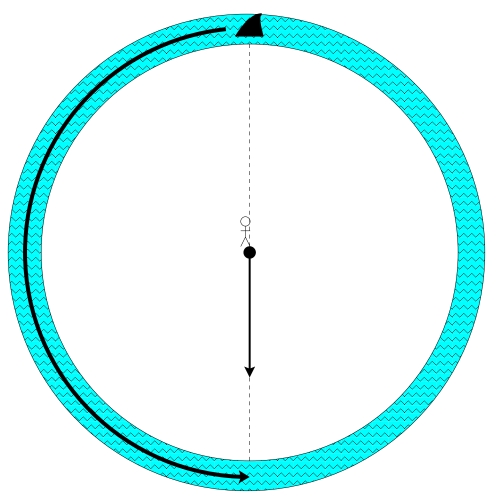
Part II. Team Round

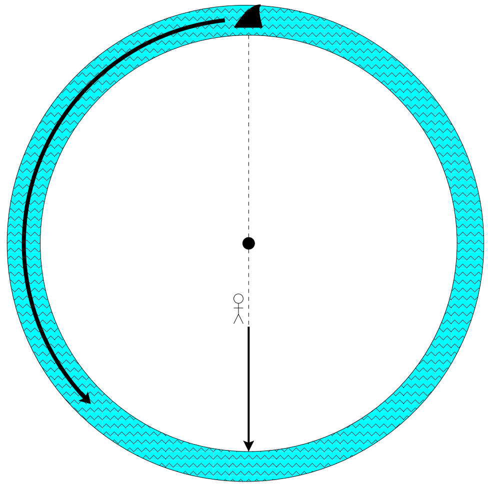
1. Oh no! OSU mascot Pistol Pete has been kidnapped, stripped of his signature cowboy hat and gun, and abandoned in the middle of the desert. Upon investigation, Pete discovers that he's on an "island" surrounded by a perfectly circular moat, which is in turn patrolled by a vicious mascot-eating shark. The moat is narrow enough for Pete to jump across, but if he tries to jump over the shark it will leap up and devour him. Unfortunately the shark swims four times as fast as Pete can run, so it is always waiting for him whenever he tries to beat it by running across the island in a straight line.

   Explain how Pete can beat the shark to a spot on the moat and escape.

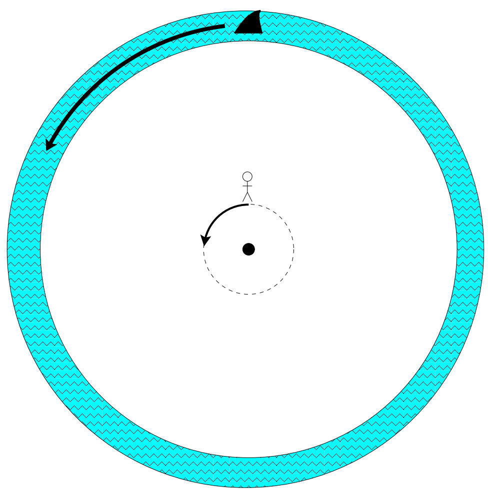
   Solution: Let us assume that the island has radius 4000 feet and that Pete can run at 1000 $\frac{\text{feet}}{\text{minute}}$. Then the shark swims at 4000 $\frac{\text{feet}}{\text{minute}}$, or one raidan per minute. (The actual distances and speeds don't matter; we just use that the shark can cover four radians in the time it takes Pete to run one radius.) Thus if Pete simply runs in a straight line from the center of the island, he needs four minutes to reach the moat, but the shark needs only π minutes to cover a semicircle.



   On the other hand, if Pete can arrange to be partway to the moat and diametrically opposite the shark, he may be able to cover the remaining distance in less than π minutes. (Specifically, Pete needs to be at least $1000(4-\pi)$ feet from the center.)



Thus, Pete wants to arrange this situation. He can run around the center of the island at an arbitrary radial speed by running in a sufficiently tight circle. Pete runs around a circle of radius r in $\frac{2\pi r}{1000}$ minutes, whereas the shark always needs 2π minutes to go all the way around the moat. Thus, if Pete runs in a circle with radius less than 1000 feet, the shark can't keep up with him.



In particular, if Pete can find a radius r which is less than 1000 feet but more than $1000(4-\pi)$ feet, he can run around at that radius until he's created the situation in the second picture, then run directly to the water. Since $4-\pi<1$, such an r exists. One possible strategy is as follows.

Step One: Mark off a circle of radius 990 feet about the center of the island.

Step Two: Run around the circle until he's diametrically opposite the shark.

Step Three: Run straight to the water.

2. The Fibonnaci numbers are defined by $F_0 = F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for all integers $n \geq 2$. Prove that $F_n \leq \left(\frac{5}{3}\right)^n$ for all $n \geq 0$.

