Team Number:

1. How many ordered pairs of nonnegative integers satisfy the equation $x^2 - y^2 = 2^{22}$? [One solution is $(x, y) = (2^{11}, 0)$].

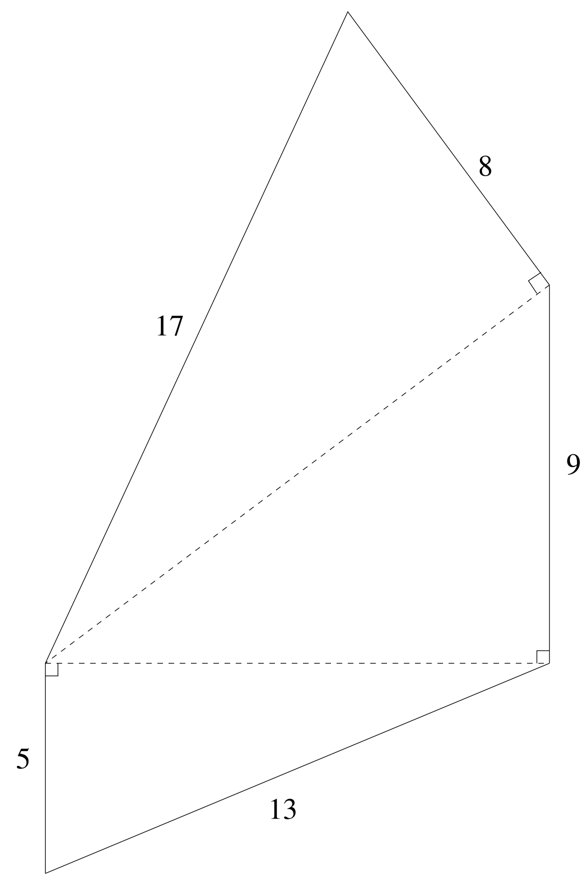
Answer: The left-hand side factors as $(x+y)(x-y)$, so $(x+y)$ and $(x-y)$ must be factors of 2^{22}. The possibilities are:

$$\begin{aligned}
x+y=2^{11}, x-y=2^{11} &\qquad (x=2^{11}, y=0),\\
x+y=2^{12}, x-y=2^{10} &\qquad (x=5(2^9), y=3(2^9)),\\
x+y=2^{13}, x-y=2^{9} &\qquad (x=(2^4+1)(2^8), y=(2^4-1)(2^8)),\\
x+y=2^{14}, x-y=2^{8} &\qquad (x=(2^6+1)(2^7), y=(2^6-1)(2^7)),\\
x+y=2^{15}, x-y=2^{7} &\qquad (x=(2^8+1)(2^6), y=(2^8-1)(2^6)),\\
x+y=2^{16}, x-y=2^{6} &\qquad (x=(2^{10}+1)(2^5), y=(2^{10}-1)(2^5)),\\
x+y=2^{17}, x-y=2^{5} &\qquad (x=(2^{12}+1)(2^4), y=(2^{12}-1)(2^4)),\\
x+y=2^{18}, x-y=2^{4} &\qquad (x=(2^{14}+1)(2^3), y=(2^{14}-1)(2^3)),\\
x+y=2^{19}, x-y=2^{3} &\qquad (x=(2^{16}+1)(2^2), y=(2^{16}-1)(2^2)),\\
x+y=2^{20}, x-y=2^{2} &\qquad (x=(2^{18}+1)(2^1), y=(2^{18}-1)(2^1)),\\
x+y=2^{21}, x-y=2^{1} &\qquad (x=(2^{20}+1)(2^0), y=(2^{20}-1)(2^0)).
\end{aligned}$$

This is $\boxed{11}$ solutions.

Team Number:

2. Find the area of the pentagon below.



Answer: We use the Pythagorean theorem to fill in the missing sides. This gives us right triangles of side-lengths 5-12-13, 9-12-15, and 8-15-17. Their areas are 30, 54, and 60, respectively. So the pentagon has area $30 + 54 + 60 = \boxed{144}$.

Team Number:

3. Find 3 consecutive positive integers such that the sum of the square of the smallest integer and the square of half of the middle integer equals the square of the largest integer.

Answer: Let x be the middle integer. Then

$$(x-1)^2 + (\frac{x}{2})^2 = (x+1)^2.$$

Simplifying, $-2x + \frac{x^2}{4} = 2x$. Thus $x^2 = 16x$, so $x = 16$. The answer is $\boxed{15, 16, 17}$.

