Math 5613

Assignment 11

Due Friday, November 7

Part one: Reading. Read through Chapter 14.2 in the textbook (more or less: see the "homework" page on the course web page for precise daily goals).

Part two: Problems to solve and write up.

I want your effort on both the problem-solving and the writeup to be collaborative. This week, you'll again be responsible for turning in writeups of three problems, but you'll also have to arrange to meet with me on Friday or early in the next week to present another orally.

Throughout, assume unless otherwise indicated that rings are commutative with 1, and that notation remains intuitive: $F \subset K$ are fields, $n \in \mathbb{Z}$, etc.

1. Compute the complete Galois theory of $\mathbb{C}/\mathbb{R}$. (That is, find all the automorphisms fixing $\mathbb{R}$, and describe the multiplication table. Also identify all subgroups and their fixed fields.)

2. Let $K = \mathbb{Q}(\sqrt{1+\sqrt{3}})$ and $L = \mathbb{Q}(\sqrt{1-\sqrt{3}})$. Compute $K \cap L$.

3. Let $K = \mathbb{Q}(\sqrt{1+\sqrt{3}})$ and $L = \mathbb{Q}(\sqrt{1-\sqrt{3}})$. Set $F = K \cap L$. Put $E = KL$. Compute the complete Galois theory of E/F. (That is, find all the automorphisms fixing $\mathbb{R}$, and describe the multiplication table. Also identify all subgroups and their fixed fields. Determine which subgroups are normal and which sub-extensions are Galois.)

4. Let K/F be Galois with cyclic Galois group $G = \sigma$, and put $n = [K : F]$. Assume that the **norm** $N_{K/F} : K \to F$, defined by the formula $N_{K/F}(\alpha) = \prod_{\tau \in G} \tau(\alpha)$, is a well-defined group homomorphism from $K^\times$ to $F^\times$.

   Let $\alpha \in K$ be such that $N_{K/F}(\alpha) = 1$. Prove that there exists $\beta \in K$ with $\alpha = \dfrac{\beta}{\sigma(\beta)}$.

   [This is Hilbert's famous Theorem 90. The hint to problem 14.2.23 at the bottom of page 583 gives you a formula for β in terms of α and σ. You are unlikely to construct it on your own, but please make a serious attempt before you look.]

Part three: Estimate the time you spent on this assignment. I will pay attention to this in writing future assignments. Meanwhile, if it's taking you longer than you think is reasonable, please talk to me so we can come up with a strategy.