Solution: We use induction on n. Observe that $F_0 = 1 \leq \left(\frac{5}{3}\right)^0$ and $F_1 = 1 \leq \left(\frac{5}{3}\right)^1$, so the statement is true for $n = 0$ and for $n = 1$.

For the inductive step, we assume that the statement is true for $n = k - 1$ and for $n = k$. That is, $F_{k-1} \leq \left(\frac{5}{3}\right)^{k-1}$, and $F_k \leq \left(\frac{5}{3}\right)^k$. Using these and the definition of the Fibonacci numbers, we have:

$$\begin{aligned}
F_{k+1} &= F_k + F_{k-1} \\
&\leq \left(\frac{5}{3}\right)^k + \left(\frac{5}{3}\right)^{k-1} \\
&= \frac{3}{5}\left(\frac{5}{3}\right)^{k+1} + \frac{9}{25}\left(\frac{5}{3}\right)^{k+1} \\
&= \frac{24}{25}\left(\frac{5}{3}\right)^{k+1} \\
&< \left(\frac{5}{3}\right)^{k+1}.
\end{aligned}$$

Thus we have shown that

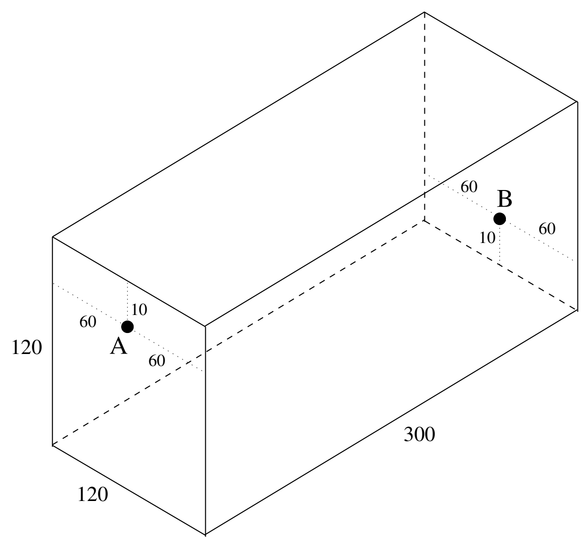
(1) $F_0 \leq \left(\frac{5}{3}\right)^0$ and $F_1 \leq \left(\frac{5}{3}\right)^1$,

and that

(2) if $F_{k-1} \leq \left(\frac{5}{3}\right)^{k-1}$ and $F_k \leq \left(\frac{5}{3}\right)^k$, then $F_{k+1} \leq \left(\frac{5}{3}\right)^{k+1}$.

It follows that $F_n \leq \left(\frac{5}{3}\right)^n$ for all positive integers n.

3. An abandoned alien spaceship is shaped like a rectangular prism with dimensions $300' \times 120' \times 120'$. An astronaut wearing magnetic shoes lands on the "front" of the spaceship at point A, $10'$ from the "top" and midway between the adjacent walls. Her shoes allow her to walk comfortably anywhere on the outside surface, and she wants to get to point B, on the "back" of the ship, $10'$ from the "bottom" and again midway between the adjacent walls. What is the length of the shortest route she can walk from point A to point B?



Solution: The astronaut has several strategic options. She can:

(a) Walk across the top of the spaceship.

(b) Walk across the side of the spaceship.

(c) Walk diagonally partway across the top, then the rest of the way along the side.

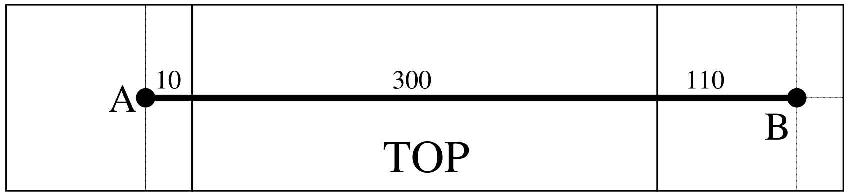
(d) Walk diagonally partway along the side, then the rest of the way across the top.

(e) Walk partway across the top, then across the side, then the rest of the way along the bottom.

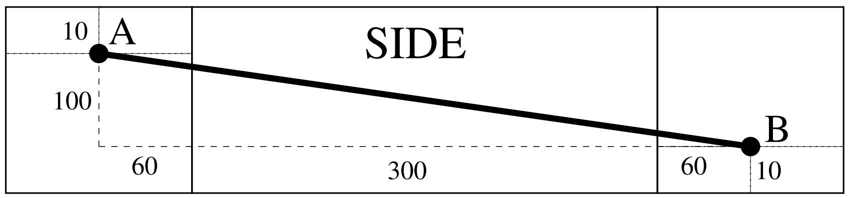
(There are other options, but they're all equal in length to one of the five listed above (because of the symmetry between A and B) or obviously longer.)

