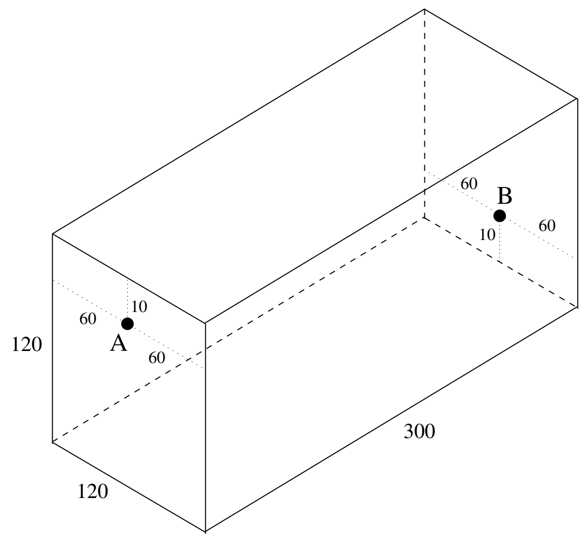
Part II. Team Round

1. Oh no! OSU mascot Pistol Pete has been kidnapped, stripped of his signature cowboy hat and gun, and abandoned in the middle of the desert. Upon investigation, Pete discovers that he's on an "island" surrounded by a perfectly circular moat, which is in turn patrolled by a vicious mascot-eating shark. The moat is narrow enough for Pete to jump across, but if he tries to jump over the shark it will leap up and devour him. Unfortunately the shark swims four times as fast as Pete can run, so it is always waiting for him whenever he tries to beat it by running across the island in a straight line.

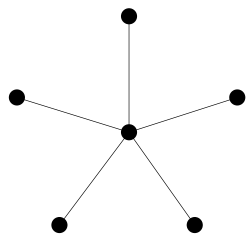
   Explain how Pete can beat the shark to a spot on the moat and escape.

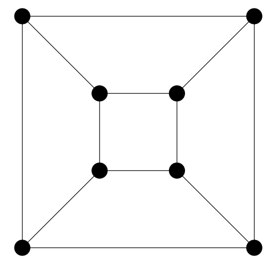
2. The Fibonnaci numbers are defined by $F_0 = F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for all integers $n \geq 2$. Prove that $F_n \leq \left(\frac{5}{3}\right)^n$ for all $n \geq 0$.

3. An abandoned alien spaceship is shaped like a rectangular prism with dimensions $300' \times 120' \times 120'$. An astronaut wearing magnetic shoes lands on the "front" of the spaceship at point A, $10'$ from the "top" and midway between the adjacent walls. Her shoes allow her to walk comfortably anywhere on the outside surface, and she wants to get to point B, on the "back" of the ship, $10'$ from the "bottom" and again midway between the adjacent walls. What is the length of the shortest route she can walk from point A to point B?



4. A *connected planar graph* is a nonempty collection of line segments (called *edges*) in the plane, with the properties that no two edges intersect except perhaps at a common endpoint, and that any two endpoints (or *vertices*) are connected by a sequence of edges. Two examples are the five-pointed star and the projection of a cube shown below.





Every connected planar graph divides the plane into a number of connected regions, called *faces*. If V, E, and F are the number of vertices, edges, and faces, then we have $(V, E, F) = (6, 5, 1)$ for the star and $(V, E, F) = (8, 12, 6)$ for the cube above. (The "outside face" counts as a face.) Prove that $V - E + F = 2$ for any connected planar graph with finitely many edges.

5. A “truncated icosahedron” or “soccer ball” is a polyhedron with pentagonal and hexagonal faces arranged according to the following rules:

   (a) No two pentagons share an edge.

   (b) The edges of each hexagon alternately meet pentagons and hexagons.

   (c) Exactly one pentagon and two hexagons meet at each vertex.

   If a truncated icosahedron has H hexagonal and P pentagonal faces, find all possible values for (H, P).