Tasty Bits of Several Complex Variables

A whirlwind tour of the subject

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More information:
See https://www.jirka.org/scv/ for more information (including contact information).

The LaTeX source for the book is available for possible modification and customization at github: https://github.com/jirilebl/scv

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Introduction

This book is a polished version of my course notes for Math 6283, Several Complex Variables, given in Spring 2014, Spring 2016, and Spring 2019 semesters at Oklahoma State University. It is meant for a semester-long course. There are quite a few exercises of various difficulty sprinkled throughout the text, and I hope a reader is at least attempting all of them. Many are required later in the text. The reader should attempt exercises in sequence as earlier exercises can help or even be required to solve later exercises.

The prerequisites are a decent knowledge of vector calculus, basic real-analysis, and a working knowledge of complex analysis in one variable. Measure theory (Lebesgue integral and its convergence theorems) is useful, but it is not essential except in a couple of places later in the book. The first two chapters and most of the third is accessible to beginning graduate students after one semester of a standard single-variable complex analysis graduate course. From time to time (e.g. proof of Baouendi–Trèves in chapter 3, and most of chapter 4, and chapter 5), basic knowledge of differential forms is useful, and in chapter 6 we use some basic ring theory from algebra. By design it can replace the second semester of complex analysis.

This book is not meant as an exhaustive reference. It is simply a whirlwind tour of several complex variables that has slightly more material than can be covered within a semester. See the end of the book for a list of books for reference and further reading. There are also appendices for a list of one-variable results, an overview of differential forms, and some basic algebra. See appendix B, appendix C, and appendix D.

0.1 Motivation, single variable, and Cauchy's formula

Let us start with some standard notation. We use $\mathbb{C}$ for the complex numbers, $\mathbb{R}$ for real numbers, $\mathbb{Z}$ for integers, $\mathbb{N} = \{1, 2, 3, \ldots\}$ for natural numbers, $i = \sqrt{-1}$. Throughout this book, we use the standard terminology of *domain* to mean connected open set. We will try to avoid using it if connectedness is not needed, but sometimes we use it just for simplicity.

As complex analysis deals with the complex numbers, perhaps we should begin with $\sqrt{-1}$. We start with the real numbers, $\mathbb{R}$, and we add $\sqrt{-1}$ into our field. We call this square root i, and we write the complex numbers, $\mathbb{C}$, by identifying $\mathbb{C}$ with $\mathbb{R}^2$ using

$$z = x + iy,$$

where $z \in \mathbb{C}$, and $(x, y) \in \mathbb{R}^2$. A subtle philosophical issue is that there are two square roots of -1. There are two chickens running around in our yard, and because we like to know which is which, we catch one and write "i" on it. If we happened to have caught the other chicken, we would have gotten an exactly equivalent theory, which we could not tell apart from the original.

Given a complex number z, its "opposite" is the *complex conjugate* of z and is defined as

$$\bar{z} \overset{\text{def}}{=} x - iy.$$

The size of z is measured by the so-called *modulus*, which is just the *Euclidean distance*:

$$|z| \overset{\text{def}}{=} \sqrt{z\bar{z}} = \sqrt{x^2 + y^2}.$$

If $z = x + iy \in \mathbb{C}$ for $x, y \in \mathbb{R}$, then x is called the *real part* and y is called the *imaginary part*. We write

$$\operatorname{Re} z = \operatorname{Re}(x + iy) = \frac{z + \bar{z}}{2} = x, \qquad \operatorname{Im} z = \operatorname{Im}(x + iy) = \frac{z - \bar{z}}{2i} = y.$$

A function $f \colon U \subset \mathbb{R}^n \to \mathbb{C}$ for an open set U is said to be continuously differentiable, or C^1 if the first (real) partial derivatives exist and are continuous. Similarly, it is C^k or C^k*-smooth* if the first k partial derivatives all exist and are differentiable. Finally, a function is said to be C^∞ or simply *smooth** if it is *infinitely differentiable*, or in other words, if it is C^k for all $k \in \mathbb{N}$.

Complex analysis is the study of holomorphic (or complex-analytic) functions. Holomorphic functions are a generalization of polynomials, and to get there one leaves the land of algebra to arrive in the realm of analysis. There is an awful lot one can do with polynomials, but sometimes they are just not enough. For example, there is no nonzero polynomial function that solves the simplest of differential equations, $f' = f$. We need the exponential function, which is holomorphic.

Let us start with polynomials. In one variable, a polynomial in z is an expression of the form

$$P(z) = \sum_{j=0}^{d} c_j\, z^j,$$

where $c_j \in \mathbb{C}$ and $c_d \neq 0$. The number d is called the *degree* of the polynomial P. We can plug in some number z and simply compute $P(z)$, so we have a function $P \colon \mathbb{C} \to \mathbb{C}$.

We try to write

$$f(z) = \sum_{j=0}^{\infty} c_j\, z^j$$

and all is very fine until we wish to know what $f(z)$ is for some number $z \in \mathbb{C}$. We usually mean

$$\sum_{j=0}^{\infty} c_j\, z^j = \lim_{d\to\infty} \sum_{j=0}^{d} c_j\, z^j.$$

As long as the limit exists, we have a function. You know all this; it is your one-variable complex analysis. We usually start with the functions and prove that we can expand into series.

Let $U \subset \mathbb{C}$ be open. A function $f \colon U \to \mathbb{C}$ is *holomorphic* (or *complex-analytic*) if it is *complex-differentiable* at every point, that is, if

$$f'(z) = \lim_{\xi \in \mathbb{C} \to 0} \frac{f(z + \xi) - f(z)}{\xi} \qquad \text{exists for all } z \in U.$$

*While C^∞ is the common definition of *smooth*, not everyone always means the same thing by the word *smooth*. I have seen it mean differentiable, C^1, piecewise-C^1, C^∞, holomorphic, …

Importantly, the limit is taken with respect to complex ξ. Another vantage point is to start with a continuously differentiable* f, and say that $f = u + i\,v$ is holomorphic if it satisfies the *Cauchy–Riemann equations*:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

The so-called *Wirtinger operators*,

$$\frac{\partial}{\partial z} \overset{\text{def}}{=} \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right), \qquad \frac{\partial}{\partial \bar{z}} \overset{\text{def}}{=} \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right),$$

provide an easier way to understand the Cauchy–Riemann equations. These operators are determined by insisting

$$\frac{\partial}{\partial z} z = 1, \quad \frac{\partial}{\partial z} \bar{z} = 0, \quad \frac{\partial}{\partial \bar{z}} z = 0, \quad \frac{\partial}{\partial \bar{z}} \bar{z} = 1.$$

The function f is holomorphic if and only if

$$\frac{\partial f}{\partial \bar{z}} = 0.$$

That seems a far nicer statement of the Cauchy–Riemann equations, and it is just one complex equation. It says a function is holomorphic if and only if it depends on z but not on $\bar{z}$ (perhaps that might not make a whole lot of sense at first glance). Let us check:

$$\frac{\partial f}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right) = \frac{1}{2}\left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} + i\frac{\partial u}{\partial y} - \frac{\partial v}{\partial y}\right) = \frac{1}{2}\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) + \frac{i}{2}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right).$$

This expression is zero if and only if the real parts and the imaginary parts are zero. In other words

$$\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} = 0, \qquad \text{and} \qquad \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} = 0.$$

That is, the Cauchy–Riemann equations are satisfied.

If f is holomorphic, the derivative in z is the standard complex derivative you know and love:

$$\frac{\partial f}{\partial z}(z_0) = f'(z_0) = \lim_{\xi \to 0} \frac{f(z_0 + \xi) - f(z_0)}{\xi}.$$

That's because

$$\begin{aligned}\frac{\partial f}{\partial z} = \frac{1}{2}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + \frac{i}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) &= \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial f}{\partial x} \\ &= \frac{1}{i}\left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right) = \frac{\partial f}{\partial (iy)}.\end{aligned}$$

A function on $\mathbb{C}$ is really a function defined on $\mathbb{R}^2$ as identified above and hence it is a function of x and y. Writing $x = \frac{z+\bar{z}}{2}$ and $y = \frac{z-\bar{z}}{2i}$, we think of it as a function of two complex variables z

*Holomorphic functions end up being infinitely differentiable anyway, so this hypothesis is not overly restrictive.

and $\bar{z}$. Pretend for a moment as if $\bar{z}$ did not depend on z. The Wirtinger operators work as if z and $\bar{z}$ really were independent variables. For example:

$$\frac{\partial}{\partial z}\left[z^2\bar{z}^3 + z^{10}\right] = 2z\bar{z}^3 + 10z^9 \qquad \text{and} \qquad \frac{\partial}{\partial \bar{z}}\left[z^2\bar{z}^3 + z^{10}\right] = z^2(3\bar{z}^2) + 0.$$

So a holomorphic function is a function not depending on $\bar{z}$.

The most important theorem in one variable is the *Cauchy integral formula*.

Theorem 0.1.1 (Cauchy integral formula)**.** *Let $U \subset \mathbb{C}$ be a bounded domain where the boundary ∂U is a piecewise smooth simple closed path (a Jordan curve). Let $f : \overline{U} \to \mathbb{C}$ be a continuous function, holomorphic in U. Orient ∂U positively (going around counter-clockwise). Then*

$$f(z) = \frac{1}{2\pi i}\int_{\partial U} \frac{f(\zeta)}{\zeta - z}\, d\zeta \qquad \textit{for all } z \in U.$$

The Cauchy formula is the essential ingredient we need from one complex variable. It follows, for example, from Green's theorem* (Stokes' theorem in two dimensions). You can look forward to Theorem 4.1.1 for a proof of a more general formula, the Cauchy–Pompeiu integral formula.

As a differential form, $dz = dx + i\,dy$. If you are uneasy about differential forms, you probably defined the path integral above directly using the Riemann–Stieltjes integral in your one-complex-variable class. Let us write down the formula in terms of the standard Riemann integral in a special case. Take the *unit disc*

$$\mathbb{D} \overset{\text{def}}{=} \left\{z \in \mathbb{C} : |z| < 1\right\}.$$

The boundary is the unit circle $\partial\mathbb{D} = \left\{z \in \mathbb{C} : |z| = 1\right\}$ oriented positively, that is, counter-clockwise. Parametrize $\partial\mathbb{D}$ by e^{it}, where t goes from 0 to 2π. If $\zeta = e^{it}$, then $d\zeta = ie^{it}dt$, and

$$f(z) = \frac{1}{2\pi i}\int_{\partial\mathbb{D}} \frac{f(\zeta)}{\zeta - z}\, d\zeta = \frac{1}{2\pi}\int_0^{2\pi} \frac{f(e^{it})e^{it}}{e^{it} - z}\, dt.$$

If you are not completely comfortable with path integrals, try to think about how you would parametrize the path, and write the integral as an integral any calculus student would recognize.

I venture a guess that 90% of what you learned in a one-variable complex analysis course (depending on who taught it) is more or less a straightforward consequence of the Cauchy integral formula. An important theorem from one variable that follows from the Cauchy formula is the *maximum modulus principle* (or just the *maximum principle*). Let us give its simplest version.

Theorem 0.1.2 (Maximum modulus principle)**.** *Suppose $U \subset \mathbb{C}$ is a domain and $f : U \to \mathbb{C}$ is holomorphic. If for some $z_0 \in U$*

$$\sup_{z\in U} |f(z)| = |f(z_0)|,$$

then f is constant, that is, $f \equiv f(z_0)$.

That is, if the supremum is attained in the interior of the domain, then the function must be constant. Another way to state the maximum principle is to say: if f extends continuously to the boundary of a domain, then the supremum of $|f(z)|$ is attained on the boundary. In one variable you learned that the maximum principle is really a property of harmonic functions.

*If you wish to feel inadequate, note that this theorem on which all of complex analysis (and all of physics) rests was proved by George Green, who was the son of a miller and had one year of formal schooling.

Theorem 0.1.3 (Maximum principle). *Let $U \subset \mathbb{C}$ be a domain and $h \colon U \to \mathbb{R}$ harmonic, that is,*

$$\nabla^2 h = \frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} = 0.$$

If for some $z_0 \in U$

$$\sup_{z \in U} h(z) = h(z_0) \qquad \text{or} \qquad \inf_{z \in U} h(z) = h(z_0),$$

then h is constant, that is, $h \equiv h(z_0)$.

