Part I. Individual Round

1. Find an integer d with the property that the multiples of d are exactly the numbers which are divisible by both 12 and 21.

   A. 4
   B. 63
   C. 84
   D. 108
   E. 252

   Answer: C.

2. Let ABC be an equilateral triangle with sides of length 2. Find the shortest possible distance from A to a point on BC.

   A. 1
   B. $\sqrt{2}$
   C. $\sqrt{3}$
   D. 2
   E. $\sqrt{5}$

   Answer: C.

3. Simplify $\sqrt{6-4\sqrt{2}}$.

   A. $2\sqrt{2}-\sqrt{3}$
   B. $\sqrt{3}-2\sqrt{2}$
   C. $\sqrt{2}-2$
   D. $2-\sqrt{2}$
   E. $\sqrt{6}-\sqrt[4]{8}$

   Answer: D.

4. Write the repeating decimal $0.\overline{987654321}$ as a fraction in lowest terms.

   A. $\dfrac{12193281}{12345697}$

   B. $\dfrac{109739369}{111111111}$

   C. $\dfrac{987654321}{999999999}$

   D. $\dfrac{48773044}{49382707}$

   E. $\dfrac{987654321}{1000000000}$

   Answer: B.

5. Square $ABCD$ is inscribed inside a circle of area π. Then another circle is inscribed inside square $ABCD$. What is the area of the smaller circle?

   A. $\frac{\pi}{4}$

   B. 1

   C. $\frac{\pi}{3}$

   D. 2

   E. $\frac{\pi}{2}$

   Answer: E.

6. Suppose we have a collection of cards, each of which has a letter printed on one side and a number printed on the other. We want the cards to follow the rule: "If there is a vowel on one side, the number on the other side must be even."

   How many of the five cards below must be turned over to verify the rule?

   $$\boxed{4}, \boxed{\text{A}}, \boxed{11}, \boxed{7}, \boxed{\text{K}}?$$

   A. None.

   B. One.

   C. Two.

   D. Three.

   E. Four.

   Answer: D.

7. What is the area of the shape bounded by the equation $x^2 + 2y^2 + 4x - 4y = 1000$?

   A. 503π

   B. 961π

   C. 1000π

   D. 1006π

   E. 2012π

   Answer: The correct answer is $503\pi\sqrt{2}$, which does not appear among these choices. We apologize for the error.

8. If two men can paint two rooms in two days, how long does it take one man to paint one room?

   A. Half a day.

   B. One day.

   C. Two days.

   D. Four days.

   E. Eight days.

   Answer: C.

9. Express $\cot(\alpha + \beta)$ in terms of $\cot\alpha$ and $\cot\beta$.

   A. $\dfrac{\cot\alpha\cot\beta - 1}{\cot\alpha + \cot\beta}$

   B. $\dfrac{\cot\alpha\cot\beta + 1}{\cot\alpha + \cot\beta}$

   C. $\cot\alpha + \cot\beta$

   D. $\dfrac{\cot\alpha + \cot\beta}{1 - \cot\alpha\cot\beta}$

   E. $\dfrac{\cot\alpha + \cot\beta}{1 + \cot\alpha\cot\beta}$

   Answer: A.

10. Find the area of a regular octagon with side-length one. (An octagon has eight sides.)

   A. $6\sqrt{2}-4$

   B. 7

   C. $2\sqrt{2}+2$

   D. $2\sqrt{6}$

   E. $8\sqrt{2}-8$

   Answer: C.

11. Given that the polynomial $x^5-2x^4-3x^3+4x^2+2x-1$ has five distinct real roots, how many of them are positive?

   A. 0

   B. 1

   C. 2

   D. 3

   E. 4

   Answer: D.

12. The positive integer x is the arithmetic mean of the twenty integers $a_1, a_2, \ldots, a_{20}$. The arithmetic mean of the twelve integers $b_1, b_2, \ldots, b_{12}$ is $5x$. What is the smallest integer that could be the arithmetic mean of the numbers $a_1, \ldots, a_{20}, b_1, \ldots, b_{12}$?

   A. 1

   B. 2

   C. 3

   D. 4

   E. 5

   Answer: E.