Team Number:

4. Say a 10 digit number $A = a_0 a_1 \cdots a_9$ is **self-referential** if each a_i is the number of digits in A equal to i. Find a self-referential number.

   Answer: The only self-referential number is $\boxed{6210001000}$.

   Suppose A is self-referential. We begin by deriving two equations. First, there are ten digits, so we have

   $$\begin{aligned} 10 &= (\text{number of 0's}) + (\text{number of 1's}) + \cdots + (\text{number of 9's}) \\ 10 &= a_0 + a_1 + \cdots + a_9 \end{aligned}$$

   Second, again since there are ten digits, we have

   $$\begin{aligned} 10 = {} & (\text{number of digits appearing once}) + 2(\text{number of digits appearing twice}) + \ldots \\ & + 8(\text{number of digits appearing eight times}) + 9(\text{number of digits appearing nine times}) \\ 10 = {} & (\text{number of 1's}) + 2(\text{number of 2's}) + \cdots + 9(\text{number of 9's}) \\ 10 = {} & a_1 + 2a_2 + 3a_3 + 4a_4 + \cdots + 9a_9. \end{aligned}$$

   We make heavy use of both equations, especially the second.

   By the second equation, a_9 is at most 1. If $a_9 = 1$, then $a_1 = 1$ and $a_0 = 8$ by the second equation. But then $a_8 \geq 1$, which is impossible. So $a_9 = 0$.

   Similarly, a_8 is also at most 1. If $a_8 = 1$, then $a_1 \geq 1$, so by the second equation $a_1 = 2$. But then $a_2 = 1$, which is impossible. So $a_8 = 0$.

   If $a_7 = 1$, then $a_1 \geq 1$. So, by the second equation, either $a_1 = 3$ (so $a_3 \geq 1$) or $a_1 = 1$ and $a_2 = 2$ (which is impossible since a_1 and a_7 are both 1's, forcing $a_1 \geq 2$). So $a_7 = 0$.

   If $a_6 = 1$, then some digit has to be a 6. By the second equation, that digit must be a_0. Now $a_1 + a_2 + a_3 + a_4 + a_5 = 3$, but $a_1 + 2a_2 + 3a_3 + 4a_4 + 5a_5 = 4$. It follows that $a_1 = 2$ and $a_2 = 1$, which yields the self-referential number 6210001000.

   Now suppose $a_6 = 0$. Observe that $a_0 \geq 4$. If $a_5 = 2$, then $a_2 \geq 1$, contradicting the second equation. If $a_5 = 0$, then $a_0 \geq 5$, so a_{a_0} must not be 0, a contradiction. So $a_5 = 1$. If $a_4 = 1$, then $a_1 \geq 2$, violating the second equation. So $a_4 = 0$, and thus $a_0 \geq 5$. By the first equation, we have $a_0 = 5$, so a_1, a_2, and a_3 must be nonzero. But this contradicts the second equation, since $a_1 + 2a_2 + 3a_3 + 5a_5 \geq 11$.

   So 6210001000 is the only self-referential number.

Team Number:

5. A number is called **very prime** if it is prime and its base-ten representation is prime when viewed as a base-eight expression.

   Some examples of very prime numbers are 2, 3, 5, 7, 13, and 2017. (13 is very prime because $13_8 = 1 \times 8 + 3 \times 1 = 11_{10}$ is prime, and 2017 is very prime because $2017_8 = 2 \times 8^3 + 0 \times 8^2 + 1 \times 8 + 7 \times 1 = 1033_{10}$ is prime. On the other hand, 11 is not very prime because $11_8 = 1 \times 8 + 1 = 9_{10}$ is composite. Numbers like 19 and 83 are not very prime, the base-eight expressions 19_8 and 83_8 don't make sense.)

   Find a very prime three-digit number.

   Answer: The first three-digit prime is 101. This is not very prime, since $101_8 = 8^2 + 1$ is divisible by 5. The next three-digit prime is 103, and $103_8 = 64 + 3 = 67$ is prime. Thus one possible answer is $\boxed{103}$.

   The complete list of very prime three-digit numbers is 103, 107, 131, 211, 227, 263, 277, 307, 337, 373, 401, 431, 433, 463, 467, 521, 541, 547, 557, 577, 631, 643, 661, 673, 701.