In one variable, if $f = u + iv$ is holomorphic for real valued u and v, then u and v are harmonic. Locally, any harmonic function is the real (or imaginary) part of a holomorphic function, so in one complex variable, studying harmonic functions is almost equivalent to studying holomorphic functions. Things are decidedly different in two or more variables.

Holomorphic functions admit a power series representation in z at each point a:

$$f(z) = \sum_{j=0}^{\infty} c_j (z-a)^j.$$

There is no $\bar{z}$ necessary there since $\frac{\partial f}{\partial \bar{z}} = 0$.

Let us see the proof using the Cauchy integral formula as we will require this computation in several variables as well. Given $a \in \mathbb{C}$ and $\rho > 0$, define the disc of radius ρ around a

$$\Delta_\rho(a) \overset{\text{def}}{=} \{ z \in \mathbb{C} : |z-a| < \rho \}.$$

Suppose $U \subset \mathbb{C}$ is open, $f \colon U \to \mathbb{C}$ is holomorphic, $a \in U$, and $\overline{\Delta_\rho(a)} \subset U$ (that is, the closure of the disc is in U, and so its boundary $\partial \Delta_\rho(a)$ is also in U).

For $z \in \Delta_\rho(a)$ and $\zeta \in \partial \Delta_\rho(a)$,

$$\left| \frac{z-a}{\zeta - a} \right| = \frac{|z-a|}{\rho} < 1.$$

In fact, if $|z-a| \leq \rho' < \rho$, then $\left| \frac{z-a}{\zeta-a} \right| \leq \frac{\rho'}{\rho} < 1$. Therefore, the geometric series

$$\sum_{j=0}^{\infty} \left(\frac{z-a}{\zeta-a} \right)^j = \frac{1}{1 - \frac{z-a}{\zeta-a}} = \frac{\zeta - a}{\zeta - z}$$

converges uniformly absolutely for $(z, \zeta) \in \overline{\Delta_{\rho'}(a)} \times \partial \Delta_\rho(a)$ (that is, $\sum_j \left| \frac{z-a}{\zeta-a} \right|^j$ converges uniformly).

Let γ be the path going around $\partial \Delta_\rho(a)$ once in the positive direction. Compute

$$\begin{aligned}
f(z) &= \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{\zeta - z} \, d\zeta \\
&= \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{\zeta - a} \frac{\zeta - a}{\zeta - z} \, d\zeta \\
&= \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{\zeta - a} \sum_{j=0}^{\infty} \left(\frac{z-a}{\zeta-a} \right)^j d\zeta \\
&= \sum_{j=0}^{\infty} \left(\frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{(\zeta-a)^{j+1}} \, d\zeta \right) (z-a)^j.
\end{aligned}$$

In the last equality, we may interchange the limit on the sum with the integral either via Fubini's theorem or via uniform convergence: z is fixed and if M is the supremum of $\left|\frac{f(\zeta)}{\zeta-a}\right| = \frac{|f(\zeta)|}{\rho}$ on $\partial\Delta_\rho(a)$, then

$$\left|\frac{f(\zeta)}{\zeta-a}\left(\frac{z-a}{\zeta-a}\right)^j\right| \leq M\left(\frac{|z-a|}{\rho}\right)^j, \qquad \text{and} \qquad \frac{|z-a|}{\rho} < 1.$$

The key point is writing the *Cauchy kernel* $\frac{1}{\zeta-z}$ as

$$\frac{1}{\zeta-z} = \frac{1}{\zeta-a}\frac{\zeta-a}{\zeta-z},$$

and then using the geometric series.

Not only have we proved that f has a power series, but we computed that the radius of convergence is at least R, where R is the maximum R such that $\Delta_R(a) \subset U$. We also obtained a formula for the coefficients

$$c_j = \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{(\zeta-a)^{j+1}}\,d\zeta.$$

For a set K, denote the *supremum norm*:

$$\|f\|_K \overset{\text{def}}{=} \sup_{z\in K}|f(z)|.$$

By a brute force estimation, we obtain the very useful *Cauchy estimates*:

$$|c_j| = \left|\frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{(\zeta-a)^{j+1}}\,d\zeta\right| \leq \frac{1}{2\pi}\int_\gamma \frac{\|f\|_\gamma}{\rho^{j+1}}\,|d\zeta| = \frac{\|f\|_\gamma}{\rho^j}.$$

We differentiate Cauchy's formula j times (using the Wirtinger $\frac{\partial}{\partial z}$ operator),

$$f^{(j)}(z) = \frac{\partial^j f}{\partial z^j}(z) = \frac{1}{2\pi i}\int_\gamma \frac{j!\,f(\zeta)}{(\zeta-z)^{j+1}}\,d\zeta,$$

and therefore

$$j!\,c_j = f^{(j)}(a) = \frac{\partial^j f}{\partial z^j}(a).$$

Hence, we can control derivatives of f by the size of the function:

$$\left|f^{(j)}(a)\right| = \left|\frac{\partial^j f}{\partial z^j}(a)\right| \leq \frac{j!\|f\|_\gamma}{\rho^j}.$$

This estimate is one of the key properties of holomorphic functions, and the reason why the correct topology for the set of holomorphic functions is the same as the topology for continuous functions. Consequently, obstructions to solving problems in complex analysis are often topological in character.

For further review of one-variable results, see appendix B.

Chapter 1

Holomorphic functions in several variables

1.1 Onto several variables

Let $\mathbb{C}^n = \overbrace{\mathbb{C} \times \mathbb{C} \times \cdots \times \mathbb{C}}^{n \text{ times}}$ denote the n-dimensional *complex Euclidean space*. Denote by $z = (z_1, z_2, \ldots, z_n)$ the coordinates of $\mathbb{C}^n$. Let $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$ denote coordinates in $\mathbb{R}^n$. We identify $\mathbb{C}^n$ with $\mathbb{R}^{2n}$ by letting $z = x + iy$, that is, $z_j = x_j + iy_j$ for every j. As in one complex variable, we write $\bar{z} = x - iy$. We call z the *holomorphic coordinates* and $\bar{z}$ the *antiholomorphic coordinates*.

Definition 1.1.1. For $\rho = (\rho_1, \rho_2, \ldots, \rho_n)$ where $\rho_j > 0$ and $a \in \mathbb{C}^n$, define a *polydisc*

$$\Delta_\rho(a) \overset{\text{def}}{=} \left\{ z \in \mathbb{C}^n : |z_j - a_j| < \rho_j \text{ for } j = 1, 2, \ldots, n \right\}.$$

We call a the *center* and ρ the *polyradius* or simply the *radius* of the polydisc $\Delta_\rho(a)$. If $\rho > 0$ is a number, then

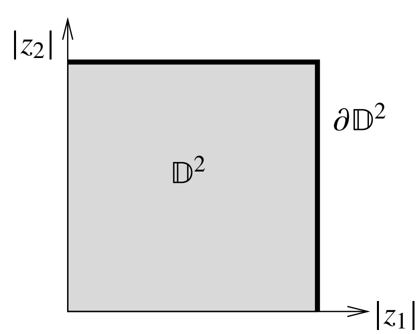
$$\Delta_\rho(a) \overset{\text{def}}{=} \left\{ z \in \mathbb{C}^n : |z_j - a_j| < \rho \text{ for } j = 1, 2, \ldots, n \right\}.$$

In two variables, a polydisc is sometimes called a *bidisc*.

As there is the unit disc $\mathbb{D}$ in one variable, so is there the *unit polydisc* in several variables:

$$\mathbb{D}^n = \mathbb{D} \times \mathbb{D} \times \cdots \times \mathbb{D} = \Delta_1(0) = \left\{ z \in \mathbb{C}^n : |z_j| < 1 \text{ for } j = 1, 2, \ldots, n \right\}.$$

In more than one complex dimension, it is difficult to draw exact pictures for lack of real dimensions on our paper. We visualize the unit polydisc in two variables by drawing the following picture by plotting just against the modulus of the variables:



Recall the *Euclidean inner product* on $\mathbb{C}^n$:

$$\langle z, w \rangle \overset{\text{def}}{=} z_1 \bar{w}_1 + z_2 \bar{w}_2 + \cdots + z_n \bar{w}_n.$$

The inner product gives us the standard *Euclidean norm* on $\mathbb{C}^n$:

$$\|z\| \overset{\text{def}}{=} \sqrt{\langle z, z \rangle} = \sqrt{|z_1|^2 + |z_2|^2 + \cdots + |z_n|^2}.$$

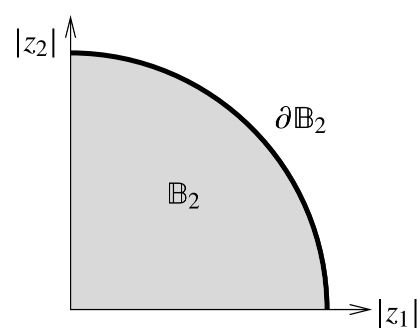
This norm agrees with the standard Euclidean norm on $\mathbb{R}^{2n}$. Define *balls* as in $\mathbb{R}^{2n}$:

$$B_\rho(a) \overset{\text{def}}{=} \{ z \in \mathbb{C}^n : \|z - a\| < \rho \},$$

And define the *unit ball* as

$$\mathbb{B}_n \overset{\text{def}}{=} B_1(0) = \{ z \in \mathbb{C}^n : \|z\| < 1 \}.$$

A ball centered at the origin can also be pictured by plotting against the modulus of the variables since the inequality defining the ball only depends on the moduli of the variables. Not every domain can be drawn like this, but those that can are called *Reinhardt domain*, more on this later. Here is a picture of $\mathbb{B}_2$:



Definition 1.1.2. Let $U \subset \mathbb{C}^n$ be an open set. A function $f : U \to \mathbb{C}$ is *holomorphic* if it is *locally bounded** and holomorphic in each variable separately. In other words, f is holomorphic if it is locally bounded and complex-differentiable in each variable separately:

$$\lim_{\xi \in \mathbb{C} \to 0} \frac{f(z_1, \ldots, z_k + \xi, \ldots, z_n) - f(z)}{\xi} \qquad \text{exists for all } z \in U \text{ and all } k = 1, 2, \ldots, n.$$

In this book, however, the words "differentiable" and "derivative" (without the "complex-") refer to plain vanilla real differentiability.

As in one variable, we define the *Wirtinger operators*

$$\frac{\partial}{\partial z_j} \overset{\text{def}}{=} \frac{1}{2} \left(\frac{\partial}{\partial x_j} - i \frac{\partial}{\partial y_j} \right), \qquad \frac{\partial}{\partial \bar{z}_j} \overset{\text{def}}{=} \frac{1}{2} \left(\frac{\partial}{\partial x_j} + i \frac{\partial}{\partial y_j} \right).$$

*For every $p \in U$, there is a neighborhood N of p such that $f|_N$ is bounded. It is a deep result of Hartogs that we might in fact just drop "locally bounded" from the definition and obtain the same set of functions.

An alternative definition is to say that a continuously differentiable function $f : U \to \mathbb{C}$ is *holomorphic* if it satisfies the *Cauchy–Riemann equations*

$$\frac{\partial f}{\partial \bar{z}_j} = 0 \qquad \text{for } j = 1, 2, \ldots, n.$$

As in one variable, if you defined partial derivatives of holomorphic functions as the limits of the definition above, then you would get the Wirtinger $\frac{\partial}{\partial z_k}$ for holomorphic functions. That is, if f is holomorphic, then

$$\frac{\partial f}{\partial z_k}(z) = \lim_{\xi \in \mathbb{C} \to 0} \frac{f(z_1, \ldots, z_k + \xi, \ldots, z_n) - f(z)}{\xi}.$$

Due to the following proposition, the alternative definition using the Cauchy–Riemann equations is just as good as the definition we gave.

