1. Pete buys 10 bags of apples, each of which contains 20 apples. If he eats 8 apples a day, how many days will it take him to eat the 10 bags of apples?

   Answer: 200 apples $\times \dfrac{1 \text{ day}}{8 \text{ apples}} = 25$ days.

2. A tortoise and a hare leave their dormitory for a study meeting in the library, a mile away, at the same time. The tortoise walks in a straight line to the library at a constant speed of two miles per hour. The hare walks at a constant speed of ten miles per hour, but decides to take a detour and stop for lunch on the way. If the hare's route from the dorm to lunch to the library is 1.5 miles long, what is the most time she can spend eating lunch if she doesn't want the tortoise to arrive at the library before her?

   Answer: The tortoise takes 30 minutes for the trip, and the hare spends nine minutes moving. So she can afford $30 - 9 = 21$ minutes at lunch.

3. The OSU football team scores four times during a game, and each score is worth 3, 6, 7, or 8 points. How many numbers could represent OSU's total score at the end of the game?

   Answer: The possible scores are $4 \times 3 = 12$, and anything between $3 \times 3 + 6 = 15$ and $4 \times 8 = 32$. This is 19 possibilities.

4. Solve for x: $3^{x+2} + 2^{x+2} + 2^x = 2^{x+5} + 3^x$.

   Answer: Rearranging, we get $3^{x+2} - 3^x = 2^{x+5} - 2^{x+2} - 2^x$, which is the same as $8(3^x) = 27(2^x)$ or $\left(\dfrac{3}{2}\right)^x = \dfrac{27}{8}$. So $x = 3$.

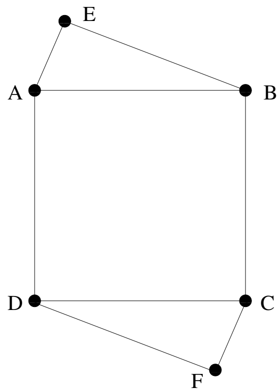
5. Order from largest to smallest: $1^{10}, 2^9, 3^8, 4^7, 5^6, 6^5, 7^4, 8^3, 9^2, 10^1$.

   Answer: $4^7 = 2^{14} = 2^4 2^{10} \approx 16(1000) = 16000$. $5^6 = 125^2 = 15625$. $6^5 = 2^5 3^5 \approx (30)(240) \approx 7200$. $3^8 = 81^2 \approx 6400$. $7^4 = 49^2 \approx 50^2 = 2500$. $2^9 = 8^3 = 512$. $9^2 = 81$. $10^1 = 10$. $1^{10} = 1$.

6. Find the smallest positive integer n with the property that $1 + 2 + 3 + \cdots + n$ is a three-digit number, and all three digits of this number are the same.

   Answer: The sum is $\dfrac{n(n+1)}{2}$, which must be divisible by $111 = (37)(3)$. So $n = 36$ or 37. But if $n = 37$, the numerator isn't divisible by 3.

7. Square $ABCD$ has sides of length 17, and triangles ABE and CDF are right triangles with $AE = CF = 8$ and $BE = DF = 15$. Find EF.



Answer: Extend the slanted sides to make a bigger square with side lengths $8+15 = 23$. EF is a diagonal of this bigger square, so its length is $23\sqrt{2}$.

8. An arithmetic series of 2017 positive integers adds to 2017,

$$a_1 + a_2 + \cdots + a_{2017} = 2017.$$

Find a_4.

Answer: The series is $1 + 1 + \cdots + 1$.

9. What is the largest two-digit number the becomes 75% greater when its digits are reversed?

Answer: If the number is AB, we get the equation $(10A + B)(1.75) = (10B + A)$. This simplifies to $B = 2A$. Since A and B have to be digits, the options are 12, 24, 36, and 48.

10. An equilateral triangle has two vertices at $(1,0)$ and $(-1,0)$, and its third vertex on the positive y-axis. It intersects the unit circle at four points: $(1,0)$, $(-1,0)$, A, and B. Find the equation of line AB.

Answer: We know the line in question is parallel to the base ($y = 0$), so it will have the form $y = C$, where C is the y-coordinate of A (hence also B). If $P = (1,0)$ then $AO = PO = 1$ and $\angle APO = 60°$, so APO is equilateral. Thus $A = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$. The answer is $y = \frac{\sqrt{3}}{2}$.

11. The volume of a cube (measured in cubic yards) is equal to its surface area (measured in square feet). Find the length of a side.

Answer: Suppose a side has length y yards, so $3y$ feet. Then the volume (in cubic yards) is y^3. The surface area (in square feet) is $6(3y)^2 = 54y^2$. We solve $y^3 = 54y^2$, and throw out the meaningless solution $y = 0$ to get $y = 54$.

12. In pentagon $ABCDE$, the measures of angles A, B, C, D, and E, in order, form an arithmetic sequence. If $\angle A = 120°$, what is $\angle C$?

    Answer: The average angle of a pentagon is $108°$. C is the middle term in this arithmetic sequence, so equal to its average.

13. Consider circle O with radius 17. Points A and B are chosen on the circle so that the distance from O to line AB is 8. Find the length of line segment AB.

    Answer: Drop a perpendicular from O to AB, and let it meet AB at P. Then AOP and BOP are 8-15-17 right triangles. $AB = AP + PB = 30$.

14. How many polynomials $p(x)$ with integer coefficients satisfy $p(3) = 20$ and $p(5) = 17$?

    Answer: Let $q(x) = p(x+3) - 20$, so $q(0) = 0$. Then, if p is a polynomial, q is also a polynomial, and is divisible by x. So $p(5) = q(2)$ must be divisible by 2. But it's not, so there are no such polynomials.

