Team Name: $\boxed{}$

1. Remember that a *palindrome* is a number (or word) that reads the same backwards and forwards. For example, 353 and 2112 are palindromes.

   Observe that the base 2 representation of 2015 is a palindrome. Find the next year with a palindromic base 2 representation. (Express your answer in base ten.)

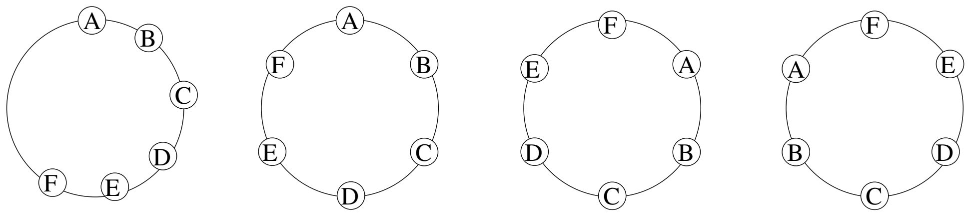
   > (The base ten representation of 2015 is 2015, because 2, 0, 1, and 5 are the coefficients in the expression
   >
   > $$2015 = 2(10^3) + 0(10^2) + 1(10^1) + 5(10^0).$$
   >
   > The base two representation is the sequence of coefficients when we replace those powers of ten with powers of two. For example, the base two representation of $17 = 2^4 + 2^0$ is 10001, and the base two representation of 2015 is 11111011111.)

   Solution: The next palindrome, written is base two, is 11111111111. Converting to base ten, this is $2^{10}+2^9+2^8+2^7+2^6+2^5+2^4+2^3+2^2+2^1+2^0$, which is a geometric series with first term 2^{10}, common ratio $\frac{1}{2}$, and first missing term 2^{-1}. Consequently, it simplifies to $\dfrac{2^{10}-2^{-1}}{\frac{1}{2}} = 2^{11}-1 = \boxed{2047}$.

Team Name:

2. A closed necklace is to be made from six different jewels. How many different types of necklaces can be made?

   (We say two necklaces are the same type if one can be obtained from the other by sliding the jewels along the chain (but not across one another), by rotating the entire necklace, or by turning it over. Thus, the four necklaces below are all of the same type.)



Solution: If we fix the positions of the jewels, as in the three necklaces on the right above, there are $6! = 720$ ways to arrange the jewels. But we can rotate any necklace into six different positions, and flip it over from any of those positions to make six more. Thus, each possible necklaces has been counted twelve times. Consequently, the answer is $\frac{720}{12} = \boxed{60}$.

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3. For any two people, we assume that there are only two options: Either the two people are strangers to each other or they are friends. What is the smallest number of people you need to invite to a party so that you are guaranteed that either (i) there are three people there who are all strangers to one another or (ii) there are three people there who are all friends?

Solution: First, the answer must be larger than five: We can arrange five people on the vertices of a pentagon; if each is friends with the people on adjacent vertices but strangers with the people on opposite vertices, the condition is not satisfied.

Next, assume that we have six people at the party, and call them A, B, C, D, E, and F. Now A either has at least three friends, or is strangers with at least three of the others. Assume that A is friends with B, C, and D. (The argument will be symmetric if they are strangers.) Then, if any two of B, C, and D are friends, it follows that they and A are three people who are all friends. But if no two of B, C, and D are friends, then they are all strangers.

Thus the answer is $\boxed{6}$.

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4. A set is *felicitous* if it satisfies two properties:

   - Its elements are all integers between 1 and 2015, inclusive.
   - The product of any two of its elements is a perfect square.

   What is the size of the largest possible felicitous set?

Solution: Suppose that x and y are both elements of the same felicitous set, and take their prime factorizations:

$$x = 2^{e_2}3^{e_3}5^{e_5}7^{e_7}\dots$$
$$y = 2^{f_2}3^{f_3}5^{f_5}7^{f_7}\dots$$

Then, since their product is a perfect square, it follows that for every prime p, $e_p + f_p$ must be even, i.e., e_p and f_p must be either both even or both odd. Let z be the product of all primes p such that e_p is even. Then x and y both factor as a perfect square times z, and every other element of our set must factor this way as well.

So the largest felicitous set is obtained by taking $z = 1$ and multiplying it by every perfect square less than 2015. Since $45^2 = 2025$ and $44^2 = 1936$, the set in question is $\{1, 4, 9, 16, \dots, 1936\} = \{1^2, 2^2, 3^2, \dots, 44^2\}$. It has $\boxed{44}$ elements.

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5. A *primitive Pythagorean quadruple* is an ordered quadruple of positive integers (a, b, c, d) satisfying $a \leq b \leq c \leq d$, $a^2 + b^2 + c^2 = d^2$ and $\gcd(a, b, c, d) = 1$. One example of a primitive Pythagorean quadruple is $(1, 2, 2, 3)$. Find three others.

Solution: One way to generate these is by chaining Pythagorean triples together. For example, since $3^2 + 4^2 = 5^2$ and $5^2 + 12^2 = 13^2$, it follows that $\boxed{(3, 4, 12, 13)}$ is a Pythagorean quadruple. Since, for any odd number n, the triple $\left(n, \frac{n^2-1}{2}, \frac{n^2+1}{2}\right)$ is a primitive Pythagorean triple, we can generate infinitely many quadruples this way. The next two after $(3, 4, 12, 13)$ are $\boxed{(5, 12, 84, 85)}$ and $\boxed{(7, 24, 312, 313)}$. But the chained triples don't have to both be primitive. For example, we could combine $(9, 12, 15)$ with $(15, 20, 25)$ to obtain $\boxed{(9, 12, 20, 25)}$.

Not every Pythagorean quadruple is obtained from two triples. In addition to $(1, 2, 2, 3)$, other small primitive quadruples are $\boxed{(2, 3, 6, 7)}$, $\boxed{(1, 4, 8, 9)}$, and $\boxed{(4, 4, 7, 9)}$.