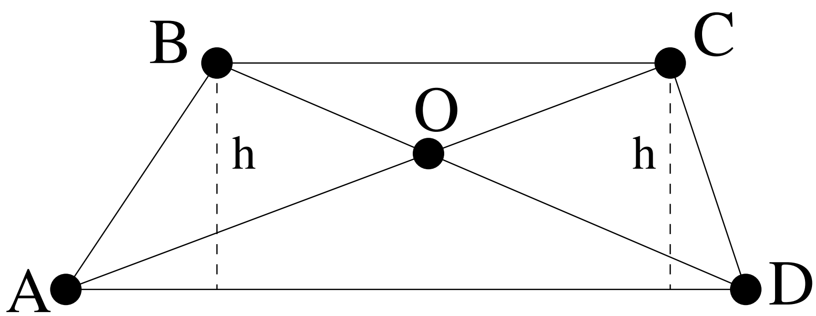
Part II. Complete solution problems

1. Diagonals AC and BD of a trapezoid $ABCD$ intersect at point O. Assuming that AD is parallel to BC, prove that the areas of triangles ABO and CDO are equal.

Solution: Let h be the height of the trapezoid, i.e., the length of a perpendicular from AD to BC. Then triangles ABD and ACD both have area equal to $\frac{1}{2}(AD)(h)$, and in particular have equal area.



Now, triangle ABD is the union of triangles ABO and ADO, and triangle ACD is the union of triangles ADO and CDO. Thus,

$$\begin{aligned} \text{area}(ABD) &= \text{area}(ACD) \\ \text{area}(ABO) + \text{area}(ADO) &= \text{area}(ADO) + \text{area}(CDO) \\ \text{area}(ABO) &= \text{area}(CDO), \end{aligned}$$

which was what we wanted.

2. Show that $a^4 + b^4 \geq 1/8$ for all real numbers a and b satisfying $a + b = 1$.

*Solution:*The Arithmetic Mean - Geometric Mean inequality states that

$$\frac{x+y}{2} \geq \sqrt{xy}$$

for all positive numbers x and y. Setting $x = X^2$ and $y = Y^2$ yields

$$\frac{X^2+Y^2}{2} \geq XY$$

for all real X and Y. (If XY is negative, then certainly $XY \leq \sqrt{X^2Y^2}$.)
Adding $\frac{X^2+Y^2}{2}$ to both sides yields

$$\begin{aligned} X^2 + Y^2 &\geq \frac{X^2 + Y^2 + 2XY}{2} \\ X^2 + Y^2 &\geq \frac{(X+Y)^2}{2}. \qquad (*) \end{aligned}$$

Applying $(*)$ with $X = a$ and $Y = b$ gives us

$$a^2 + b^2 \geq \frac{(a+b)^2}{2} = \frac{1^2}{2} = \frac{1}{2},$$

so $a^2 + b^2 \geq \frac{1}{2}$. Applying $(*)$ again, this time with $X = a^2$ and $Y = b^2$, gives us

$$a^4 + b^4 \geq \frac{(a^2+b^2)^2}{2} \geq \frac{(\frac{1}{2})^2}{2} = \frac{1}{8},$$

as desired.

3. Let $A = (c, 0)$ and $B = (-c, 0)$ be distinct points in the plane, and let k be a positive number different from 1. Show that all points P such that $|\overline{AP}|/|\overline{BP}| = k$ lie on a circle with center on the line through A and B. What is the location of points P when $k = 1$?

Solution: We may assume without loss of generality that $c = 1$ (if not, we can divide all coordinates by c; this divides all distances by c and preserves circles and ratios between distances).

Let such a point P be given, and write $P = (x, y)$. Then we have

$$\begin{aligned}
\frac{\sqrt{(x-1)^2 + y^2}}{\sqrt{(x+1)^2 + y^2}} &= k \\
\sqrt{(x-1)^2 + y^2} &= k\sqrt{(x+1)^2 + y^2} \\
(x-1)^2 + y^2 &= k^2(x+1)^2 + k^2y^2 \\
(k^2-1)x^2 + (2k^2+2)x + (k^2-1) + (k^2-1)y^2 &= 0
\end{aligned}$$

Now we must handle the cases $k = 1$ and $k \neq 1$ separately. If $k = 1$, this equation becomes $4x = 0$, telling us that P lies on the y-axis (i.e., the perpendicular bisector of $\overline{AB}$). If $k \neq 1$, we can divide by $k^2 - 1$, yielding

$$\begin{aligned}
x^2 + \frac{2(k^2+1)}{k^2-1}x + 1 + y^2 &= 0 \\
x^2 + \frac{2(k^2+1)}{k^2-1}x + \left(\frac{k^2+1}{k^2-1}\right)^2 + y^2 &= \left(\frac{k^2+1}{k^2-1}\right)^2 - 1 \\
\left(x + \frac{k^2+1}{k^2-1}\right)^2 + y^2 &= \left(\frac{k^2+1}{k^2-1}\right)^2 - 1,
\end{aligned}$$

which is the equation of a circle centered at $C = (-\frac{k^2+1}{k^2-1}, 0)$; this point is on the line connecting A and B.

(Strictly speaking, one must verify that the right-hand side, $\left(\frac{k^2+1}{k^2-1}\right)^2 - 1$, is non-negative. This can be done algebraically, but it is simpler to observe that the point P satisfies the equation; since the left-hand side is non-negative at P, the right-hand side must be as well. Since the right-hand side is constant, we conclude that it's non-negative, so we have the equation of a circle.)

4. What are the last 2015 digits of 2010^{2010}? (For example, the last three digits of 2010 are 010.)

Solution: We factor $2010^{2010} = 201^{2010}(10^{2010})$. Thus, the last 2010 digits are all zeroes, and we need to find the last five digits of 201^{2010}.

By the binomial theorem,

$$\begin{aligned} 201^{2010} &= (1+200)^{2010} \\ &= 1^{2010} + \binom{2010}{1}(1^{2009})(200) + \binom{2010}{2}(1^{2008})(200^2) + \text{other terms.} \end{aligned}$$

All the other terms are divisible by $(200)^3$, so they end in at least six zeroes and have no effect on our computation. We compute the last five digits of

$$\begin{aligned} &1^{2010} + \binom{2010}{1}(1^{2009})(200) + \binom{2010}{2}(1^{2008})(200^2) \\ &= 1 + (2010)(200) + \frac{2010 \cdot 2009}{2}(200)(200) \\ &= 1 + 402000 + (2010)(2009)(100)(200). \end{aligned}$$

The third term is divisible by 10^5, so, restricting our attention to the last five digits, we are looking for

$$1 + 02000 + 00000 = 02001.$$

Thus the last 2015 digits of 2010^{2010} are 02001 followed by 2010 zeroes.

5. Consider an 8×8 chessboard with two squares in diagonally opposite corners removed. Is it possible to tile this board with 2×1 dominoes?

Solution: Color the squares orange and black in the standard checkerboard pattern (that is, each orange square shares edges only with black squares, and vice versa). Then there are 32 squares of each color, but diagonally opposite squares have the same color, so after the two squares are removed, there are 32 squares of one color and 30 squares of the other. Every domino covers one square of each color, so after the first 30 dominoes are placed, the last two squares will share a color and in particular will not be adjacent. Thus no tiling is possible.