Team Number:

1. Find all the four-digit palindromes that are divisible by 18.

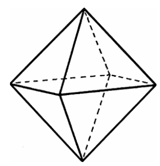
   [A palindrome is a number that reads the same from left-to-right as from right-to-left. For example, 1221 is a four-digit palindrome, but it isn't divisible by 18.]

   [A four-digit number is a number between 1000 and 9999 (inclusive). For example, the number 0000 does not count as a four-digit palindrome.]

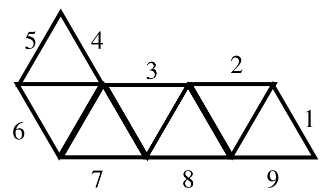
   Solution: The palindrome must have the form $ABBA$, with $A \neq 0$. It must be divisible by 2, so A must be even. And it must be divisible by 9, so $A + B + B + A = 2A + 2B$ must be divisible by 9, so $A + B$ must be a multiple of 9. Thus the possible pairs (A, B) are $(2, 7)$, $(4, 5)$, $(6, 3)$, and $(8, 1)$. So there are four such palindromes, namely 2772, 4554, 6336, and 8118.

Team Number:

2. An **octahedron** is a regular polyhedron with eight triangular sides, as in the picture.



A **net** of an octahedron is an arrangement of eight equilateral triangles in the plane that can be folded to form an octahedron. The following diagram of seven equilateral triangles can be extended to a net by adding one more triangle along which edge(s)?



Solution: Label the faces of our partial net A through G, and the interior edges 10 through 15, so that face A has edges $4, 5, 10$; face B has edges $6, 10, 11$; face C has edges $7, 11, 12$; face D has edges $3, 12, 13$; face E has edges $8, 13, 14$; face F has edges $2, 14, 15$; and face G has edges $1, 9, 15$.

Every edge is contained in two faces, which means each exterior edge is duplicated (or will be an edge of the missing face). Every vertex is shared by four faces and four edges, which means that the common vertex to faces A, B, C, D is contained in edges 10, 11, 12, and $3 = 4$. Let N be the vertex where these all meet; let us refer to N as the "north pole".

If we set O equal to the vertex where edges 4 and 5 meet, P the vertex where edges 5 and 6 meet, Q the vertex where 6 and 7 meet, and R the vertex where 7 and 8 meet, then $OPQR$ forms an equator of the octahedron, with faces A, B, C, D in the "northern hemisphere" and E, F, G in the "south". These faces meet at the "south pole", S, which appears at the intersection of edges 8 and 9.

Now face E is ORS, since it contains the endpoint of edge 3 (equal to edge 4) and the south pole. Thus edge 8 is OR and edge 11 is OS. We conclude that face F is OPS (since P is diametrically opposite R), so edge 2 is OP (and equal to edge 5), and edge 10 is PS. Finally, face G is PQS, so edge 1 is PQ (and equal to edge 6) and edge 9 is QS.

The missing face is thus QRS, and its edges are QR, RS, and SQ. These can be glued to the diagram along edge 7, 8, or 9, respectively.

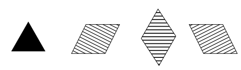
Team Number:

3. There are fourteen ordered pairs (a, b) with the property that the polynomial $x^3 - ax^2 + bx - 2024$ has three distinct positive integer zeros. What is the smallest possible value of a?

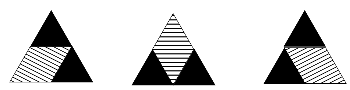
Solution: If the zeroes are r, s, and t, then we have $a = r + s + t$ and $2024 = rst$. To minimize the sum of three positive numbers with a fixed product, we want to minimize the largest factor and maximize the smallest. Since r, s, and t must be integers, and 2024 has prime factorization $2^3 \cdot 11 \cdot 23$, we must in fact have $\{r, s, t\} = \{8, 11, 23\}$. So $a = 42$ (and $b = (8)(11) + 8(23) + (11)(23) = 525$).

