Team Number: $\boxed{}$

1. Goldbach's conjecture asserts that every even number greater than 4 can be written as the sum of two odd primes. It is not known whether this statement is true or false.

   On the other hand, it is known that every sufficiently large even number can be written as a sum of two odd *composite* numbers.

   How many positive even numbers cannot be written as a sum of two positive odd composite numbers?

   [0 is not positive, and 1 is neither prime nor composite.]

   Solution:

   Observe that the smallest positive odd composite number divisible by 3 is 9, but that every multiple of 3 greater than 9 is composite.

   Also observe that the smallest positive odd composite number that leaves a remainder of 1 on division by 3 is 25, and the smallest such that leaves a remainder of 2 on division by 3 is 35.

   Thus, for any n, we can write:

   - If n is a multiple of 3, then $n = 9 + (n - 9)$. As long as $n - 9 > 3$, i.e., $n > 12$, this decomposes n as a sum of two positive odd composites.
   - If n is one more than a multiple of 3, then $n = 25 + (n - 25)$. As long as $n - 25 > 3$, i.e., $n > 28$, this decomposes n as a sum of two positive odd composites.
   - If n is two more than a multiple of 3, then $n = 35 + (n - 35)$. As long as $n - 35 > 3$, i.e., $n > 38$, this decomposes n as a sum of two positive odd composites.

   The remaining even numbers are 6, 12; 4, 10, 16, 22, 28; 2, 8, 14, 20, 26, 32, and 38.

   None of these can be written as the sum of a positive odd multiple of three and another positive odd composite. Since the smallest sum of two positive odd composites that aren't multiples of 3 is $25 + 25 = 50$, none of them can be written as a sum of positive odd composites in any other way either.

   Counting, there are $\boxed{14}$ numbers with the desired property.

Team Number:

2. Remember that a *palindrome* is a number (or word) that reads the same backwards and forwards. For example, 353 and 2112 are palindromes.

   Frank begins driving his car, which has a six-digit odometer (which tells him how many miles the car has driven). He notices that the last four digits of the odometer form a palindrome. After driving one mile, he notices that the last five digits of the odometer form a palindrome. He drives another mile and notices that the middle four digits of the odometer are a palindrome. Finally, he drives one more mile, and all six digits of the odometer form a palindrome. There are two possible initial odometer readings that could have given Frank these observations. Find one of them.

   Solution: $\boxed{\text{199999 or 198888}}$.

   Let n be the initial odometer reading, and write $n = ABCDEF$.

   Since the last four digits are a palindrome, we in fact have $n = ABCDDC$.

   After Frank drives one mile, the odometer reads $ABCDDc$ (where $c = C + 1$) if $C \neq 9$, or $AB9Dd0$ (if $C = 9$ and $D \neq 9$), or $Ab0000$ (if $C = D = 9$ and $B \neq 9$), or $a00000$ (if $B = C = D = 9$ and $A \neq 9$), or 000000 if $A = B = C = D = 9$. We treat all five cases separately.

   First, suppose $C \neq 9$. Then BCDDc is a palindrome, so $B = c = C+1$ and $C = D$. Thus the original reading was $ABCCCC$. After Frank drives two miles, the reading is either $A98890$ (if $C = 8$) or $ABCCC\Gamma$ (if $C \neq 8$). Since the middle four digits form a palindrome and $B \neq C$, we conclude that $C = 8$, so B=9, and the original reading was $A98888$. After Frank drives three miles, the reading is $A98891$, which must be a palindrome. We conclude that $A = 1$, so the original reading in this case was 198888.

   Second, suppose $C = 9$ and $D \neq 9$. After one mile, the last five digits, $B9Dd0$, form a palindrome. Thus $B = 0$ and $9 = d$, so $D = 8$, and the original reading was $A09889$. After two miles, the reading is $A09891$, and the middle four digits, 0989, do not form a palindrome.

   Third, suppose $C = D = 9$ and $B \neq 9$. After one mile, the last five digits, $b0000$, form a palindrome, so $b = 0$ and $B = 9$, a contradiction.

   Fourth, suppose $B = C = D = 9$ and $A \neq 9$. After one mile, the reading is $a00000$, and the last five digits, 00000, do form a palindrome. After two miles, the reading is $a00001$, and the middle four digits, 0000, do form a palindrome. After three miles, the reading is $a00002$ and is supposed to be a palindrome. Thus $a = 2$, so $A = 1$. The initial reading was 199999.

