Hints for the 2024 OSU individual competition

1. To be a multiple of 2024, the number must be a multiple of both 8 and 11. To check for divisibility by 8, ignore all put the last three digits. To check for divisibility by 11, take the alternting sum of the digits.

2. A diagonal of the small square is a side of the big square.The ratio between the diagonal and side of a square is $\sqrt{2}$.

3. There are 16 home fields, each of which hosts 15 games.

4. If we draw radii to the two points where the circle meets any side, we'll find equilateral triangles. The area inside the circle and outside the square is four copies of a 60-degree arc, minus four equilateral triangles.

5. The total number of dances involving a man and a woman is (12 men) times ((3 dances per man).

6. Each square of toilet paper has a (small, but nonzero) thickness, so removing each square has the same effect on the cross-section of the roll shown in the diagram. The width is the diameter of the roll. Don't forget the cardboard tube in the middle of the roll!

7. The probability of (Orange then Black) is $\frac{5}{11} \times \frac{6}{10}$. We also need the probability of (Black then Orange).

8. We know the values of $\frac{a_1 + a_2 + \cdots + a_{2023}}{2023}$ and $\frac{a_1 + a_2 + \cdots + a_{2024}}{2024}$. If we get rid of the denominators and subtract, we should be left with a_{2024}.

9. This is a variant of the AM-GM inequality. (If we know the perimeter of a rectangle, how should we maximize its area?)

10. This is a geometric series with first term 2024 and common ratio $\frac{3}{11}$.

11. Clearing the denominators yields $x^2 - 7x + 6 = x^2 - (2+k)x + (2k)$, which simplifies to a linear equation. It can have no solution only if the x coefficient is zero. But clearing the denominators risks introducing a false solution of $x = 2$ or $x = 6$, so we have to avoid those possibilities as well.

12. Set $b_1 = a_1 + a_2 + \cdots + a_{22}$, $b_2 = a_{23} + a_{24} + \cdots + a_{44}$, and so on. Then we are told the values of b_1 and $b_1 + b_2$, and asked for $b_1 + \cdots + b_{92}$.

13. By the change-of-base formula, this is $\frac{\frac{\ln 253}{\ln 4}}{\frac{\ln 253}{\ln 8}}$.

14. The diagram consists of three semicircles and a triangle. Refer to the areas of the smaller semicircles as X and Y, the area of the larger semicircle as Z, and the area of the triangle as T. If we were to color the triangle in orange, the remaining white area would be $Z - T$, and the shaded area would be $X + Y -$ (the white area). To finish, we'll also need the Pythagorean theorem.

15. The three angles at B must add to 2π. We know two of the interior angles: $\frac{2\pi}{4}$ and $\frac{3\pi}{5}$. The third angle is $\frac{(n-2)\pi}{n}$.

16. One of the five factors must be divisible by 3, and one of them must be divisible by 5. Given any other prime p, we can choose n so that none of the factors is divisible by p (e.g., $n = p$ works for all p except 7.) The last step is to check that the product doesn't have to be a multiple of 9 or 25.

17. Line CX is the perpendicular bisector of AB, say at P. Then APX is a 30-60-90 triangle. Setting $r = AP$ tells us PX and hence CP in terms of r. Applying the Pythagorean theorem to ACX reveals that ACX is also a 30-60-90 triangle, which means that A, C, and Y are collinear.

18. If no heads are tossed (probability $\frac{1}{4}$), then the sum is automatically even. Otherwise, the probability of an odd sum is $\frac{1}{2}$.

19. Let s stand for the length of a side. Then the area of the triangle is $\frac{s^2\sqrt{3}}{4}$. But the area is also the sum of the areas of triangles ABP, ACP, and BCP.

20. We can compute $f(f(f(x))) = x$. Since $f^3(x) = x$, we also have $f^{2022}(x) = x$.

21. Since $2024 = 2^3 \cdot 11^1 \cdot 23^1$, it has $(3+1)(1+1)(1+1) = 16$ positive integer factors. Observe that we may also write $16 = (1+1)(1+1)(1+1)(1+1)$.

22. Drop perpendiculars from the incenter O to each of the sides, meeting at X, Y, and Z. Observe that $BXOY$ is a square, and that $AZ = AY$ and $CZ = CX$.

23. It is enough to find the remainder upon division by both 8 and 125. Since 2024 is already divisible by 8, so is 2024^{2024}. Meanwhile, the remainder on dividing 2024 by 125 is $24 = 5^2 - 1$. By the binomial theorem,

$$2024^{2024} = (\text{Things divisible by } 5^3) - 2024(5^2) + 1^{2024},$$

which has a remainder of 26 when divided by 125. It remains to find a number between 000 and 999 that has a remainder of 0 when divided by 8 and a remainder of 26 when divided by 125.

24. The power of a point theorem tells us that $(AP)(BP) = (CP)(DP)$.

25. Notice that

$$\begin{aligned} S &= (1!2!)(3!4!)\cdots(99!100!) \\ &= (2\cdot(1!)^2)(4\cdot(3!)^2)\cdots(100\cdot(99!)^2) \\ &= (\text{a square})(2\cdot 4\cdot\cdots\cdot 100) \\ &= (\text{a square})(2^{50})(1\cdot 2\cdot\cdots\cdot 50) \\ &= (\text{a square})(50!). \end{aligned}$$

This tells us right away that $k = 50$ works. To eliminate everthing else, first observe that counting the copies of 53 forces $k \leq 52$ and counting the factors of 47 forces $k \geq 47$. Counting the copies of 2 or 3 eliminates the remaining candidates.