Proposition 1.1.3. *Let $U \subset \mathbb{C}^n$ be an open set and suppose $f : U \to \mathbb{C}$ is holomorphic. Then f is infinitely differentiable.*

Proof. Suppose $\Delta = \Delta_\rho(a) = \Delta_1 \times \cdots \times \Delta_n$ is a polydisc centered at a, where each Δ_j is a disc and suppose $\overline{\Delta} \subset U$, that is, f is holomorphic on a neighborhood of the closure of Δ. Let z be in Δ. Orient $\partial \Delta_1$ positively and apply the Cauchy formula (after all f is holomorphic in z_1):

$$f(z) = \frac{1}{2\pi i} \int_{\partial \Delta_1} \frac{f(\zeta_1, z_2, \ldots, z_n)}{\zeta_1 - z_1} \, d\zeta_1.$$

Apply it again on the second variable, again orienting $\partial \Delta_2$ positively:

$$f(z) = \frac{1}{(2\pi i)^2} \int_{\partial \Delta_1} \int_{\partial \Delta_2} \frac{f(\zeta_1, \zeta_2, z_3, \ldots, z_n)}{(\zeta_1 - z_1)(\zeta_2 - z_2)} \, d\zeta_2 \, d\zeta_1.$$

Applying the formula n times we obtain

$$f(z) = \frac{1}{(2\pi i)^n} \int_{\partial \Delta_1} \int_{\partial \Delta_2} \cdots \int_{\partial \Delta_n} \frac{f(\zeta_1, \zeta_2, \ldots, \zeta_n)}{(\zeta_1 - z_1)(\zeta_2 - z_2) \cdots (\zeta_n - z_n)} \, d\zeta_n \cdots d\zeta_2 \, d\zeta_1. \tag{1.1}$$

As f is bounded on the compact set $\partial \Delta_1 \times \cdots \times \partial \Delta_n$, we find that f is continuous in Δ. We may differentiate underneath the integral, because f is bounded on $\partial \Delta_1 \times \cdots \times \partial \Delta_n$ and so are the derivatives of the integrand with respect to x_j and y_j, where $z_j = x_j + i y_j$, as long as z is a positive distance away from $\partial \Delta_1 \times \cdots \times \partial \Delta_n$. We may differentiate as many times as we wish. □

Above in (1.1), we derived the Cauchy integral formula in several variables. To write the formula more concisely we apply the Fubini's theorem to write it as a single integral. We will write it down using differential forms. If you are unfamiliar with differential forms, think of the integral as the iterated integral above, and you can read the next few paragraphs a little lightly. It is enough to understand real differential forms; we simply allow complex coefficients. For a good introduction to differential forms see Rudin [R2].

Recall that given real coordinates $x = (x_1, \ldots, x_n)$, a one-form dx_j is a linear functional on tangent vectors such that when $dx_j\big(\frac{\partial}{\partial x_j}\big) = 1$ and $dx_j\big(\frac{\partial}{\partial x_k}\big) = 0$ if $j \neq k$. Because $z_j = x_j + i y_j$ and $\bar{z}_j = x_j - i y_j$,

$$dz_j = dx_j + i \, dy_j, \qquad d\bar{z}_j = dx_j - i \, dy_j.$$

As expected,

$$dz_j\left(\frac{\partial}{\partial z_k}\right) = \delta_j^k, \qquad dz_j\left(\frac{\partial}{\partial \bar{z}_k}\right) = 0, \qquad d\bar{z}_j\left(\frac{\partial}{\partial z_k}\right) = 0, \qquad d\bar{z}_j\left(\frac{\partial}{\partial \bar{z}_k}\right) = \delta_j^k,$$

where δ_j^k is the Kronecker delta, that is, $\delta_j^j = 1$, and $\delta_j^k = 0$ if $j \neq k$. One-forms are objects of the form

$$\sum_{j=1}^n \alpha_j\, dz_j + \beta_j\, d\bar{z}_j,$$

where α_j and β_j are functions (of z). Also recall the wedge product $\omega \wedge \eta$ is anti-commutative on one-forms, that is, for one-forms ω and η, $\omega \wedge \eta = -\eta \wedge \omega$. A wedge product of two one-forms is a two-form, and so on. A k-form is an object that can be integrated on a so-called k-chain, for example, a k-dimensional surface. The wedge product takes care of the orientation.

At this point, we need to talk about orientation in $\mathbb{C}^n$. There are two natural real-linear isomorphisms of $\mathbb{C}^n$ and $\mathbb{R}^{2n}$. That is, we identify $z = x + iy$ as

$$(x, y) = (x_1, \ldots, x_n, y_1, \ldots, y_n) \qquad \text{or} \qquad (x_1, y_1, x_2, y_2, \ldots, x_n, y_n).$$

If we take the natural orientation of $\mathbb{R}^{2n}$, it is possible that we obtain (if n is even) two opposite orientations on $\mathbb{C}^n$. The one we take as the natural orientation of $\mathbb{C}^n$ (in this book) corresponds to the second ordering above, that is, $(x_1, y_1, \ldots, x_n, y_n)$. Both isomorphisms may be used in computation as long as they are used consistently, and the underlying orientation is kept in mind.

Theorem 1.1.4 (Cauchy integral formula)**.** *Let $\Delta \subset \mathbb{C}^n$ be a polydisc. Suppose $f : \overline{\Delta} \to \mathbb{C}$ is a continuous function holomorphic in Δ. Write $\Gamma = \partial\Delta_1 \times \cdots \times \partial\Delta_n$ oriented appropriately (each $\partial\Delta_j$ has positive orientation). Then for $z \in \Delta$*

$$f(z) = \frac{1}{(2\pi i)^n} \int_\Gamma \frac{f(\zeta_1, \zeta_2, \ldots, \zeta_n)}{(\zeta_1 - z_1)(\zeta_2 - z_2)\cdots(\zeta_n - z_n)}\, d\zeta_1 \wedge d\zeta_2 \wedge \cdots \wedge d\zeta_n.$$

We stated a more general result where f is only continuous on $\overline{\Delta}$ and holomorphic in Δ. The proof of this slight generalization is contained within the next two exercises.

Exercise* 1.1.1*: *Suppose $f : \overline{\mathbb{D}^2} \to \mathbb{C}$ is continuous and holomorphic on $\mathbb{D}^2$. For any $\theta \in \mathbb{R}$, prove*

$$g_1(\xi) = f(\xi, e^{i\theta}) \qquad \text{and} \qquad g_2(\xi) = f(e^{i\theta}, \xi)$$

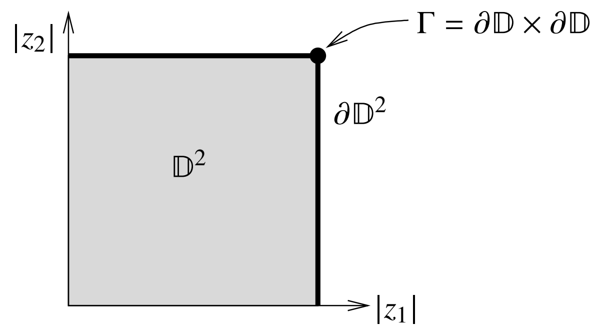
are holomorphic in $\mathbb{D}$.

Exercise* 1.1.2*: *Prove the theorem above, that is, the slightly more general Cauchy integral formula given f is only continuous on $\overline{\Delta}$ and holomorphic in Δ.*

The Cauchy integral formula shows an important and subtle point about holomorphic functions in several variables: the value of the function f on Δ is completely determined by the values of f

on the set Γ, which is much smaller than the boundary of the polydisc $\partial\Delta$. In fact, the Γ is of real dimension n, while the boundary of the polydisc has real dimension $2n - 1$.

The set $\Gamma = \partial\Delta_1 \times \cdots \times \partial\Delta_n$ is called the *distinguished boundary*. For the unit bidisc we have:



The set Γ is a 2-dimensional torus, like the surface of a donut. Whereas the set $\partial\mathbb{D}^2 = (\partial\mathbb{D} \times \overline{\mathbb{D}}) \cup (\overline{\mathbb{D}} \times \partial\mathbb{D})$ is the union of two filled donuts, or more precisely it is both the inside and the outside of the donut put together, and these two things meet on the surface of the donut. So the set Γ is quite small in comparison to the entire boundary.

Exercise* 1.1.3*: *Suppose Δ is a polydisc, Γ its distinguished boundary, and $f : \overline{\Delta} \to \mathbb{C}$ is continuous on $\overline{\Delta}$ and holomorphic on Δ. Prove $|f(z)|$ achieves its maximum on Γ.*

Exercise* 1.1.4*: *A ball is different from a polydisc. Prove that for any $z \in \partial\mathbb{B}_n$ there exists a continuous $f : \overline{\mathbb{B}_n} \to \mathbb{C}$, holomorphic on $\mathbb{B}_n$ such that $|f(z)|$ achieves a strict maximum at z.*

Exercise* 1.1.5*: *Show that in the real setting, differentiable in each variable separately does not imply differentiable even if the function is locally bounded. Let $f(x, y) = \frac{xy}{x^2+y^2}$ outside the origin and $f(0,0) = 0$. Prove that f is a locally bounded function in $\mathbb{R}^2$, which is differentiable in each variable separately (all partial derivatives exist), but the function is not even continuous. There is something very special about the holomorphic category.*

Exercise* 1.1.6*: *Suppose $U \subset \mathbb{C}^n$ is open. Prove that $f : U \to \mathbb{C}$ is holomorphic, if and only if, f is locally bounded and for every $a, b \in \mathbb{C}^n$, the function $\zeta \mapsto f(\zeta a + b)$ is holomorphic on the open set $\{\zeta \in \mathbb{C} : \zeta a + b \in U\}$.*

Exercise* 1.1.7*: *Prove a several complex variables version of Morera's theorem (see* Theorem B.4*). A triangle $T \subset \mathbb{C}^n$ is the closed convex hull of three points, so including the inside. Orient T in some way (will not matter which way) and orient ∂T accordingly. Suppose $U \subset \mathbb{C}^n$ is open and $f : U \to \mathbb{C}$ is continuous. Prove that f is holomorphic if and only if*

$$\int_{\partial T} f(z)\, dz_k = 0$$

for every triangle $T \subset U$ and every $k = 1, 2, \ldots, n$. Hint: The previous exercise may be useful.

1.2 Power series representation

As you noticed, writing out all the components can be a pain. It would become even more painful later on. Just as we write vectors as z instead of $(z_1, z_2, \ldots, z_n)$, we similarly define the so-called *multi-index notation* to deal with formulas such as the ones above.

Let $\alpha \in \mathbb{N}_0^n$ be a vector of nonnegative integers (where $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$). We write

$$z^\alpha \overset{\text{def}}{=} z_1^{\alpha_1} z_2^{\alpha_2} \cdots z_n^{\alpha_n}, \qquad \frac{1}{z} \overset{\text{def}}{=} \frac{1}{z_1 z_2 \cdots z_n}, \qquad \frac{z}{w} \overset{\text{def}}{=} \left(\frac{z_1}{w_1}, \frac{z_2}{w_2}, \ldots, \frac{z_n}{w_n} \right),$$

$$\lvert z\rvert^\alpha \overset{\text{def}}{=} \lvert z_1\rvert^{\alpha_1} \lvert z_2\rvert^{\alpha_2} \cdots \lvert z_n\rvert^{\alpha_n}, \qquad \frac{\partial^{\lvert\alpha\rvert}}{\partial z^\alpha} \overset{\text{def}}{=} \frac{\partial^{\alpha_1}}{\partial z_1^{\alpha_1}} \frac{\partial^{\alpha_2}}{\partial z_2^{\alpha_2}} \cdots \frac{\partial^{\alpha_n}}{\partial z_n^{\alpha_n}}, \qquad dz \overset{\text{def}}{=} dz_1 \wedge dz_2 \wedge \cdots \wedge dz_n,$$

$$\lvert\alpha\rvert \overset{\text{def}}{=} \alpha_1 + \alpha_2 + \cdots + \alpha_n, \qquad \alpha! \overset{\text{def}}{=} \alpha_1! \alpha_2! \cdots \alpha_n!.$$

We can also make sense of this notation, especially the notation z^α, if $\alpha \in \mathbb{Z}^n$, that is, if it includes negative integers, although usually α is assumed to be in $\mathbb{N}_0^n$. Furthermore, when we use 1 as a vector it means $(1, 1, \ldots, 1)$. For example, if $z \in \mathbb{C}^n$ then,

$$1 - z = (1 - z_1, 1 - z_2, \ldots, 1 - z_n), \qquad \text{or} \qquad z^{\alpha+1} = z_1^{\alpha_1+1} z_2^{\alpha_2+1} \cdots z_n^{\alpha_n+1}.$$

It goes without saying that when using this notation it is important to be careful to always realize which symbol lives where, and most of all, to not get carried away. For example one could interpret $\frac{1}{z}$ in two different ways depending on if we interpret 1 as a vector or not, and if we are expecting a vector or a number. Best to just keep to the limited set of cases as given above.

