^3\} &\to \{a^3b, ab^3\}, \\ \{a^3b, a^2b^2, ab^3, b^4\} &\to \{a^3b, ab^3, b^4\}, & \{a^3b, a^2b^2, b^4\} &\to \{a^3b, b^4\}, \\ & & \{a^2b^2, ab^3, b^4\} &\to \{a^2b^2, b^4\}. \end{aligned}$$

is in fact a Morse-closed Morse matching. □

Yaknow what? I think we should just make another section in between §3 and §4 where we classify the Morse-closed powers of primes.

Proposition 5.7. *Let P be a prime of height one. Then P^n is Morse-closed for all n.*

Proof. P^n is principal, so the empty matching is Morse-closed. □

Theorem 5.8. *Let P be a monomial prime. Then P^n is Morse-closed if and only if one of the following holds.*

- $n = 1$.
- *P has height one.*
- *P has height two and $n \leq 4$.*

We are now ready to prove our main result of this subsection.

I think we can get it for disconnected graphs. Two disjoint edges will work for $n \leq 4$ because that's also a complete intersection. Three or more components will fail for the same reason as a star of size three. So all that's left is small powers of $P_3 \coprod P_2$ and $P_3 \coprod P_3$. It looks to me like both squares are going to fail from disjoint triples

Theorem 5.9. *Let G be a connected graph with at least one edge and no isolated vertex, and $n \geq 2$ an integer. Then I_G^n is Morse-closed if and only if one of the following holds:*

1. *G has one edge;*
2. *$G = P_3$ and $n \leq 4$.*

Proof. A combination of all the results in this section. □

6 Morse elements

Think this should go in the prose/examples after Lemma 3.9 rather than get its own section. And the definition should be "a thing like in the lemma" rather than the below.

Definition 6.1. Let I be a monomial ideal. A monomial $m \in \operatorname{gens}(I)$ is called a *Morse element* of I if the Morse matching

$$\{\sigma \cup \{m\} \to \sigma \colon \sigma \in 2^{\operatorname{gens}(I)}, m \notin \sigma, \text{ and } m \mid \operatorname{lcm}(\sigma)\}$$

induces the minimal free resolution of I.

The following is straightforward.

Lemma 6.2. *If a monomial ideal I has a Morse element, then I is Morse-closed.*

A natural class of monomial ideals with a Morse element is 1-semidominant ideals. However, it is noteworthy that there are they are not the only ones, e.g., edge ideals of Lyubeznik graphs.

Remark. Some connections to pruned resolution?

7 Further questions

weak Morse-closed-ness is almost hopeless. We do not even have an answer for edge ideals in general . A reason for this is that unlike Morse-closed-ness, we do not have easy-to-find obstructions.

This makes it a good further question, I think. But it does make the appropriateness of the definition somewhat less clear.