Math 5613

Assignment 9

Due Friday, October 24

Part one: Reading. Read Chapters 9 and 13.1 in the textbook (more or less: see the "homework" page on the course web page for precise daily goals).

Part two: Problems to solve and write up.

I want your effort on both the problem-solving and the writeup to be collaborative. This week, you'll again be responsible for turning in writeups of three problems, but you'll also have to arrange to meet with me on Friday or early in the next week to present another orally.

Throughout, assume unless otherwise indicated that rings are commutative with 1, and that notation remains intuitive: R and S are rings, $n \in \mathbb{Z}$, etc.

1. Is $f = x^2 + x + 1$ irreducible when viewed as an element of $\mathbb{Z}[x]$? $\mathbb{Q}[x]$? $\mathbb{C}[x]$? $\mathbb{F}_2[x]$? Prove your answers.

2. Factor $x^4 - 4x^3 + 6 \in \mathbb{Q}[x]$ as a product of irreducibles, or prove that it is already irreducible.

3. Fix a prime p, and set $f = (x+2)^p - 2^p \in \mathbb{Z}[x]$. Factor f as a product of irreducibles. (The answer could depend on p.)

4. Let $n \in \mathbb{N}$ and put $f = (x-1)(x-2)\dots(x-n) - 1 \in \mathbb{Z}[x]$. Prove that f is irreducible.

   [You may assume standard results from undergraduate pre-calculus, calculus, and analysis.]

   [If $f = gh$, what can we say about $g(1)$ and $h(1)$? About $g(2)$ and $h(2)$?]

5. Let $n \in \mathbb{N}$ and put $f = (x-1)(x-2)\dots(x-n) + 1 \in \mathbb{Z}[x]$. Factor f as a product of irreducibles. (The factorization may depend on n.)

   [You may assume standard results from undergraduate pre-calculus, calculus, and analysis.]

   [If $f = gh$, what can we say about $g(1)$ and $h(1)$? About $g(2)$ and $h(2)$?]

6. Let $f = x^4 - 4x^2 + 8x + 2 \in \mathbb{Q}(\sqrt{-2})[x]$. Prove that f is irreducible.

7. Verify that $x = (10x-9)(15x+16)(6x-5)$ in $\dfrac{\mathbb{Z}}{30\mathbb{Z}}[x]$. Then find all ways to factor x as a product of three linear factors in this ring.

8. Recall that, for a ring R, the *nilradical* is $\sqrt{R} = \{x \in R : x^n = 0$ for some n. Let F be a field and $f \in F[x]$, and write f as a product of irreducibles, $f = \prod p_i^{e_i}$. What is $\sqrt{\dfrac{F[x]}{(f)}}$?

9. Let ϕ be the Euler ϕ-function, $\phi(n) = \left| \left(\frac{\mathbb{Z}}{n\mathbb{Z}} \right)^{\times} \right|$. Prove that

$$n = \sum_{d \text{ divides } n} \phi(d).$$

10. Define a function $\alpha : \mathbb{Q}(\sqrt{2}) \to \mathbb{Q}(\sqrt{2})$ by the rule $\alpha(a + b\sqrt{2}) = a - b\sqrt{2}$ for $a, b \in \mathbb{Q}$. Prove that α is an automorphism.

    [α is defined as a function. You may not assume it is a homomorphism.]

11. Suppose that $f \in \mathbb{Q}[x]$ is a monic polynomial, and $q \in \mathbb{Q}$ is a root of f. Prove that in fact $q \in \mathbb{Z}$.

    [This follows immediately from the rational roots theorem, so you may not simply cite it. You may use the rational roots theorem if you first prove it, or you may opt to prove it another way.]

12. Let $a \in \mathbb{Z}$ and put $f = x^5 + ax - 1 \in \mathbb{Z}[x]$. Factor f as a product of irreducibles. (The answer may depend on a.)

Part three: Estimate the time you spent on this assignment. I will pay attention to this in writing future assignments. Meanwhile, if it's taking you longer than you think is reasonable, please talk to me so we can come up with a strategy.