In this notation, the Cauchy formula becomes the perhaps deceptively simple

$$f(z) = \frac{1}{(2\pi i)^n} \int_\Gamma \frac{f(\zeta)}{\zeta - z} \, d\zeta.$$

Let us move to power series. For simplicity we start with power series at the origin. Using the multinomial notation we write such a series as

$$\sum_{\alpha \in \mathbb{N}_0^n} c_\alpha z^\alpha.$$

You have to admit that the above is far nicer to write than, for example, for 3 variables writing

$$\sum_{j=0}^\infty \sum_{k=0}^\infty \sum_{\ell=0}^\infty c_{jk\ell} z_1^j z_2^k z_3^\ell, \tag{1.2}$$

which is not even exactly the definition of the series sum (see below). When it is clear from context that we are talking about a power series and all the powers are nonnegative, we often write just

$$\sum_\alpha c_\alpha z^\alpha.$$

It is important to note what this means. The sum does not have a natural ordering. We are summing over $\alpha \in \mathbb{N}_0^n$, and there is no natural ordering of $\mathbb{N}_0^n$. So it makes no sense to talk about conditional convergence. When we say the series *converges*, we mean absolutely. Fortunately power series converge absolutely, and so the ordering does not matter. So if you want to write down the limit in terms of partial sums, you pick some ordering of the multiindices: $\alpha(1), \alpha(2), \ldots$, and then

$$\sum_{\alpha} c_\alpha z^\alpha = \lim_{m\to\infty} \sum_{k=1}^{m} c_{\alpha(k)} z^{\alpha(k)}.$$

By the Fubini theorem (for sums) this is also equal to the iterated sum such as in (1.2).

We say a power series $\sum_\alpha c_\alpha z^\alpha$ *converges uniformly absolutely* for $z \in X$ when $\sum_\alpha \lvert c_\alpha z^\alpha \rvert$ converges uniformly for $z \in X$.

As in one variable, we need the *geometric series in several variables*. For $z \in \mathbb{D}^n$ (unit polydisc),

$$\begin{aligned}\frac{1}{1-z} &= \frac{1}{(1-z_1)(1-z_2)\cdots(1-z_n)} = \left(\sum_{j=0}^{\infty} {z_1}^j\right)\left(\sum_{j=0}^{\infty} {z_2}^j\right)\cdots\left(\sum_{j=0}^{\infty} {z_n}^j\right) \\ &= \sum_{j_1=0}^{\infty}\sum_{j_2=0}^{\infty}\cdots\sum_{j_n=0}^{\infty}\left({z_1}^{j_1}{z_n}^{j_2}\cdots {z_n}^{j_n}\right) = \sum_{\alpha} z^\alpha.\end{aligned}$$

The series converges uniformly absolutely on all compact subsets of the unit polydisc: Any compact set in the unit polydisc is contained in a closed polydisc $\overline{\Delta}$ centered at 0 of radius $1-\epsilon$ for some $\epsilon > 0$. The convergence is uniformly absolute on $\overline{\Delta}$. This claim follows by simply noting the same fact for each factor is true in one dimension.

We now prove that holomorphic functions are precisely those having a power series expansion.

Theorem 1.2.1. *Let $\Delta = \Delta_\rho(a) \subset \mathbb{C}^n$ be a polydisc. Suppose $f \colon \overline{\Delta} \to \mathbb{C}$ is a continuous function holomorphic in Δ. Then on Δ, f is equal to a power series converging uniformly absolutely on compact subsets of Δ:*

$$f(z) = \sum_{\alpha} c_\alpha (z-a)^\alpha. \tag{1.3}$$

Conversely, if $f \colon \Delta \to \mathbb{C}$ is defined by (1.3) converging uniformly absolutely on compact subsets of Δ, then f is holomorphic on Δ.

Proof. First assume a continuous $f \colon \overline{\Delta} \to \mathbb{C}$ is holomorphic on Δ. Write $\Gamma = \partial\Delta_1 \times \cdots \times \partial\Delta_n$ and orient it positively. Take $z \in \Delta$ and $\zeta \in \Gamma$. As in one variable, write the Cauchy kernel as

$$\frac{1}{\zeta - z} = \frac{1}{\zeta - a}\,\frac{1}{\left(1 - \frac{z-a}{\zeta-a}\right)} = \frac{1}{\zeta - a}\sum_{\alpha}\left(\frac{z-a}{\zeta-a}\right)^{\alpha}.$$

Here, make sure you interpret the formulas as $\frac{1}{\zeta - z} = \frac{1}{(\zeta_1 - z_1)\cdots(\zeta_n - z_n)}$, $\frac{1}{\zeta-a} = \frac{1}{(\zeta_1-a_1)\cdots(\zeta_n-a_n)}$ and $\frac{z-a}{\zeta-a} = \left(\frac{z_1-a_1}{\zeta_1-a_1}, \ldots, \frac{z_n-a_n}{\zeta_n-a_n}\right)$.

The multivariable geometric series is a product of geometric series in one variable, and geometric series in one variable are uniformly absolutely convergent on compact subsets of the unit disc. So the series above converges uniformly absolutely for $(z, \zeta) \in K \times \Gamma$ for any compact subset K of Δ.

Compute

$$\begin{aligned} f(z) &= \frac{1}{(2\pi i)^n} \int_\Gamma \frac{f(\zeta)}{\zeta - z} d\zeta \\ &= \frac{1}{(2\pi i)^n} \int_\Gamma \frac{f(\zeta)}{\zeta - a} \sum_\alpha \left(\frac{z-a}{\zeta - a}\right)^\alpha d\zeta \\ &= \sum_\alpha \left(\frac{1}{(2\pi i)^n} \int_\Gamma \frac{f(\zeta)}{(\zeta - a)^{\alpha+1}} \, d\zeta \right) (z-a)^\alpha . \end{aligned}$$

The last equality follows by Fubini or uniform convergence just as it does in one variable.

Uniform absolute convergence (as z moves) on compact subsets of the final series follows from the uniform absolute convergence of the geometric series. It is also a direct consequence of the Cauchy estimates below.

We have shown that

$$f(z) = \sum_\alpha c_\alpha (z-a)^\alpha ,$$

where

$$c_\alpha = \frac{1}{(2\pi i)^n} \int_\Gamma \frac{f(\zeta)}{(\zeta - z)^{\alpha+1}} \, d\zeta .$$

Notice how strikingly similar the computation is to one variable.

To prove the converse statement, we note that the limit is continuous as it is a uniform-on-compact-sets limit of continuous functions, and hence it is locally bounded in Δ. Second we restrict to each variable in turn, that is,

$$z_j \mapsto \sum_\alpha c_\alpha (z-a)^\alpha ,$$

and this function is holomorphic via the corresponding one-variable argument. □

The converse statement also follows by applying the Cauchy–Riemann equations to the series term-wise. First one must show that the term-by-term derivative series also converges uniformly absolutely on compact subsets. It is left as an exercise. Then you apply the theorem from real analysis about derivatives of limits: if a sequence of functions and its derivatives converges uniformly, then the derivatives converge to the derivative of the limit.

Exercise* 1.2.1*: *Prove the claim above that if a power series converges uniformly absolutely on compact subsets of a polydisc Δ, then the term-by-term derivative converges. Do the proof without using the analogous result for single variable series.*

A third way to prove the converse statement of the theorem is to note that partial sums are holomorphic and write them using the Cauchy formula. Uniform convergence shows that the limit also satisfies the Cauchy formula, and differentiating under the integral obtains the result.

> ***Exercise* 1.2.2:** *Follow the logic above to prove the converse of the theorem. Do the proof without using the analogous result for single variable series. Hint: Let $\Delta'' \subset \Delta' \subset \Delta$ be two polydiscs with the same center a such that $\overline{\Delta''} \subset \Delta'$ and $\overline{\Delta'} \subset \Delta$. Apply Cauchy formula on Δ' for $z \in \overline{\Delta''}$.*

Next we organize some immediate consequences of the theorem and the calculation in the proof.

Proposition 1.2.2. *Let $\Delta = \Delta_\rho(a) \subset \mathbb{C}^n$ be a polydisc, and Γ its distinguished boundary. Suppose $f : \overline{\Delta} \to \mathbb{C}$ is a continuous function holomorphic in Δ. Then, for $z \in \Delta$,*

$$\frac{\partial^{|\alpha|} f}{\partial z^\alpha}(z) = \frac{1}{(2\pi i)^n} \int_\Gamma \frac{\alpha!\, f(\zeta)}{(\zeta - z)^{\alpha+1}}\, d\zeta.$$

In particular if f is given by (1.3),

$$c_\alpha = \frac{1}{\alpha!} \frac{\partial^{|\alpha|} f}{\partial z^\alpha}(a),$$

and we have the Cauchy estimates*:*

$$|c_\alpha| = \frac{\|f\|_\Gamma}{\rho^\alpha}.$$

In particular the coefficients of the power series only depend on derivatives of f at a (and so the values of f in an arbitrarily small neighborhood of a) and not the specific polydisc used in the theorem.

Proof. Using the Leibniz rule (taking derivatives under the integral), as long as $z \in \Delta$ and not on the boundary, we can differentiate under the integral. We are talking regular real partial differentiation, and we use it to apply the Wirtinger operator. The point is that

$$\frac{\partial}{\partial z_j} \left[\frac{1}{(\zeta_j - z_j)^k} \right] = \frac{k}{(\zeta_j - z_j)^{k+1}}.$$

Let us do a single derivative to get the idea:

$$\begin{aligned} \frac{\partial f}{\partial z_1}(z) &= \frac{\partial}{\partial z_1} \left[\frac{1}{(2\pi i)^n} \int_\Gamma \frac{f(\zeta_1, \zeta_2, \ldots, \zeta_n)}{(\zeta_1 - z_1)(\zeta_2 - z_2) \cdots (\zeta_n - z_n)}\, d\zeta_1 \wedge d\zeta_2 \wedge \cdots \wedge d\zeta_n \right] \\ &= \frac{1}{(2\pi i)^n} \int_\Gamma \frac{f(\zeta_1, \zeta_2, \ldots, \zeta_n)}{(\zeta_1 - z_1)^2(\zeta_2 - z_2) \cdots (\zeta_n - z_n)}\, d\zeta_1 \wedge d\zeta_2 \wedge \cdots \wedge d\zeta_n. \end{aligned}$$

How about we do it a second time:

$$\frac{\partial^2 f}{\partial z_1^2}(z) = \frac{1}{(2\pi i)^n} \int_\Gamma \frac{2 f(\zeta_1, \zeta_2, \ldots, \zeta_n)}{(\zeta_1 - z_1)^3(\zeta_2 - z_2) \cdots (\zeta_n - z_n)}\, d\zeta_1 \wedge d\zeta_2 \wedge \cdots \wedge d\zeta_n.$$

Notice the 2 before the f. Next time 3 is coming out, so after j derivatives in z_1 you will get the constant $j!$. It is exactly the same thing that is happening in one variable. A moment's thought will convince you that the following formula is correct for $\alpha \in \mathbb{N}_0^n$:

$$\frac{\partial^{|\alpha|} f}{\partial z^\alpha}(z) = \frac{1}{(2\pi i)^n} \int_\Gamma \frac{\alpha!\, f(\zeta)}{(\zeta - z)^{\alpha+1}}\, d\zeta.$$

Therefore

$$\alpha!\, c_\alpha = \frac{\partial^{|\alpha|} f}{\partial z^\alpha}(a).$$

And we obtain the Cauchy estimates as before:

$$\left|\frac{\partial^{|\alpha|} f}{\partial z^\alpha}(a)\right| = \left|\frac{1}{(2\pi i)^n}\int_\Gamma \frac{\alpha! f(\zeta)}{(\zeta-a)^{\alpha+1}}\,d\zeta\right| \leq \frac{1}{(2\pi)^n}\int_\Gamma \frac{\alpha!|f(\zeta)|}{\rho^{\alpha+1}}\,|d\zeta| \leq \frac{\alpha!}{\rho^\alpha}\|f\|_\Gamma. \qquad \square$$

As in one-variable theory, the Cauchy estimates prove the following proposition.