We determine the minimum length of a path by unfolding the spaceship in a way that leaves the appropriate faces connected.

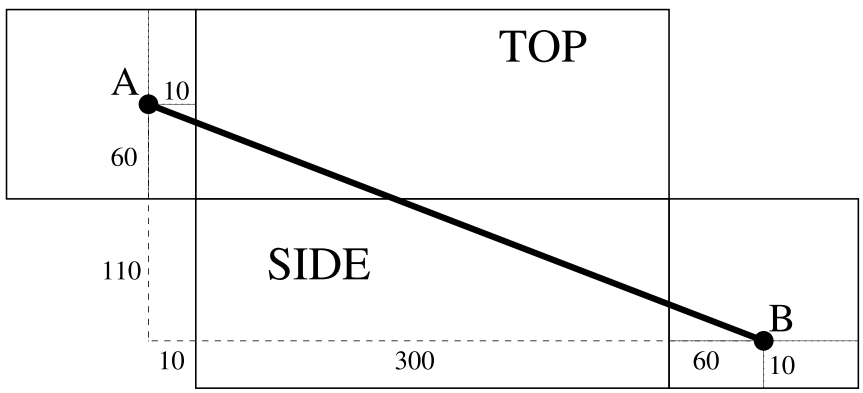
Strategy (a) requires the astronaut to walk $420 = \sqrt{176400}$ feet:



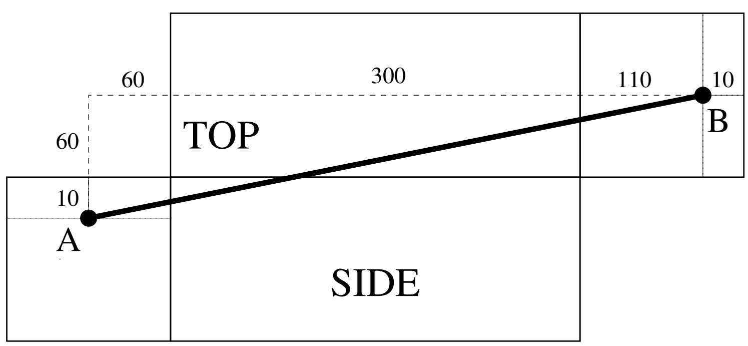
Strategy (b) requires the astronaut to walk $\sqrt{370^2 + 170^2} = \sqrt{165800}$ feet:



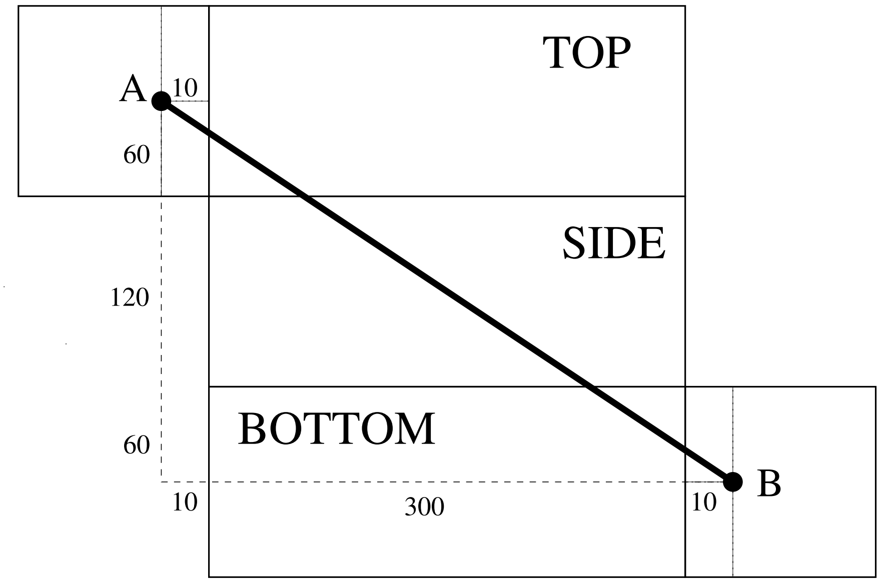
Strategy (c) requires the astronaut to walk $\sqrt{420^2 + 100^2} = \sqrt{186400}$ feet:



Strategy (d) requires the astronaut to walk $\sqrt{470^2 + 70^2} = \sqrt{225800}$ feet:

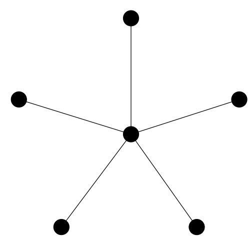


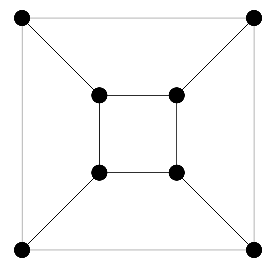
Strategy (e) requires the astronaut to walk $\sqrt{320^2 + 240^2} = \sqrt{160000}$ feet:



The answer is the shortest of these, $\sqrt{160000} = \boxed{400 \text{ feet}}$.