Team Number:

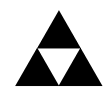
4. A triangular region can be subdivided into smaller triangles and then tiled with translates of the three diamond tiles and the upward-pointing triangle:



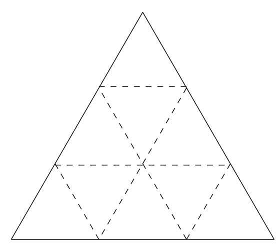
There are exactly three ways to tile an equilateral triangle of side-length two:



(Rotations aren't allowed, so the downward-pointing triangle isn't a tile, which means that arrangements like the picture on the right don't count as tilings.)



There are exactly eighteen ways to tile the equilateral triangle of side-length three:



If we were to build all eighteen of these tilings simultaneously, how many copies of the upward-pointing triangle would we need?

(If the question were about the triangle with side 2, the answer would be six, since we want to display all the tilings at once.)

Solution: The big triangle contains six upward-pointing triangles and three downward-pointing triangle. Since each of the diamond tiles covers one upward-pointing triangle and one downward-pointing triangle (and a diamond tile is necessary to cover each downward-pointing triangle), every tiling will require exactly three diamonds. This leaves exactly three upward-pointing triangles uncovered; we must use the triangle tile on each.

We conclude that each of the 18 tilings uses exactly three upward-pointing triangles, so we need $3 \cdot 18 = 54$ such tiles.

Team Number: [ ]

5. At halftime of the homecoming game, Pistol Pete stations 101 giant foam cowboy hats on the football field, one on each yard line (and the two goal lines). One lucky fan is chosen from the student section, and told to pick a hat. Before the fan picks the hat, Pete explains that each hat covers some hundred-dollar bills: The hat on the n-yard line contains a number of bills equal to the number of hats containing exactly n hundred-dollar bills.

   (The OSU goal line is considered the 0-yard line, and the opposing goal line is considered the 100-yard line. Similarly, the yard lines between the 50-yard line and the 100-yard line count upward, so for example the opposing 30-yard line is considered the 70-yard line, and so on.)

   Which hat should the fan pick? If she chooses her hat wisely, how much money does she get?

Solution: She should take the hat on the 0-yard line, which contains \$9700.

First observe that every hat contains between 0 and 101 bills. For i between 0 and 101, let c_i be the number of bills under the hat on the i-yard line. Then c_i is also the number of hats containing i bills, so

$$\sum_{i=0}^{101} c_i = (\text{the number of hats containing between 0 and 101 bills}) = 101. \qquad (\dagger)$$

Meanwhile, ic_i is the total number of bills inside hats containing exactly i bills, so

$$\sum_{i=0}^{101} ic_i = (\text{the number of bills inside hats}) = 101. \qquad (\ddagger)$$

Let $c_0 = a$, and assume $a < 100$. Then there are $101 - a$ nonempty hats, including hats 0 and a. This leaves $99 - a$ other nonempty hats, and $(\ddagger)$ says

$$101 = \sum_{i=0}^{101} ic_i \geq 0(a) + (a)(c_a) + \sum_{i=1}^{99-a} (i)(1) \geq a + \frac{(99-a)(100-a)}{2}. \qquad (*)$$

Some algebra reveals $a > 96$. If $a = 97$, then $c_{97} = 1$ so $c_1 \geq 1$ and there are two more bills to place. If they go in hats r and s we must have $r + s = 2$, so $r = 1$ and $s = 2$. In this case, we have $c_0 = 97$, $c_1 = 2$, $c_2 = 1$, $c_{97} = 1$, and all other $c_i = 0$.

If $a = 98$, then $c_{98} = 1$, so $c_1 \geq 1$ and all other $c_i = 0$. By $(\dagger)$, $c_1 = 2$. But then $c_2 \geq 0$, a contradiction.

Finally, if $a \geq 99$, then $c_a = 1$, so $c_1 \geq 1$, so $c_0 \leq 98$, a contradiction.