Proposition 1.2.3. *Let $U \subset \mathbb{C}^n$ be an open set. Suppose $f_j \colon U \to \mathbb{C}$ converge uniformly on compact subsets to $f \colon U \to \mathbb{C}$. If every f_j is holomorphic, then f is holomorphic and $\frac{\partial^{|\alpha|} f_j}{\partial z^\alpha}$ converge to $\frac{\partial^{|\alpha|} f}{\partial z^\alpha}$ uniformly on compact subsets.*

***Exercise* 1.2.3:** *Prove the above proposition.*

Let $W \subset \mathbb{C}^n$ be the set where a power series converges such that it diverges on the complement. The interior of W is called the *domain of convergence*. In one variable, every domain of convergence is a disc, and hence it is described with a single number (the radius). In several variables, the domain of convergence is not as easy to describe. For the multivariable geometric series the domain of convergence is the unit polydisc, but more complicated examples are easy to find.

Example 1.2.4: In $\mathbb{C}^2$, the power series

$$\sum_{k=0}^{\infty} z_1 z_2^k$$

converges absolutely on the set

$$\left\{z \in \mathbb{C}^2 : |z_2| < 1\right\} \cup \left\{z \in \mathbb{C}^2 : z_1 = 0\right\},$$

and nowhere else. This set is not quite a polydisc. It is neither an open set nor a closed set, and its closure is not the closure of the domain of convergence, which is the set $\left\{z \in \mathbb{C}^2 : |z_2| < 1\right\}$.

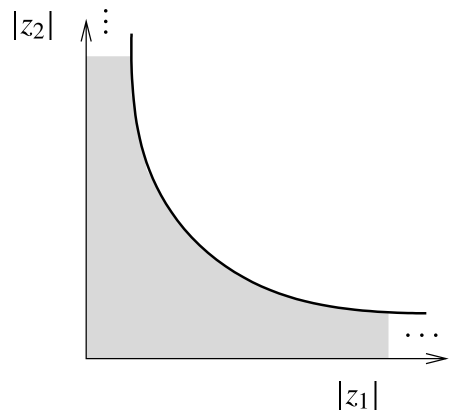
Example 1.2.5: The power series

$$\sum_{k=0}^{\infty} z_1^k z_2^k$$

converges absolutely exactly on the set

$$\left\{z \in \mathbb{C}^2 : |z_1 z_2| < 1\right\}.$$

The picture is definitely more complicated than a polydisc:



***Exercise* 1.2.4:** *Find the domain of convergence of $\sum_{j,k} \frac{1}{k!} z_1^j z_2^k$ and draw the corresponding picture.*

***Exercise* 1.2.5:** *Find the domain of convergence of $\sum_{j,k} c_{j,k} z_1^j z_2^k$ and draw the corresponding picture if $c_{k,k} = 2^k$, $c_{0,k} = c_{k,0} = 1$ and $c_{j,k} = 0$ otherwise.*

***Exercise* 1.2.6:** *Suppose a power series in two variables can be written as a sum of a power series in z_1 and a power series in z_2. Show that the domain of convergence is a polydisc.*

Suppose $U \subset \mathbb{C}^n$ is a domain is such that if $z \in U$ and $|z_j| = |w_j|$ for all j, then $w \in U$. Such a U is called a *Reinhardt domain*. The domains we were drawing so far are Reinhardt domains, they are exactly the domains that you can draw by plotting what happens for the moduli of the variables. A domain is called a *complete Reinhardt domain* if whenever $z \in U$ then for $r = (r_1, \ldots, r_n)$ where $r_j = |z_j|$ for all j, we have that the whole polydisc $\overline{\Delta_r(0)} \subset U$. So a complete Reinhardt domain is a union (possibly infinite) of polydiscs centered at the origin.

***Exercise* 1.2.7:** *Let $W \subset \mathbb{C}^n$ be the set where a certain power series at the origin converges. Show that the interior of W is a complete Reinhardt domain.*

Theorem 1.2.6 (Identity theorem)**.** *Let $U \subset \mathbb{C}^n$ be a domain (connected open set) and let $f : U \to \mathbb{C}$ be holomorphic. Suppose $f|_N \equiv 0$ for a nonempty open subset $N \subset U$. Then $f \equiv 0$.*

Proof. Let Z be the set where all derivatives of f are zero; then $N \subset Z$, so Z is nonempty. The set Z is closed in U as all derivatives are continuous. Take an arbitrary $a \in Z$. Expand f in a power series around a converging to f in a polydisc $\Delta_\rho(a) \subset U$. As the coefficients are given by derivatives of f, the power series is the zero series. Hence, f is identically zero in $\Delta_\rho(a)$. Therefore Z is open. As Z is also closed and nonempty, and U is connected, we have $Z = U$. □

The theorem is often used to show that if two holomorphic functions f and g are equal on a small open set, then $f \equiv g$.

Theorem 1.2.7 (Maximum principle)**.** *Let $U \subset \mathbb{C}^n$ be a domain (connected open set). Let $f : U \to \mathbb{C}$ be holomorphic and suppose $|f(z)|$ attains a local maximum at some $a \in U$. Then $f \equiv f(a)$.*

Proof. Suppose $|f(z)|$ attains a local maximum at $a \in U$. Consider a polydisc $\Delta = \Delta_1 \times \cdots \times \Delta_n \subset U$ centered at a. The function

$$z_1 \mapsto f(z_1, a_2, \ldots, a_n)$$

is holomorphic on the disc Δ_1 and its modulus attains the maximum at the center. Therefore it is constant by maximum principle in one variable, that is, $f(z_1, a_2, \ldots, a_n) = f(a)$ for all $z_1 \in \Delta_1$. For any fixed $z_1 \in \Delta_1$, consider the function

$$z_2 \mapsto f(z_1, z_2, a_3, \ldots, a_n).$$

This function, holomorphic on the disc Δ_2, again attains its maximum modulus at the center of Δ_2 and hence is constant on Δ_2. Iterating this procedure we obtain that $f(z) = f(a)$ for all $z \in \Delta$. By the identity theorem we have that $f(z) = f(a)$ for all $z \in U$. □

Exercise* 1.2.8*: *Let V be the volume measure on $\mathbb{R}^{2n}$ and hence on $\mathbb{C}^n$. Suppose Δ centered at $a \in \mathbb{C}^n$, and f is a function holomorphic on a neighborhood of $\overline{\Delta}$. Prove*

$$f(a) = \frac{1}{V(\Delta)} \int_\Delta f(\zeta)\, dV(\zeta),$$

where $V(\Delta)$ is the volume of Δ and dV is the volume measure. That is, $f(a)$ is an average of the values on a polydisc centered at a.

Exercise* 1.2.9*: *Prove the maximum principle by using the Cauchy formula instead. (Hint: Use the previous exercise)*

Exercise* 1.2.10*: *Prove a several variables analogue of the* Schwarz's lemma*: Suppose f is holomorphic in a neighborhood of $\overline{\mathbb{D}^n}$, $f(0) = 0$, and for some $k \in \mathbb{N}$ we have $\frac{\partial^{|\alpha|} f}{\partial z^\alpha}(0) = 0$ whenever $|\alpha| < k$. Further suppose for all $z \in \mathbb{D}^n$, $|f(z)| \leq M$ for some M. Show that*

$$|f(z)| \leq M \|z\|^k \qquad \text{for all } z \in \overline{\mathbb{D}^n}.$$

Exercise* 1.2.11*: *Apply the one-variable Liouville's theorem to prove it for several variables. That is, suppose $f : \mathbb{C}^n \to \mathbb{C}$ is holomorphic and bounded. Prove f is constant.*

Exercise* 1.2.12*: *Improve Liouville's theorem slightly in $\mathbb{C}^2$. A complex line though the origin is the image of a linear map $L : \mathbb{C} \to \mathbb{C}^n$.*

a) Prove that for any collection of finitely many complex lines through the origin, there exists an entire nonconstant holomorphic function ($n \geq 2$) bounded (hence constant) on these complex lines.

b) Prove that if an entire holomorphic function in $\mathbb{C}^2$ is bounded on countably many distinct complex lines through the origin then it is constant.

c) Find a nonconstant entire holomorphic function in $\mathbb{C}^3$ that is bounded on countably many distinct complex lines through the origin.

Exercise* 1.2.13*: *Prove the several variables version of* Montel's theorem*: Suppose $\{f_k\}$ is a sequence of holomorphic functions on $U \subset \mathbb{C}^n$ that is uniformly bounded. Show that there exists a subsequence $\{f_{k_j}\}$ that converges uniformly on compact subsets to some holomorphic function f. Hint: Mimic the one-variable proof.*

Exercise* 1.2.14*: *Prove a several variables version of* Hurwitz's theorem*: Suppose $\{f_k\}$ is a sequence of nowhere zero holomorphic functions on a domain $U \subset \mathbb{C}^n$ converging uniformly on compact subsets to a function f. Show that either f is identically zero, or that f is nowhere zero. Hint: Feel free to use the one-variable result.*

Exercise* 1.2.15*: *Suppose $p \in \mathbb{C}^n$ is a point and $D \subset \mathbb{C}^n$ is a ball with $p \in D$. A holomorphic function $f : D \to \mathbb{C}$ can be* analytically continued *along a path $\gamma : [0,1] \to \mathbb{C}^n$ if for every $t \in [0,1]$ there exists a ball D_t centered at $\gamma(t)$ and a holomorphic function $f_t : D_t \to \mathbb{C}$ and for each $t_0 \in [0,1]$ there is an $\epsilon > 0$ such that if $|t - t_0| < \epsilon$, then $f_t = f_{t_0}$ in $D_t \cap D_{t_0}$. Prove a several variables version of the* Monodromy theorem*: If $U \subset \mathbb{C}^n$ is a simply connected domain, $D \subset U$ a ball and $f : D \to \mathbb{C}$ a holomorphic function that can be analytically continued from $p \in D$ to any $q \in U$, then there exists a unique holomorphic function $F : U \to \mathbb{C}$ such that $F|_D = f$.*

Let us define notation for the set of holomorphic functions. At the same time, we notice that the set of holomorphic functions is a commutative ring under pointwise addition and multiplication.

Definition 1.2.8. Let $U \subset \mathbb{C}^n$ be an open set. Define $\mathcal{O}(U)$ to be the *ring of holomorphic functions*. The letter $\mathcal{O}$ is used to recognize the fundamental contribution to several complex variables by Kiyoshi Oka*.

Exercise* 1.2.16*: *Prove that $\mathcal{O}(U)$ is actually a commutative ring with the operations*

$$(f+g)(z) = f(z) + g(z), \qquad (fg)(z) = f(z)g(z).$$

Exercise* 1.2.17*: *Show that $\mathcal{O}(U)$ is an integral domain (has no zero divisors) if and only if U is connected. That is, show that U being connected is equivalent to showing that if $h(z) = f(z)g(z)$ is identically zero for $f, g \in \mathcal{O}(U)$, then either $f(z)$ or $g(z)$ are identically zero.*

A function F defined on a dense open subset of U is *meromorphic* if locally near every $p \in U$, $F = f/g$ for f and g holomorphic in some neighborhood of p. We remark that it follows from a deep result of Oka that for domains $U \subset \mathbb{C}^n$, every meromorphic function can be represented as f/g globally. That is, the ring of meromorphic functions is the field of fractions of $\mathcal{O}(U)$. This problem is the so-called *Poincarè problem*, and its solution is no longer positive once we generalize U to complex manifolds.

The points where $g = 0$ are called the *poles* of F. Unlike in one variable we do not require poles to be isolated, and in fact poles are never isolated in more than one variable. There is also a new type of singular point for meromorphic functions in more than one variable:

*See https://en.wikipedia.org/wiki/Kiyoshi_Oka

Exercise* 1.2.18*: *In two variables one can no longer think of a meromorphic function F simply taking on the value of ∞, when the denominator vanishes. Show that $F(z,w) = z/w$ achieves all values of $\mathbb{C}$ in every neighborhood of the origin. The origin is called a* point of indeterminacy.