4. A *connected planar graph* is a nonempty collection of line segments (called *edges*) in the plane, with the properties that no two edges intersect except perhaps at a common endpoint, and that any two endpoints (or *vertices*) are connected by a sequence of edges. Two examples are the five-pointed star and the projection of a cube shown below.





Every connected planar graph divides the plane into a number of connected regions, called *faces*. If V, E, and F are the number of vertices, edges, and faces, then we have $(V, E, F) = (6, 5, 1)$ for the star and $(V, E, F) = (8, 12, 6)$ for the cube above. (The "outside face" counts as a face.) Prove that $V - E + F = 2$ for any connected planar graph with finitely many edges.

Solution: We work by induction on E.

If $E = 1$, we have a graph with one edge and two vertices. There is only one face: the entire plane. Thus $V = 2$, $E = 1$, and $F = 1$, so $V - E + F = 2$.

For the inductive step, let G be a connected graph with E_G edges, and assume we know that $V_H - E_H + F_H = 2$ for every connected graph H with at most $E_G - 1$ edges.

First, suppose that G has a vertex v which is an endpoint of only one edge e. Let H be the graph obtained by deleting v and e from G. Then $V_H = V_G - 1$, $E_H = E_G - 1$, and $F_H = F_G$. Furthermore, H is still connected. Thus $V_H - E_H + F_H = 2$ by the inductive hypothesis, so $V_G - E_G + F_G = 2$ as well.

If G has no such vertex, then every vertex is a common endpoint of two or more edges. We claim that G contains a loop: Choose any vertex v_1 and any edge e_1 having v_1 as an endpoint. Then let v_2 be the other endpoint of e_1, and let e_2 be any other edge having v_2 as an endpoint. Let v_3 be the other endpoint of e_2, and continue in this way until a vertex repeats. (This has to happen because there are only finitely many vertices.) Suppose $v_a = v_b$ is the repeated vertex. Then the edges $e_1, \ldots, e_{b-1}$ form a loop, which divides the plane into two regions (inside and outside).

G may have many faces inside the loop, and many faces outside it, but every face is completely contained in either the interior or the exterior of our loop. Let e be any edge on the boundary of the loop. Then the two faces bordering e are distinct, since one is inside the loop and the other is outside. If we delete e, these two faces will merge, but we will not lose any vertices.

Let H be the graph obtained by deleting e from G. Then $E_H = E_G - 1$, $V_H = V_G$, and $F_H = F_G - 1$. Also, H is connected (since any connection that went through the deleted edge e can detour around the loop). Since $E_H < E_G$, we again have $V_H - E_H + F_H = 2$ from the inductive hypothesis. It follows that $V_G - E_G + F_G = 2$ as well.

Thus we have proved that

(1) $V - E + F = 2$ if $E = 1$.

and that

(2) If $V - E + F = 2$ whenever $E = n - 1$, then $V - E + F = 2$ whenever $E = n$.

Consequently, $V - E + F = 2$ whenever E is any positive integer.

5. A "truncated icosahedron" or "soccer ball" is a polyhedron with pentagonal and hexagonal faces arranged according to the following rules:

   (a) No two pentagons share an edge.

   (b) The edges of each hexagon alternately meet pentagons and hexagons.

   (c) Exactly one pentagon and two hexagons meet at each vertex.

   If a truncated icosahedron has H hexagonal and P pentagonal faces, find all possible values for (H, P).

Solution: Let V, E, and F be the number of vertices, edges, and faces, respectively.

Immediately, we have

$$F = H + P. \tag{1}$$

From the statement of problem 4 (if puncture any face of a polyhedron and pull it flat, or, equivalently, take the stereographic projection from a point just outside, the edges form a connected planar graph), we have

$$V - E + F = 2. \tag{2}$$

If we count the vertices of each pentagon, (c) tells us that we count each vertex once. On the other hand, if we count the vertices of each hexagon, (c) tells us that that we count each vertex twice. Thus

$$5P = V \tag{3}$$
$$6H = 2V. \tag{4}$$

Finally, if we count all the edges of each pentagon and all the edges of each hexagon, we will have counted every edge twice. Thus

$$5P + 6H = 2E. \tag{5}$$

This is five equations in five variables, so we should be able to solve.

From (3), (4), and (5), we have $3V = 2E$. From (1), (3), and (4), we have $15F = 8V$. Substituting these into (2) yields $V - \frac{3}{2}V + \frac{8}{15}V = 2$. Thus $V = 60$, $E = 90$, $F = 32$, $P = 12$, and $H = 20$.

The answer is $\boxed{(20, 12)}$.