   Finally, suppose that $A = B = C = D = 9$, so that the initial reading was 999999. After one mile, the reading is 000000, and the last five digits, 00000, form a palindrome. After two miles, the reading is 000001, and the middle four digits, 0000, do form a palindrome. After three miles, the reading is 000002, which is not a palindrome.

Team Number: $\boxed{}$

3. The equation

$$2023 = 9^3 + 8^3 + 7^3 + 6^3 + 5^3 + 4^3 + 3^3 + 2^3 + 1^3$$

is false, but can be made true by changing some of the plus signs to minus signs. How many signs need to be changed?

Solution: The answer is $\boxed{1}$.

The right-hand side of the original expression adds to $\left(\frac{(9)(9+1)}{2}\right)^2 = 2025$, which exceeds the left-hand side by 2. Thus we need to reverse the sign on summands adding to $\frac{2}{2} = 1$. The unique way to do this is to change the right-hand side to

$$9^3 + 8^3 + 7^3 + 6^3 + 5^3 + 4^3 + 3^3 + 2^3 - 1^3$$

Team Number:

4. King Arthur's round table has 2023 chairs, arranged in a circle. K of Arthur's knights sit around the table in such a way that no two knights are in adjacent chairs, but a $(K+1)^{\text{st}}$ knight would not be able to join without sitting next to somebody.

   How many values of K are possible?

   Solution: If we want to pack as many knights as possible around the table, we need to leave one empty seat between each knight. The easiest way to do this is to put the knights at the even-numbered seats, $\{2, 4, 6, \ldots, 2022\}$. We conclude that the largest possible value of K is $\frac{2022}{2} = 1011$.

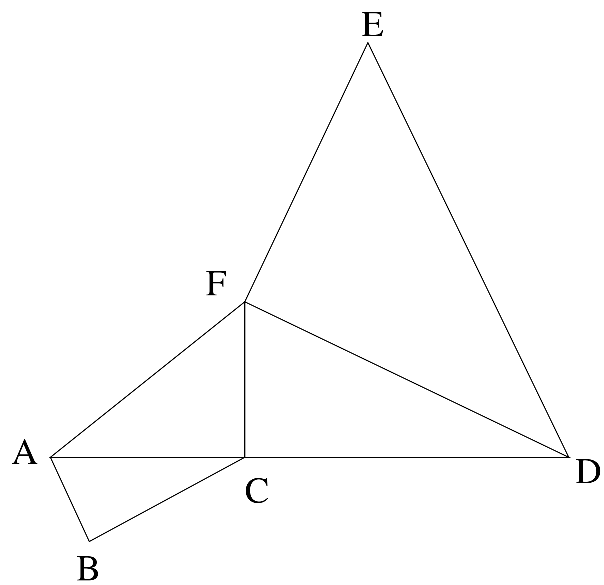
   If instead we want to minimize the number of knights, we need to leave two seats between each knight. The easiest way to do this is to put them at seats numbered with a multiple of three, $\{3, 6, 9, \ldots, 2022\}$. This accounts for $\frac{2022}{3} = 674$ knights. But the table is circular, and seats 2023, 1, and 2 are left empty, so a 675^{th} knight can be placed at seat 1.

   Thus K could be any number between 675 and 1011, inclusive. There are $1011 - 675 + 1 = \boxed{337}$ possible values of K.

Team Number:

5. In the figure (not drawn to scale), triangles ABC, ACF, CDF and DEF are all right triangles (with right angles at B, C, C, and F, respectively).

   If all sides of all four triangles have integer length, and $AB = 3$, there are several possibilities for the length of DE. Find two of them.



Solution: The possible values of DE are $\boxed{\text{17, 25, 29, 39, 52, 85, 101, 113, and 685}}$.

Since $AB = 3$, we have $BC = 4$ and $AC = 5$. Then $AF = 13$ and $CF = 12$.

There are four Pythagorean triples with 12 as a leg, namely $(5, 12, 13)$, $(9, 12, 15)$, $(12, 16, 20)$, and $(12, 35, 37)$. Thus DF is equal to 13, 15, 20, or 37.

If $DF = 13$, the sides of DEF are $(13, 84, 85)$.

If $DF = 15$, the sides of DEF are $(8, 15, 17)$, $(15, 20, 25)$, $(15, 36, 39)$, or $(15, 112, 113)$.

If $DF = 20$, the sides of DEF are $(15, 20, 25)$, $(20, 21, 29)$, $(20, 48, 52)$, or $(20, 99, 101)$.

If $DF = 37$, the sides of DEF are $(37, 684, 685)$.