15. Find the remainder when $2018^{2017} - 2016^{2017}$ is divided by 2017.

    Answer: 2018 has remainder 1, and 2016 has remainder -1. So $2018^{2017} - 2016^{2017}$ has the same remainder as $1^{2017} - (-1)^{2017} = 2$.

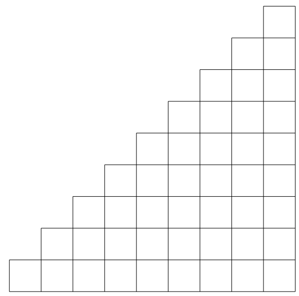
16. The circles $x^2 + y^2 = 289$ and $(x-5)^2 + (y-12)^2 = 169$ intersect in two points P and Q. Find the equation of line PQ.

    Answer: The second equation simplifies to $x^2 + y^2 - 10x - 24y = 0$. Subtracting from the first equation gives us $10x + 24y = 289$.

17. Find the last digit of 2017^{2017}.

    Answer: The digits repeat in the pattern 7, 9, 3, 1, 7, 9, 3, 1, which has length four. 2017 has a remainder of 1 upon division by 4, so its last digit is the first in this sequence: 7.

18. A **Triangle Sudoku** puzzle is defined as follows: nine 9's, eight 8's, seven 7's, six 6's, five 5's, four 4's, three 3's, two 2's, and one 1 must be placed in the triangular array of boxes below in such a way that no row or column contains two copies of the same number.



How many solutions are there to the triangle sudoku puzzle?

Answer: The only way to make the 9's work is to put them along the diagonal. Then the 8's must go along the subdiagonal, and so on. So there's one solution.

19. Consider triangle ABC with side lengths a, b, c. Given that $\angle A + \angle B = \angle C$ and $4a^2 = c^2$, find the ratio $\angle B : \angle A$.

Answer: The first equation tells us that ABC is a right triangle with right angle C. Now the second tells us that $b = a\sqrt{3}$. Thus ABC is a 30-60-90 triangle with $A = 30°$ and $B = 60°$.

20. When Pistol Pete walks his dog, he always walks around a perfect square with side length 2017 feet. If Pete holds the dog on a five-foot leash, find the area through which the dog could travel during the walk.

Answer: The dog can travel within five feet of the square. The area it can go through consists of four 2017×10 rectangles (with each of the interior 5×5 corners double-counted), together with four quarter-circles of radius 5. So the answer is $4(20170) - 4(25) + 25\pi$.

21. Let ABC be an isosceles triangle with $AB = AC$. Draw altitudes AD, BE, and CF. Which, if any, of the statements below must be true?

    (I) $\frac{1}{AD} = \frac{1}{BE} + \frac{1}{CF}$.

    (II) D lies between B and C.

    (III) E lies between A and C.

    (IV) $2 \times AD \geq BE$.

    (V) $AD \leq BE$.

    Answer: (I) is false for an equilateral triangle, so likely to be false in general. (II) is true: D is the midpoint of BC. (III) is false any time A is obtuse. (IV) is true: $(BC)(AD) = (AC * BE)$ because both are equal to twice the area, and $BC < 2AC$ by the triangle inequality. (V) is false, since it's possible to have $BC < AC$, and $(BC)(AD) = (AC * BE)$.

22. A grasshopper starts at the origin and wants to move to the point $(20, 17)$. It can move only by jumping east or north, and can only jump distances of exactly three or five units. How many distinct sequences of jumps can it take to reach its destination?

    Answer: Northward, the grasshopper must take one long jump and four short jumps. Eastward, it may take four long jumps or one long jump and five short jumps. So the possible paths correspond to arrangements of the letters $NnnnnEEEE$ or $NnnnnEeeeee$. The first string can be arranged in $\frac{9!}{1!4!4!}$ ways, and the second string can be arranged in $\frac{11!}{1!4!1!5!}$ ways.

23. Let x, y, z be real numbers such that $x+y+z = 2$, $x^2+y^2+z^2 = 30$, $x^3+y^3+z^3 = 116$. Compute xyz.

    Answer: Since $(x+y+z)^2 = 4$ and $x^2+y^2+z^2 = 30$, we conclude $(xy+xz+yz) = -13$. Now $(x^2+y^2+z^2)(x+y+z) = 60$, so $(x^2y+x^2z+y^2x+y^2z+z^2x+z^2y) = -56$. Finally, $8 = (x+y+z)^3 = (x^3+y^3+z^3)+3(x^2y+x^2z+y^2x+y^2z+z^2x+z^2y)+6xyz$, so $xyz = 10$. (In fact, the numbers are -1, -2, and 5.)

24. Let $f(x)$ be a function defined on the real numbers with the properties

    - $f(a+b) = f(a)f(b)$ for all real numbers a and b.
    - $f(1) = 2$.

    Find $f(2017)$.

    Answer: The first equation tells us that $f(x)$ is the exponential b^x for some base b. The second tells us $b = 2$.

25. How many distinct real roots does the equation $x^7 - 7x^4 - 16 = 0$ have?

Answer: By Descartes' rule of signs, there's one sign change in the coefficients, so at most one positive root. Meanwhile, $f(-x) = -x^7 - 7x^4 - 16$ has no sign changes, so no positive roots.

Therefore $f(x) = x^7 - 7x^4 - 16$ has at most one positive root and no negative roots. But it's a polynomial of odd degree, so it has at least one root.