Team Number:

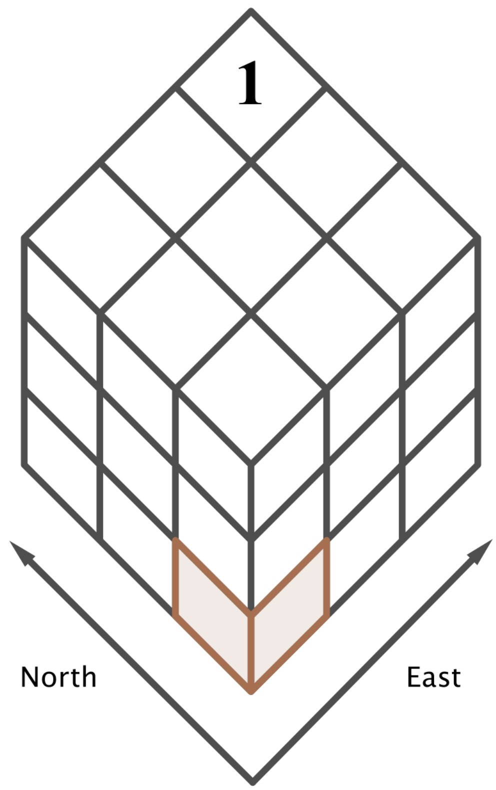
1. Find two positive numbers $x \neq y$ with the property that $x^y = y^x$.

Solution: While there are infinitely many solutions, the simplest are $x = 2$, $y = 4$ and $x = 4, y = 2$.

Team Number:

2. A mouse-house is made up of 27 rooms arranged in a $3 \times 3 \times 3$ cube, as shown in the figure below. A mouse begins on the lowest floor, in the southwest corner room. For each move, he can travel to the room above, the room to the east, or the room to the north.

   The mouse wishes to travel from his starting room (shaded in the figure) to the room in the northeast corner of the top floor (marked "1"). How many ways are there for him to do this?



Solution: The mouse must make a total of six steps, two in each of the directions North, East, and Up. In other words, every path corresponds to a rearrangement of the letters NNEEUU.

There are $\binom{6}{2,2,2} = \frac{6!}{2!2!2!} = 90$ possible ways to accomplish this.

Team Number:

3. A seating section in a corner of Boone Pickens Stadium is arranged as a trapezoid: There are thirty rows, each of which contains one more seat than the row immediately in front of it. If row 7 contains 12 seats, how many seats are in the section?

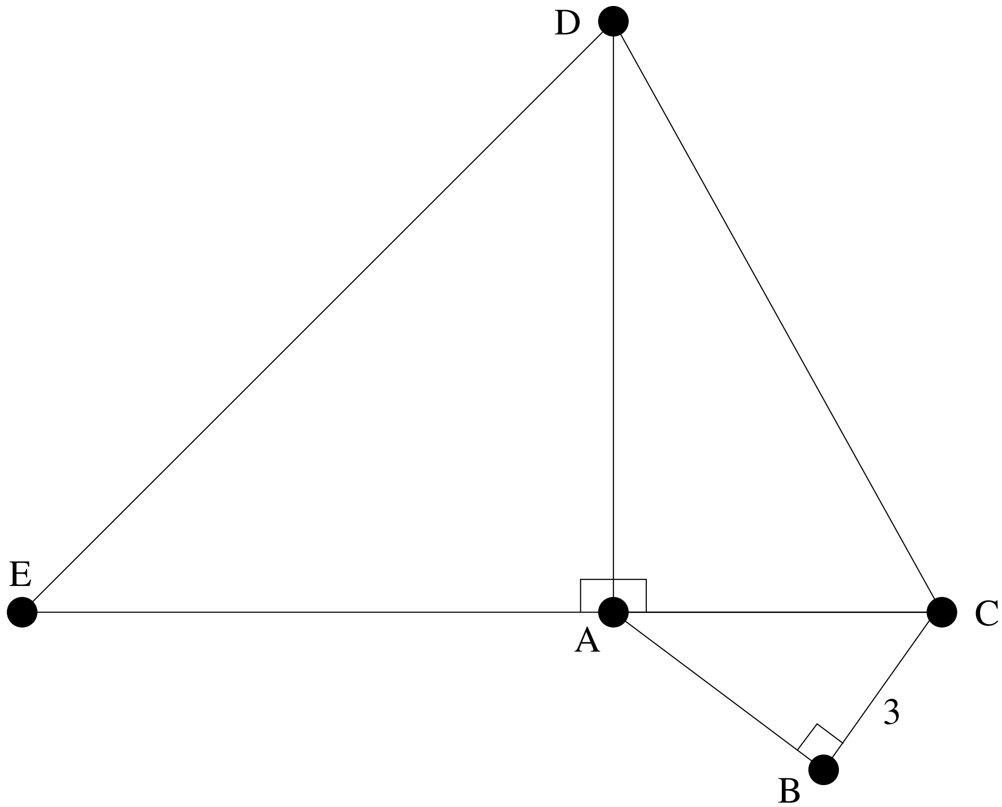
Solution: Apparently, row n will contain $n + 5$ seats, so Row 1 has 6 seats and Row 30 has 35 seats.

The total number of seats is thus

$$6 + 7 + 8 + \cdots + 35 = 30\left(\frac{6 + 35}{2}\right) = 615.$$

Team Number:

4. In the figure below, all the line segments (AB, AC, AD, AE, BC, CD, and DE) have integer length, and angles ABC, CAD, and DAE are right angles. If $BC = 3$, find a possible value of DE. (There is more than one correct answer.)



Set $AB = a$ and $AC = b$. Then, by the Pythagorean theorem, $9 = b^2 - a^2 = (b+a)(b-a)$. So either $b + a = b - a = 3$ or $b + a = 9$, $b - a = 1$. The only integer solution is $a = 4$, $b = 5$.

Now set $AD = c$ and $CD = d$. A similar argument yields $d^2 - c^2 = 25$, which has unique integer solution $c = 12$, $d = 13$.

Finally, set $AE = e$, $DE = f$. We have $f^2 - e^2 = 144$, which has several solutions (corresponding to the factorizations of 144 as a product of two distinct even numbers: If $f - e = 2, f + e = 72$, then $e = 35, f = 37$. If $f - e = 4, f + e = 36$, then $e = 16, f = 20$. If $f - e = 6, f + e = 24$, then $e = 9, f = 15$. If $f - e = 8, f + e = 18$, then $e = 5, f = 13$. So the possible answers are 13, 15, 20, and 37.

Team Number:

5. Call a set of natural numbers *self-referential* if it contains its size. For example, $\{1, 3, 4\}$ is self-referential because it contains 3, but $\{1, 5\}$ is not, because it does not contain 2. How many self-referential subsets of $\{1, 2, 3, 4, 5\}$ are there?

Solution: There are $\binom{4}{0} = 1$ self-referential subsets of size 1: We must have 1 and zero of the other four numbers.

There are $\binom{4}{1} = 4$ self-referential subsets of size 2: We must have 2 and one of the other four numbers.

There are $\binom{4}{2} = 6$ self-referential subsets of size 3: We must have 3 and two of the other four numbers.

There are $\binom{4}{3} = 4$ self-referential subsets of size 4: We must have 4 and three of the other four numbers.

There are $\binom{4}{4} = 1$ self-referential subsets of size 5: We must have 5 and four of the other four numbers.

Consequently, $\{1, 2, 3, 4, 5\}$ has $\binom{4}{0} + \binom{4}{1} + \binom{4}{2} + \binom{4}{3} + \binom{4}{4} = 2^4 = 16$ self-referential subsets.