Exercise* 1.2.19*: *Prove that zeros (and so poles) are never isolated in $\mathbb{C}^n$ for $n \geq 2$. In particular, consider $z_1 \mapsto f(z_1, z_2, \dots, z_n)$ as you move $z_2, \dots, z_n$ around, and use for example Hurwitz.*

1.3 Derivatives

Given a function $f = u + iv$, the complex conjugate is $\bar{f} = u - iv$, defined simply by $z \mapsto \overline{f(z)}$. When f is holomorphic, then $\bar{f}$ is called an *antiholomorphic function*. An antiholomorphic function is a function that does not depend on z, but only on $\bar{z}$. So if we write the variable, we write $\bar{f}$ as $\bar{f}(\bar{z})$. Let us see why this makes sense. Using the definitions of the Wirtinger operators,

$$\frac{\partial \bar{f}}{\partial z_j} = \overline{\frac{\partial f}{\partial \bar{z}_j}} = 0, \qquad \frac{\partial \bar{f}}{\partial \bar{z}_j} = \overline{\left(\frac{\partial f}{\partial z_j}\right)}, \qquad \text{for all } j = 1, \dots, n.$$

For functions that are neither holomorphic or anti-holomorphic we pretend they depend on both z and $\bar{z}$. Since we want to write functions in terms of z and $\bar{z}$, let us figure out how the chain rule works for Wirtinger derivatives, rather than writing derivatives in terms of x and y, which gets tedious very quick.

Proposition 1.3.1 (Complex chain rule)**.** *Suppose $U \subset \mathbb{C}^n$ and $V \subset \mathbb{C}^m$ are open sets and suppose $f\colon U \to V$, and $g\colon V \to \mathbb{C}$ are (real) differentiable functions (mappings). Write the variables as $z = (z_1, \dots, z_n) \in U \subset \mathbb{C}^n$ and $w = (w_1, \dots, w_m) \in V \subset \mathbb{C}^m$. Then for any $j = 1, \dots, n$,*

$$\frac{\partial}{\partial z_j}[g \circ f] = \sum_{\ell=1}^{m} \left(\frac{\partial g}{\partial w_\ell} \frac{\partial f_\ell}{\partial z_j} + \frac{\partial g}{\partial \bar{w}_\ell} \frac{\partial \bar{f}_\ell}{\partial z_j} \right), \qquad \frac{\partial}{\partial \bar{z}_j}[g \circ f] = \sum_{\ell=1}^{m} \left(\frac{\partial g}{\partial w_\ell} \frac{\partial f_\ell}{\partial \bar{z}_j} + \frac{\partial g}{\partial \bar{w}_\ell} \frac{\partial \bar{f}_\ell}{\partial \bar{z}_j} \right). \tag{1.4}$$

Proof. Write $f = u + iv$, $z = x + iy$, $w = s + it$, f is a function of z, and g is a function of w. The composition plugs in f for w, and so it plugs in u for s, and v for t. Using the standard chain rule,

$$\begin{aligned}
\frac{\partial}{\partial z_j}[g \circ f] &= \frac{1}{2}\left(\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j}\right)[g \circ f] \\
&= \frac{1}{2}\sum_{\ell=1}^{m}\left(\frac{\partial g}{\partial s_\ell}\frac{\partial u_\ell}{\partial x_j} + \frac{\partial g}{\partial t_\ell}\frac{\partial v_\ell}{\partial x_j} - i\left(\frac{\partial g}{\partial s_\ell}\frac{\partial u_\ell}{\partial y_j} + \frac{\partial g}{\partial t_\ell}\frac{\partial v_\ell}{\partial y_j}\right)\right) \\
&= \sum_{\ell=1}^{m}\left(\frac{\partial g}{\partial s_\ell}\frac{1}{2}\left(\frac{\partial u_\ell}{\partial x_j} - i\frac{\partial u_\ell}{\partial y_j}\right) + \frac{\partial g}{\partial t_\ell}\frac{1}{2}\left(\frac{\partial v_\ell}{\partial x_j} - i\frac{\partial v_\ell}{\partial y_j}\right)\right) \\
&= \sum_{\ell=1}^{m}\left(\frac{\partial g}{\partial s_\ell}\frac{\partial u_\ell}{\partial z_j} + \frac{\partial g}{\partial t_\ell}\frac{\partial v_\ell}{\partial z_j}\right).
\end{aligned}$$

For $\ell = 1, \dots, m$,

$$\frac{\partial}{\partial s_\ell} = \frac{\partial}{\partial w_\ell} + \frac{\partial}{\partial \bar{w}_\ell}, \qquad \frac{\partial}{\partial t_\ell} = i\left(\frac{\partial}{\partial w_\ell} - \frac{\partial}{\partial \bar{w}_\ell}\right).$$

Continuing:

$$
\begin{aligned}
\frac{\partial}{\partial z_j}\left[g \circ f\right] &= \sum_{\ell=1}^{m}\left(\frac{\partial g}{\partial s_\ell}\frac{\partial u_\ell}{\partial z_j} + \frac{\partial g}{\partial t_\ell}\frac{\partial v_\ell}{\partial z_j}\right) \\
&= \sum_{\ell=1}^{m}\left(\left(\frac{\partial g}{\partial w_\ell}\frac{\partial u_\ell}{\partial z_j} + \frac{\partial g}{\partial \bar{w}_\ell}\frac{\partial u_\ell}{\partial z_j}\right) + i\left(\frac{\partial g}{\partial w_\ell}\frac{\partial v_\ell}{\partial z_j} - \frac{\partial g}{\partial \bar{w}_\ell}\frac{\partial v_\ell}{\partial z_j}\right)\right) \\
&= \sum_{\ell=1}^{m}\left(\frac{\partial g}{\partial w_\ell}\left(\frac{\partial u_\ell}{\partial z_j} + i\frac{\partial v_\ell}{\partial z_j}\right) + \frac{\partial g}{\partial \bar{w}_\ell}\left(\frac{\partial u_\ell}{\partial z_j} - i\frac{\partial v_\ell}{\partial z_j}\right)\right) \\
&= \sum_{\ell=1}^{m}\left(\frac{\partial g}{\partial w_\ell}\frac{\partial f_\ell}{\partial z_j} + \frac{\partial g}{\partial \bar{w}_\ell}\frac{\partial \bar{f}_\ell}{\partial z_j}\right).
\end{aligned}
$$

The $\bar{z}$ derivative works similarly. □

Because of the proposition, when we deal with an arbitrary, possibly non-holomorphic, function f, we often write $f(z, \bar{z})$ and treat f as a function of z and $\bar{z}$.

Remark 1.3.2. It is good to notice the subtlety of what we just said. Formally it seems as if we are treating z and $\bar{z}$ as independent variables when taking derivatives, but in reality they are not independent if we actually wish to evaluate the function. Under the hood, a smooth function that is not necessarily holomorphic is really a function of the real variables x and y, where $z = x + iy$.

Remark 1.3.3. Another remark is that we could have swapped z and $\bar{z}$, by flipping the bars everywhere. There is no difference between the two, they are twins in effect. We just need to know which one is which. After all, it all starts with taking the two square roots of -1 and deciding which one is i (remember the chickens?). There is no "natural choice" for that, but once we make that choice we must be consistent. And once we picked which root is i, we also picked what is holomorphic and what is antiholomorphic. This is a subtle philosophical as much as a mathematical point.

Definition 1.3.4. Let $U \subset \mathbb{C}^n$ be an open set. A mapping $f : U \to \mathbb{C}^m$ is said to be holomorphic if each component is holomorphic. That is, if $f = (f_1, \ldots, f_m)$ then each f_j is a holomorphic function.

As in one variable, the composition of holomorphic functions (mappings) is holomorphic.

Theorem 1.3.5. *Let $U \subset \mathbb{C}^n$ and $V \subset \mathbb{C}^m$ be open sets and suppose $f : U \to V$ and $g : V \to \mathbb{C}^k$ are both holomorphic. Then the composition $g \circ f$ is holomorphic.*

Proof. The proof is almost trivial by chain rule. Again let g be a function of $w \in V$ and f be a function of $z \in U$. For any $j = 1, \ldots, n$ and any $p = 1, \ldots, k$, we compute

$$
\frac{\partial}{\partial \bar{z}_j}\left[g_p \circ f\right] = \sum_{\ell=1}^{m}\left(\frac{\partial g_p}{\partial w_\ell}\cancelto{0}{\frac{\partial f_\ell}{\partial \bar{z}_j}} + \cancelto{0}{\frac{\partial g_p}{\partial \bar{w}_\ell}}\frac{\partial \bar{f}_\ell}{\partial \bar{z}_j}\right) = 0.
$$
□

For holomorphic functions the chain rule simplifies, and it formally looks like the familiar vector calculus rule. Suppose again $U \subset \mathbb{C}^n$ and $V \subset \mathbb{C}^m$ are open sets and $f : U \to V$, and $g : V \to \mathbb{C}$ are holomorphic. Again name the variables $z = (z_1, \ldots, z_n) \in U \subset \mathbb{C}^n$ and $w = (w_1, \ldots, w_m) \in V \subset \mathbb{C}^m$.

In formula (1.4) for the z_j derivative, the $\bar{w}_j$ derivative of g is zero and the z_j derivative of $\bar{f_\ell}$ is also zero because f and g are holomorphic. Therefore for any $j = 1, \ldots, n$,

$$\frac{\partial}{\partial z_j} [g \circ f] = \sum_{\ell=1}^{m} \frac{\partial g}{\partial w_\ell} \frac{\partial f_\ell}{\partial z_j}.$$

***Exercise* 1.3.1:** *Prove using only the Wirtinger derivatives that a holomorphic function that is real-valued must be constant.*

***Exercise* 1.3.2:** *Let f be a holomorphic function on $\mathbb{C}^n$. When we write $\bar{f}$ we mean the function $z \mapsto \overline{f(z)}$ and we usually write $\bar{f}(\bar{z})$ as the function is antiholomorphic. However if we write $\bar{f}(z)$ we really mean $z \mapsto \overline{f(\bar{z})}$, that is composing both the function and the argument with conjugation. Prove $z \mapsto \bar{f}(z)$ is holomorphic and prove f is real-valued on $\mathbb{R}^n$ (when $y = 0$) if and only if $f(z) = \bar{f}(z)$ for all z.*

For a $U \subset \mathbb{C}^n$, a holomorphic mapping $f : U \to \mathbb{C}^m$, and a point $p \in U$, define the holomorphic derivative, sometimes called the *Jacobian matrix*,

$$Df(p) \overset{\text{def}}{=} \left[\frac{\partial f_j}{\partial z_k}(p) \right]_{jk}.$$

The notation $f'(p) = Df(p)$ is also used.

***Exercise* 1.3.3:** *Suppose $U \subset \mathbb{C}^n$ is open, $\mathbb{R}^n$ is naturally embedded in $\mathbb{C}^n$. Consider a holomorphic mapping $f : U \to \mathbb{C}^m$ and suppose that $f|_{U \cap \mathbb{R}^n}$ maps into $\mathbb{R}^m \subset \mathbb{C}^m$. Prove that for any $p \in U \cap \mathbb{R}^n$, the real Jacobian matrix of the map $f|_{U\cap\mathbb{R}^n} : U \cap \mathbb{R}^n \to \mathbb{R}^m$ is equal to the holomorphic Jacobian matrix of the map f at p. In particular, $Df(p)$ is a matrix with real entries.*

Using the holomorphic chain rule above, as in the theory of real functions, the derivative of the composition is the composition of derivatives (multiplied as matrices).

Proposition 1.3.6 (Chain rule for holomorphic mappings)**.** *Let $U \subset \mathbb{C}^n$ and $V \subset \mathbb{C}^m$ be open sets. Suppose $f : U \to V$ and $g : V \to \mathbb{C}^k$ are both holomorphic, and $p \in U$. Then*

$$D(g \circ f)(p) = Dg\big(f(p)\big)\, Df(p).$$

In short-hand we often simply write $D(g \circ f) = DgDf$.

***Exercise* 1.3.4:** *Prove the proposition.*

Suppose $U \subset \mathbb{C}^n$, $p \in U$, and $f : U \to \mathbb{C}^m$ is a differentiable function at p. Since $\mathbb{C}^n$ is identified with $\mathbb{R}^{2n}$, the mapping f takes $U \subset \mathbb{R}^{2n}$ to $\mathbb{R}^{2m}$. The normal vector calculus Jacobian at p of this mapping (a $2m \times 2n$ real matrix) is called the *real Jacobian*, and we write it as $D_{\mathbb{R}} f(p)$.

