Math 5613

Assignment 8

Due Friday, October 17

Part one: Reading. Read Chapters 7.5 and 9 in the textbook (more or less: see the "homework" page on the course web page for precise daily goals).

Part two: Problems to solve and write up.
I want your effort on both the problem-solving and the writeup to be collaborative. This week, you'll again be responsible for turning in writeups of three problems, but you'll also have to arrange to meet with me on Friday or early in the next week to present another orally.

Throughout, assume unless otherwise indicated that rings are commutative with 1, and that notation remains intuitive: R and S are rings, $n \in \mathbb{Z}$, etc.

1. Let R be an integral domain, $D \subset R$ a multiplicatively closed set, and put $Q = D^{-1}R$ and $\iota : R \to Q$ be the map given by $\iota(r) = \dfrac{r}{1}$ for all $r \in R$. Prove that (Q, ι) are the unique ring and map (up to isomorphism) satisfying the universal property:

   > Suppose S is a ring, and $\phi : R \to S$ is a homomorphism such that $\phi(d)$ is a unit in S for all $d \in D$. Then there is a unique homomorphism $\phi^* : Q \to S$ such that $\phi = \phi^* \circ \iota$.

   [There are some standard tricks for dealing with proofs of universal properties. If you haven't seen them before, you are likely to make this problem considerably harder than it needs to be. Start early, and don't be afraid to ask for help after you've grappled a bit.]

2. Let R be an integral domain with field of fractions F, and $D \subset R$ a multiplicatively closed set. Prove that $D^{-1}R \subset F$.

3. Let F be a field. Prove that either Q or $\dfrac{\mathbb{Z}}{p\mathbb{Z}}$ is a subfield of F.

   [Attempt to construct $\mathbb{Z}$ as a subring. What happens?]

4. Let F be a field, and consider the rings $R = F[[x]] = \{\sum_{i=0}^{\infty} c_i x^i : c_i \in F\}$ and $S = F((x)) = \bigcup_N \{\sum_{i=-N}^{\infty} c_i x^i : c_i \in F\}$ of "formal power series" and "formal Laurent series". Prove that S is the field of fractions of R. If F is the field of fractions of some ring A, must S also be the field of fractions of $A[[x]]$?

5. Prove that $\mathbb{R}$, like $\mathbb{Q}$, is the field of fractions of a proper subring.

   [Hint: Use Zorn's Lemma to establish the existence of a subring $A \subset \mathbb{R}$ that is maximal with respect to the property of not containing $\frac{1}{2}$.]

6. Prove that $\mathbb{Z}[2\sqrt{2}]$ is not a UFD.

   [Find a quadratic that factors in a way that contradict Gauss's Lemma.]

7. Let $R = \{f \in \mathbb{Q}[x] : \frac{df}{dx}(0) = 0\}$. (That is, R is the polynomials with rational coefficients and no linear term.) Determine, with proof, whether R is a UFD.

8. Let $R = \{f \in \mathbb{Q}[x] : f(0) \in \mathbb{Z}\}$. Prove that $x \in R$ does not factor as a product of irreducibles.

9. Let $R = \{f \in \mathbb{Q}[x] : f(0) \in \mathbb{Z}\}$. For $f, g \in R$, define n to be maximal such that $f, g \in (x)^n\mathbb{Q}$. (That is, all lower-degree coefficients on both polynomials are zero, but without loss the x^n coefficient on f is nonzero. However, we do not require that this coefficient is an integer.) Prove that $\gcd(f, g)$ exists and is contained in $(x)^n\mathbb{Q}$.

10. Let $R = \{f \in \mathbb{Q}[x] : f(0) \in \mathbb{Z}\}$. For $f, g \in R$, prove that $\gcd(f, g)$ can be chosen to be an R-linear combination of f and g.

11. Let $R = \{f \in \mathbb{Q}[x] : f(0) \in \mathbb{Z}\}$. Prove R is Bezout. (Recall that a Bezout domain is one where all finitely-generated ideals are principal.)

12. Let $R = \{f \in \mathbb{Q}[x] : f(0) \in \mathbb{Z}\}$. Prove R is not a PID.

    [There are two ways to attack this. One is to explicitly construct an ideal that isn't principal, which you've come close to doing above. The other is to show that R is not a UFD. In any event, you may not use the equivalence of the two definitions of Noetherian without proof; if you're inclined to write out that proof you should notice yourself constructing a non-principal ideal.]

Part three: Estimate the time you spent on this assignment. I will pay attention to this in writing future assignments. Meanwhile, if it's taking you longer than you think is reasonable, please talk to me so we can come up with a strategy.