Math 5613

Assignment 12

Due Friday, November 21

Part one: Reading. Read through Chapter 14.6 in the textbook (more or less: see the "homework" page on the course web page for precise daily goals).

Part two: Problems to solve and write up.

I want your effort on both the problem-solving and the writeup to be collaborative. This week, you'll again be responsible for turning in writeups of three problems, but you'll also have to arrange to meet with me on Friday or early in the next week to present another orally.

Throughout, assume unless otherwise indicated that rings are commutative with 1, and that notation remains intuitive: $F \subset K$ are fields, $n \in \mathbb{Z}$, etc.

1. Let F be a cyclotomic extension (that is, the splitting field of $x^n - 1$ for some n) of $\mathbb{Q}$. Prove that $\sqrt[3]{2} \notin F$.

2. Read the construction of the 17-gon described in Exercise 14.5.17 on pages 604–605. Assume the results of part (a), i.e., that the lines ℓ_1 and ℓ_2 have equations $(1+\sqrt{17})x+4y = 4$ and $4x-(1+\sqrt{17})y = -(1+\sqrt{17})$. "Solve" part (b) by finding the values of s and t.

3. Let $f \in \mathbb{Q}[x]$ be a degree d polynomial, and assume that f has a repeated root in $\mathbb{C}$. For what d, if any, may we conclude that f has a root (repeated or otherwise) in $\mathbb{Q}$?

4. Determine the isomorphism type of $\mathrm{Gal}(x^4 - 25/\mathbb{Q})$. Within reason, also describe its Galois theory — What are the generators, automorphisms, intermediate fields?

5. Determine the isomorphism type of $\mathrm{Gal}(x^5+x-1/\mathbb{Q})$. Within reason, also describe its Galois theory — What are the generators, automorphisms, intermediate fields?

6. Let $p \neq q$ be odd primes. Prove that $\mathrm{Gal}(x^4 - px^x + q) \cong D_8$.

   [The text gives a recipe for quartics, which you may of course follow. Alternatively, the footnotes at the bottom of page 618 suggest a *Monthly* article from 1989. I haven't hunted it down myself, but it might suggest a more simple approach.]

7. Let $f \in \mathbb{Q}[x]$ be irreducible of degree 4 and suppose the discriminant of f is negative. Prove that $\mathrm{Gal}(f/\mathbb{Q}) \not\cong C_4$.

   [Hint: Let K be the splitting field. Think about the relationship between complex conjugation as an element of $\mathrm{Gal}(K/\mathbb{Q})$ and of $\mathrm{Gal}(\mathbb{Q}(\sqrt{D})/\mathbb{Q}$.]

8. Let ω_p be a primitive p^{th} root of unity. Prove that $\sqrt{p} \in \mathbb{Q}(\omega_p)$ if $p \equiv 1 \pmod 4$, and $\sqrt{-p} \in \mathbb{Q}(\omega_p)$ if $p \equiv 3 \pmod 4$.

   [Compute the discriminant of the appropriate cyclotomic polynomial.]

9. Let $f = x^6 - 2x^3 - 2 \in \mathbb{Q}[x]$, and set K equal to the splitting field of f and $F = \mathbb{Q}(i, \sqrt{3}, \sqrt[3]{2})$. (You don't have to prove that $F \subseteq K$, but you should convince yourself by finding the roots of f and playing around with them. Determine $[F : \mathbb{Q}]$, and prove that $[K : F]$ divides 3.

10. Let $f = x^6 - 2x^3 - 2$. Prove that $\mathrm{Gal}(f/\mathbb{Q}) \cong S_3 \times S_3$.

    [This is problem 14.6.48.f on page 623. You should take (a) through (e) as hints. But prove what you use.]

11. Suppose $f_1, \ldots, f_n \in \mathbb{R}[x]$ are polynomials. For any α, define the associated **sign sequence** to be the sequence $(\mathrm{sgn}(f_1(\alpha)), \ldots, \mathrm{sgn}(f_n(\alpha)))$, where $\mathrm{sgn}(f_i(\alpha))$ is equal to $+$, $-$, or 0 in the obvious way. Set $V(\alpha)$ equal to the number of sign changes in the sign sequence. (Sign changes are defined by ignoring any zeroes. That is, a sign change is positions $i < j$ such that $\mathrm{sgn}(f_k(\alpha)) = 0$ for all $i < k < j$ and either $\mathrm{sgn}(f_i(\alpha)) = +$, $\mathrm{sgn}(f_j(\alpha)) = -$ or $\mathrm{sgn}(f_i(\alpha)) = -$, $\mathrm{sgn}(f_j(\alpha)) = +$.)

    Suppose that no f_i is equal to zero on the closed interval $[a, b]$. Prove that $V(a) = V(b)$.

    [Despite our best efforts, algebraists sometimes use analysis.]

12. Prove **Sturm's theorem**: Let $f \in \mathbb{R}[x]$ be squarefree, and define the **Sturm sequence** of f by $f_0 = f$, $f_1 = \frac{df}{dx}$, and recursively applying the Division Algorithm to f_i, f_{i-1} to write $f_{i-1} = f_i Q + R$, set $f_{i+1} = -R$. For all $\alpha \in \mathbb{R}$, define $V(\alpha)$ as above.

    Then the number of roots of f in the interval $[a, b]$ is given by $V(a) - V(b)$.

    [This is Problem 14.6.51.g on page 624. You should take (a) through (f) as hints (but prove what you use).]

Part three: Estimate the time you spent on this assignment. I will pay attention to this in writing future assignments. Meanwhile, if it's taking you longer than you think is reasonable, please talk to me so we can come up with a strategy.