13. If you answered all 25 questions on this contest at random, what is the probability that you would get exactly one question right?

A. $\left(\frac{4}{5}\right)^{24}$

B. $\frac{1}{5}+\left(\frac{4}{5}\right)^{24}$

C. $25\left(\frac{4}{5}\right)^{25}$

D. $5\left(\frac{4}{5}\right)^{24}$

E. $4\left(\frac{4}{5}\right)^{25}$

Answer: D.

14. Suppose $ABCD$ is a quadrilateral with perpendicular diagonals. If $AB = 6$, $BC = 2$, and $CD = 7$, find AD.

A. 8

B. 9

C. 10

D. 11

E. 12

Answer: B.

15. Find the largest integer less than 2012 with an odd number of positive divisors.

A. 1440

B. 1936

C. 1944

D. 2000

E. 2011

Answer: B.

16. Suppose that $x^{\frac{1}{5}}$ and $x^{\frac{1}{7}}$ are integers. Which of the following must also be an integer?

A. $x^{\frac{1}{2}}$

B. $x^{\frac{1}{12}}$

C. $x^{\frac{1}{35}}$

D. All of the above.

E. None of the above.

Answer: C.

17. An athlete runs three laps around a quarter-mile track at 9 miles per hour. How fast must she run the fourth lap in order to complete the mile in 6 minutes?

A. It can't be done.

B. 6 mph

C. 10 mph

D. 12 mph

E. 15 mph

Answer: E.

18. What is the shape of the graph of $\big|x-y\big| + \big||x|-x\big| = 0$?

A. A line segment.

B. A line.

C. A rectangle.

D. A single point.

E. A ray.

Answer: E.

19. Find $\cos\frac{\pi}{12}$.

A. $\dfrac{\sqrt{2-\sqrt{3}}}{2}$

B. $\dfrac{\sqrt{2+\sqrt{3}}}{2}$

C. $\dfrac{\sqrt{2}+\sqrt{3}}{2}$

D. $\dfrac{\sqrt{3}-\sqrt{2}}{2}$

E. $\dfrac{\sqrt{\sqrt{2}+\sqrt{3}}}{2}$

Answer: B.

20. What is the area of the pentagon with vertices (in order) at $(0, 0)$, $(12, 2)$, $(14, 15)$, $(7, 16)$, and $(2, 14)$?

   A. $\frac{333}{2}$

   B. $\frac{335}{2}$

   C. $\frac{337}{2}$

   D. $\frac{339}{2}$

   E. $\frac{341}{2}$

   Answer: C.

21. Find all the functions on $\mathbb{R}$ that satisfy the condition $f(x) > 0$ for all x and the equation $f(x + y) = \dfrac{f(x)}{f(y)}$ for all x and y.

   A. The power functions $f(x) = x^n$ for even integers n.

   B. The constant function $f(x) = 1$.

   C. The logarithmic functions $f(x) = C \ln x$ for positive numbers C.

   D. No functions satisfy these conditions.

   E. The exponential functions $f(x) = b^x$ for positive numbers b.

   Answer: B.

22. How many digits are in the decimal representation of 2012^{2012}?

   A. 6047

   B. 6197

   C. 6347

   D. 6497

   E. 6647

   Answer: E.

23. Ten dots are marked on a circle. In how many ways can five chords be drawn so that each chord connects two of the dots, but no two chords intersect?

A. 42

B. 60

C. 84

D. 120

E. 1920

Answer: A.

24. Given $A = (0, 1)$ and $C = (2, 3)$, let $B = (x, 0)$ be the point on the x-axis that minimizes the perimeter of triangle ABC. Find x.

A. $\frac{1}{3}$

B. $\frac{1}{2}$

C. $\frac{2}{3}$

D. $\frac{5}{6}$

E. 1

Answer: B.

25. Given a triangle ABC, let D be a point on AB such that $AD : DB = 2 : 3$, and let E be a point on BC such that $BE : EC = 5 : 7$. If AE meets CD at F, find $\frac{AF}{EF}$.

A. $\frac{7}{8}$

B. $\frac{14}{15}$

C. 1

D. $\frac{15}{14}$

E. $\frac{8}{7}$

Answer: E.