Proposition 1.3.7. *Let $U \subset \mathbb{C}^n$ be an open set, $p \in U$, and $f \colon U \to \mathbb{C}^n$ be holomorphic. Then*

$$\left|\det Df(p)\right|^2 = \det D_{\mathbb{R}} f(p).$$

The expression $\det Df(p)$ is called the *Jacobian determinant* and clearly it is important to know if we are talking about the holomorphic Jacobian determinant or the standard real Jacobian determinant $\det D_{\mathbb{R}} f(p)$. Recall from vector calculus that if the real Jacobian determinant $\det D_{\mathbb{R}} f(p)$ of a smooth mapping is positive, then the mapping preserves orientation. In particular, the proposition says that holomorphic mappings preserve orientation.

Proof. The real mapping using our identification is $(\operatorname{Re} f_1, \operatorname{Im} f_1, \ldots, \operatorname{Re} f_n, \operatorname{Im} f_n)$ as a function of $(x_1, y_1, \ldots, x_n, y_n)$. The statement is about the two Jacobians, that is the derivatives at p, and hence, we can assume that $p = 0$ and f is complex linear, that is $f(z) = Az$ for some $n \times n$ matrix A. It is just a statement about matrices. So A is the (holomorphic) Jacobian matrix of f, and let B be the real Jacobian matrix of f.

Let us change basis of B to be $(z, \bar{z})$ using $z = x + iy$ and $\bar{z} = x - iy$, on both the target and the source. In other words, there is some invertible complex matrix M such that $M^{-1}BM$ (the real Jacobian matrix B in these new coordinates) is a matrix of the derivatives of $(f_1, \ldots, f_n, \bar{f}_1, \ldots, \bar{f}_n)$ in terms of $(z_1, \ldots, z_n, \bar{z}_1, \ldots, \bar{z}_n)$. In other words,

$$M^{-1}BM = \begin{bmatrix} A & 0 \\ 0 & \overline{A} \end{bmatrix}.$$

Thus

$$\det(B) = \det(M^{-1}MB) = \det(M^{-1}BM) = \det(A)\det(\overline{A}) = \det(A)\,\overline{\det(A)} = \left|\det(A)\right|^2. \qquad \square.$$

The regular implicit function theorem and the chain rule give that the implicit function theorem holds in the holomorphic setting. The main thing to check is to check that the solution given by the standard implicit function theorem is holomorphic, which follows by the chain rule.

Theorem 1.3.8 (Implicit function theorem)**.** *Let $U \subset \mathbb{C}^n \times \mathbb{C}^m$ be an open set, let $(z, w) \in \mathbb{C}^n \times \mathbb{C}^m$ be our coordinates, and let $f \colon U \to \mathbb{C}^m$ be a holomorphic mapping. Let $(z^0, w^0) \in U$ be a point such that $f(z^0, w^0) = 0$ and such that the $m \times m$ matrix*

$$\left[\frac{\partial f_j}{\partial w_k}(z^0, w^0) \right]_{jk}$$

is invertible. Then there exists an open set $V \subset \mathbb{C}^n$ with $z^0 \in V$, open set $W \subset \mathbb{C}^m$ with $w^0 \in W$, $V \times W \subset U$, and a holomorphic mapping $g \colon V \to W$, with $g(z^0) = w^0$ such that for every $z \in V$, the point $g(z)$ is the unique point in W such that

$$f\bigl(z, g(z)\bigr) = 0.$$

Exercise* 1.3.5*: *Prove the holomorphic implicit function theorem above. Hint: Check that the normal implicit function theorem for C^1 functions applies, and then show that the g you obtain is holomorphic.*

Exercise* 1.3.6*: *State and prove a holomorphic version of the inverse function theorem.*

1.4 Inequivalence of ball and polydisc

Definition 1.4.1. Two domains $U \subset \mathbb{C}^n$ and $V \subset \mathbb{C}^n$ are said to be *biholomorphic* or *biholomorphically equivalent* if there exists a one-to-one and onto holomorphic map $f : U \to V$ such that the inverse $f^{-1} : V \to U$ is holomorphic. The mapping f is said to be a *biholomorphic map* or a *biholomorphism*.

As function theory on two biholomorphic domains is the same, one of the main questions in complex analysis is to classify domains up to biholomorphic transformations. In one variable, there is the rather striking theorem due to Riemann:

Theorem 1.4.2 (Riemann mapping theorem)**.** *If $U \subset \mathbb{C}$ is a nonempty simply connected domain such that $U \neq \mathbb{C}$, then U is biholomorphic to $\mathbb{D}$.*

In one variable, a topological property on U is enough to classify a whole class of domains. It is one of the reasons why studying the disc is so important in one variable, and why many theorems are stated for the disc only. There is simply no such theorem in several variables. We will show momentarily that the unit ball and the polydisc,

$$\mathbb{B}_n = \left\{ z \in \mathbb{C}^n : \|z\| < 1 \right\} \qquad \text{and} \qquad \mathbb{D}^n = \left\{ z \in \mathbb{C}^n : |z_j| < 1 \text{ for } j = 1, \ldots, n \right\},$$

are *not* biholomorphically equivalent. Both are simply connected (have no holes), and they are the two most obvious generalizations of the disc to several variables. They are homeomorphic, that is, topology does not see any difference.

> ***Exercise* 1.4.1*:*** *Prove that there exists a* homeomorphism $f : \mathbb{B}_n \to \mathbb{D}^n$*, that is, f is a bijection, and both f and f^{-1} are continuous.*

Let us stick with $n = 2$. Instead of proving that $\mathbb{B}_2$ and $\mathbb{D}^2$ are biholomorphically inequivalent we will prove a stronger theorem. First a definition.

Definition 1.4.3. Suppose $f : X \to Y$ is a continuous map between two topological spaces. Then f is a *proper map* if for every compact $K \subset\subset Y$, the set $f^{-1}(K)$ is compact.

The notation "$\subset\subset$" is a common notation for relatively compact subset, that is, the closure is compact in the relative (subspace) topology. Often the distinction between compact and relatively compact is not important, for example, in the above definition we can replace compact with relatively compact. So it is also sometimes used if just "compact" is meant.

Vaguely, "proper" means that "boundary goes to the boundary." As a continuous map, f pushes compacts to compacts; a proper map is one where the inverse does so too. If the inverse is a continuous function, then clearly f is proper, but not every proper map is invertible. For example, the map $f : \mathbb{D} \to \mathbb{D}$ given by $f(z) = z^2$ is proper, but not invertible. The codomain of f is important. If we replace f by $g : \mathbb{D} \to \mathbb{C}$, still given by $g(z) = z^2$, then the map is no longer proper. Let us state the main result of this section.

Theorem 1.4.4 (Rothstein 1935)**.** *There exists no proper holomorphic mapping of the unit bidisc $\mathbb{D}^2 = \mathbb{D} \times \mathbb{D} \subset \mathbb{C}^2$ to the unit ball $\mathbb{B}_2 \subset \mathbb{C}^2$.*

As a biholomorphic mapping is proper, the unit bidisc is not biholomorphically equivalent to the unit ball in $\mathbb{C}^2$. This fact was first proved by Poincaré by computing the automorphism groups of $\mathbb{D}^2$ and $\mathbb{B}_2$, although his proof assumed the maps extended past the boundary. The first complete proof was by Henri Cartan in 1931, though popularly the theorem is attributed to Poincaré. It seems standard practice that any general audience talk about several complex variables contains a mention of Poincaré, and often the reference is to this exact theorem.

We need some lemmas before we get to the proof of the result. First, a certain one-dimensional object plays an important role in the geometry of several complex variables. It allows us to apply one-variable results in several variables. It is especially important in understanding the boundary behavior of holomorphic functions. It also prominently appears in complex geometry.

Definition 1.4.5. A non-constant holomorphic mapping $\varphi\colon \mathbb{D} \to \mathbb{C}^n$ is called an *analytic disc*. If the mapping φ extends continuously to the closed unit disc $\overline{\mathbb{D}}$, then the mapping $\varphi\colon \overline{\mathbb{D}} \to \mathbb{C}^n$ is called a *closed analytic disc*.

Often we call the image $\Delta = \varphi(\mathbb{D})$ the analytic disc rather than the mapping. For a closed analytic disc we write $\partial\Delta = \varphi(\partial\mathbb{D})$ and call it the boundary of the analytic disc.

In some sense, analytic discs play the role of line segments in $\mathbb{C}^n$. It is important to always have in mind that there is a mapping defining the disc, even if we are more interested in the set. Obviously for a given image, the mapping φ is not unique.

Let us consider the boundaries of the unit bidisc $\mathbb{D} \times \mathbb{D} \subset \mathbb{C}^2$ and the unit ball $\mathbb{B}_2 \subset \mathbb{C}^2$. We notice the boundary of the unit bidisc contains analytic discs $\{p\} \times \mathbb{D}$ and $\mathbb{D} \times \{p\}$ for $p \in \partial\mathbb{D}$. That is, through every point in the boundary, except for the distinguished boundary $\partial\mathbb{D} \times \partial\mathbb{D}$, there exists an analytic disc lying entirely inside the boundary. On the other hand, the ball contains no analytic discs in its boundary.

Proposition 1.4.6. *The unit sphere $S^{2n-1} = \partial\mathbb{B}_n \subset \mathbb{C}^n$ contains no analytic discs.*

Proof. Suppose there is a holomorphic function $g\colon \mathbb{D} \to \mathbb{C}^n$ such that the image $g(\mathbb{D})$ is inside the unit sphere. In other words

$$\|g(z)\|^2 = |g_1(z)|^2 + |g_2(z)|^2 + \cdots + |g_n(z)|^2 = 1$$

for all $z \in \mathbb{D}$. Without loss of generality (after composing with a unitary matrix) assume that $g(0) = (1, 0, 0, \ldots, 0)$. Consider the first component and notice that $g_1(0) = 1$. If a sum of positive numbers is less than or equal to 1, then they all are, and hence $|g_1(z)| \leq 1$. Maximum principle says that $g_1(z) = 1$ for all $z \in \mathbb{D}$. But then $g_j(z) = 0$ for all $j = 2, \ldots, n$ and all $z \in \mathbb{D}$. Therefore g is constant and thus not an analytic disc. $\square$

The fact that the sphere contains no analytic discs is the most important geometric distinction between the boundary of the polydisc and the sphere.

***Exercise* 1.4.2:** *Modify the proof to show some stronger results.*

a) Let Δ be an analytic disc and $\Delta \cap \partial\mathbb{B}_n \neq \emptyset$. Prove Δ contains points not in $\overline{\mathbb{B}_n}$.

b) Let Δ be an analytic disc. Prove that $\Delta \cap \partial\mathbb{B}_n$ is nowhere dense in Δ.

c) Find an analytic disc, such that $(1, 0) \in \Delta$, $\Delta \cap \mathbb{B}_n = \emptyset$, and locally near $(1, 0) \in \partial\mathbb{B}_2$, the set $\Delta \cap \partial\mathbb{B}_2$ is the curve defined by $\operatorname{Im} z_1 = 0$, $\operatorname{Im} z_2 = 0$, $(\operatorname{Re} z_1)^2 + (\operatorname{Re} z_2)^2 = 1$.

Before we prove the theorem let us prove a lemma making the statement about proper maps taking boundary to boundary precise.

Lemma 1.4.7. *Let $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^m$ be bounded domains and let $f\colon U \to V$ be continuous. Then f is proper if and only if for every sequence $\{p_k\}$ in U such that $p_k \to p \in \partial U$, the set of limit points of $\{f(p_k)\}$ lies in ∂V.*

Proof. First suppose f is proper. Take a sequence $\{p_k\}$ in U such that $p_k \to p \in \partial U$. Then take any convergent subsequence $\{f(p_{k_j})\}$ of $\{f(p_k)\}$ converging to some $q \in \overline{V}$. Consider $E = \{f(p_{k_j})\}$ as a set. Let $\overline{E}$ be the closure of E in V (subspace topology). If $q \in V$, then $\overline{E} = E \cup \{q\}$ and $\overline{E}$ is compact. Otherwise, if $q \notin V$, then $\overline{E} = E$. The inverse image $f^{-1}(\overline{E})$ is not compact (it contains a sequence going to $p \in \partial U$) and hence $\overline{E}$ is not compact either as f is proper. Thus $q \notin V$, and hence $q \in \partial V$. As we took an arbitrary subsequence of $\{f(p_k)\}$, q was an arbitrary limit point. Therefore, all limit points are in ∂V.

Let us prove the converse. Suppose that for every sequence $\{p_k\}$ in U such that $p_k \to p \in \partial U$, the set of limit points of $\{f(p_k)\}$ lies in ∂V. Take a closed set $E \subset V$ (subspace topology) and look at $f^{-1}(E)$. If $f^{-1}(E)$ is not compact, then there exists a sequence $\{p_k\}$ in $f^{-1}(E)$ such that $p_k \to p \in \partial U$. That is because $f^{-1}(E)$ is closed (in U) but not compact. The hypothesis then says that the limit points of $\{f(p_k)\}$ are in ∂V. Hence E has limit points in ∂V and is not compact. □

***Exercise* 1.4.3:** *Let $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^m$ be bounded domains and let $f\colon \overline{U} \to \overline{V}$ be continuous. Suppose $f(U) \subset V$, and $g\colon U \to V$ is defined by $g(x) = f(x)$ for all $x \in U$. Prove that g is proper if and only if $f(\partial U) \subset \partial V$.*

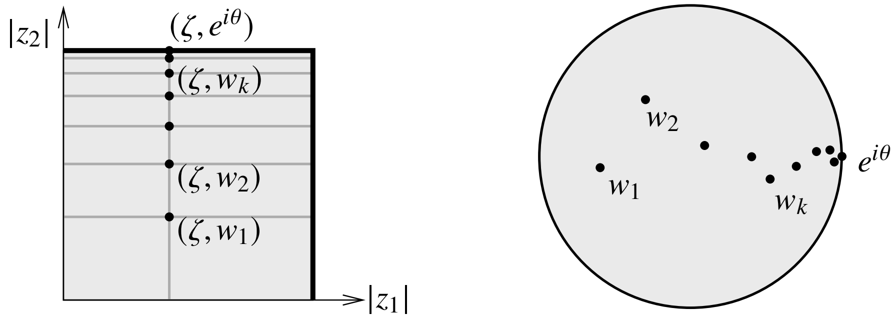
***Exercise* 1.4.4:** *Let $f\colon X \to Y$ be a continuous function of locally compact Hausdorff topological spaces. Let X_∞ and Y_∞ be the one-point compactifications of X and Y. Then f is a proper map if and only if it extends as a continuous map $f_\infty\colon X_\infty \to Y_\infty$ by letting $f_\infty|_X = f$ and $f_\infty(\infty) = \infty$.*

We now have all the lemmas needed to prove the theorem of Rothstein.

Proof of Theorem 1.4.4. Suppose there is a proper holomorphic map $f\colon \mathbb{D}^2 \to \mathbb{B}_2$. Fix some $e^{i\theta}$ in the boundary of the disc $\mathbb{D}$. Take a sequence $w_k \in \mathbb{D}$ such that $w_k \to e^{i\theta}$. The functions $g_k(\zeta) = f(\zeta, w_k)$ map the unit disc into $\mathbb{B}_2$. By the standard Montel's theorem, by passing to a subsequence we assume that the sequence of functions converges (uniformly on compact subsets) to a limit $g\colon \mathbb{D} \to \overline{\mathbb{B}_2}$. As $(\zeta, w_k) \to (\zeta, e^{i\theta}) \in \partial\mathbb{D}^2$, then by Lemma 1.4.7 we have that $g(\mathbb{D}) \subset \partial\mathbb{B}_2$ and hence g must be constant.

Let g_k' denote the derivative (we differentiate each component). The functions g_k' converge to $g' = 0$. So for any fixed $\zeta \in \mathbb{D}$, $\frac{\partial f}{\partial z_1}(\zeta, w_k) \to 0$. This limit holds for all $e^{i\theta}$ and some subsequence of an arbitrary sequence $\{w_k\}$ where $w_k \to e^{i\theta}$. The holomorphic mapping $w \mapsto \frac{\partial f}{\partial z_1}(\zeta, w)$ therefore extends continuously to the closure $\overline{\mathbb{D}}$ and is zero on $\partial\mathbb{D}$. We apply the maximum principle or the Cauchy formula and the fact that ζ was arbitrary to find $\frac{\partial f}{\partial z_1} \equiv 0$. By symmetry $\frac{\partial f}{\partial z_2} \equiv 0$. Therefore f is constant, which is a contradiction as f was proper.

The proof is illustrated in the following picture:



In the picture, on the left hand side is the bidisc, and we restrict f to the horizontal gray lines (where the second component is fixed to be w_k) and take a limit to produce an analytic disc in the boundary of $\mathbb{B}_2$. We then show that $\frac{\partial f}{\partial z_1} = 0$ on the vertical gray line (where the first component is fixed to be ζ). The right hand side shows the disc where $z_1 = \zeta$ is fixed, which corresponds to the vertical gray line on the left. □

The proof says that the reason why there is not even a proper mapping is the fact that the boundary of the polydisc contains analytic discs, while the sphere does not. The proof extends easily to higher dimensions as well, and the proof of the generalization is left as an exercise.

Theorem 1.4.8. *Let $U = U' \times U'' \subset \mathbb{C}^n \times \mathbb{C}^k$, $n, k \geq 1$, and $V \subset \mathbb{C}^m$, $m \geq 1$, be bounded domains such that ∂V contains no analytic discs. Then there exist no proper holomorphic mapping $f \colon U \to V$.*

***Exercise* 1.4.5:** *Prove Theorem 1.4.8.*

The key take-away from this section is that in several variables, to see if two domains are equivalent, the geometry of the boundaries makes a difference, not just the topology of the domains.

There is a fun exercise in one dimension about proper maps of discs:

***Exercise* 1.4.6:** *Let $f \colon \mathbb{D} \to \mathbb{D}$ be a proper holomorphic map. Then*

$$f(z) = e^{i\theta} \prod_{j=1}^{k} \frac{z - a_j}{1 - \bar{a}_j z},$$

for some real θ and some $a_j \in \mathbb{D}$ (that is, f is a finite Blaschke product). Hint: Consider $f^{-1}(0)$.

In several variables, when $\mathbb{D}$ is replaced by a ball, this question (what are the proper maps) becomes far more involved, and if the dimensions of the balls are different, it is not solved in general.

***Exercise* 1.4.7:** *Suppose $f: U \to \mathbb{D}$ be a proper holomorphic map where $U \subset \mathbb{C}^n$ is a nonempty domain. Prove that $n = 1$. Hint: Consider the same idea as in Exercise 1.2.19.*

***Exercise* 1.4.8:** *Suppose $f: \overline{\mathbb{B}_n} \to \mathbb{C}^m$ is a nonconstant continuous map such that $f|_{\mathbb{B}_n}$ is holomorphic and $\lVert f(z) \rVert = 1$ whenever $\lVert z \rVert = 1$. Prove that $f|_{\mathbb{B}_n}$ maps into $\mathbb{B}_m$ and furthermore that this map is proper.*

1.5 Cartan's uniqueness theorem

The following theorem is another analogue of Schwarz's lemma to several variables. It says that for a bounded domain, it is enough to know that a self mapping is the identity at a single point to show that it is the identity everywhere. As there are quite a few theorems named for Cartan, this one is often referred to as the *Cartan's uniqueness theorem*. It is very useful in computing the automorphism groups of certain domains. An *automorphism* of U is a biholomorphic map from U onto U. Automorphisms form a group under composition, called the *automorphism group*. As exercises, you will use the theorem to compute the automorphism groups of $\mathbb{B}_n$ and $\mathbb{D}^n$.

Theorem 1.5.1 (Cartan)**.** *Suppose $U \subset \mathbb{C}^n$ is a bounded domain, $a \in U$, $f: U \to U$ is a holomorphic mapping, $f(a) = a$, and $Df(a)$ is the identity. Then $f(z) = z$ for all $z \in U$.*

***Exercise* 1.5.1:** *Find a counterexample to the theorem if U is unbounded. Hint: For simplicity take $a = 0$ and $U = \mathbb{C}^n$.*

Before we get into the proof, let us write down the Taylor series of a function in a nicer way, splitting it up into parts of different degree.

A polynomial $P: \mathbb{C}^n \to \mathbb{C}$ is *homogeneous* of degree d if

$$P(sz) = s^d P(z)$$

for all $s \in \mathbb{C}$ and $z \in \mathbb{C}^n$. A homogeneous polynomial of degree d is a polynomial whose every monomial is of total degree d. For example, $z^2 w - iz^3 + 9zw^2$ is homogeneous of degree 3 in the variables $(z, w) \in \mathbb{C}^2$. A polynomial vector-valued mapping is homogeneous, if each component is. If f is holomorphic near $a \in \mathbb{C}^n$, then write the power series of f at a as

$$\sum_{j=0}^{\infty} f_j(z - a),$$

where f_j is a homogeneous polynomial of degree j. The f_j is called the *degree d homogeneous part* of f at a. The f_j would be vector-valued if f is vector-valued, such as in the statement of the theorem. In the proof, we will require the vector-valued Cauchy estimates (exercise below)*.

*The normal Cauchy estimates could also be used in the proof of Cartan by applying them componentwise.

***Exercise* 1.5.2:** *Prove a vector-valued version of the Cauchy estimates. Suppose* $f \colon \overline{\Delta_r(a)} \to \mathbb{C}^m$ *is continuous function holomorphic on a polydisc* $\Delta_r(a) \subset \mathbb{C}^n$*. Let* Γ *denote the distinguished boundary of* Δ*. Show that for any multi-index* α *we get*

$$\left\lVert \frac{\partial^{|\alpha|} f}{\partial z^\alpha}(a) \right\rVert \leq \frac{\alpha!}{r^\alpha} \sup_{z \in \Gamma} \lVert f(z) \rVert .$$

Proof of Cartan's uniqueness theorem. Without loss of generality, assume $a = 0$. Write f as a power series at the origin, written in homogeneous parts:

$$f(z) = z + f_k(z) + \sum_{j=k+1}^{\infty} f_j(z),$$

where $k \geq 2$ is an integer such that $f_j(z)$ is zero for all $2 \leq j < k$. The degree-one homogeneous part is simply the vector z, because the derivative of the mapping at the origin is the identity. Compose f with itself ℓ times:

$$f^\ell(z) = \underbrace{f \circ f \circ \cdots \circ f}_{\ell \text{ times}}(z).$$

As $f(U) \subset U$, then f^ℓ is a holomorphic map of U to U. As U is bounded, there is an M such that $\lVert z \rVert \leq M$ for all $z \in U$. Therefore $\lVert f(z) \rVert \leq M$ for all $z \in U$, and $\lVert f^\ell(z) \rVert \leq M$ for all $z \in U$.

Notice that if we plug in $z + f_k(z) +$ higher order terms into f_k, we get $f_k(z) +$ some other higher order terms. Therefore $f^2(z) = z + 2 f_k(z) +$ higher order terms. Continuing this procedure,

$$f^\ell(z) = z + \ell f_k(z) + \sum_{j=k+1}^{\infty} \tilde{f}_j(z),$$

for some other degree j homogeneous polynomials $\tilde{f}_j$. Suppose $\Delta_r(0)$ is a polydisc whose closure is in U. Via Cauchy estimates, for any multinomial α with $|\alpha| = k$,

$$\frac{\alpha!}{r^\alpha} M \geq \left\lVert \frac{\partial^{|\alpha|} f^\ell}{\partial z^\alpha}(0) \right\rVert = \ell \left\lVert \frac{\partial^{|\alpha|} f}{\partial z^\alpha}(0) \right\rVert .$$

The inequality holds for all $\ell \in \mathbb{N}$, and so $\frac{\partial^{|\alpha|} f}{\partial z^\alpha}(0) = 0$. Therefore $f_k \equiv 0$. Hence on the domain of convergence of the expansion we get $f(z) = z$, as there is no other nonzero homogeneous part in the expansion of f. As U is connected, then the identity theorem says $f(z) = z$ for all $z \in U$. ◻

As an application, let us classify all biholomorphisms of all bounded circular domains that fix a point. A *circular domain* is a domain $U \subset \mathbb{C}^n$ such that if $z \in U$, then $e^{i\theta} z \in U$ for all $\theta \in \mathbb{R}$.

Corollary 1.5.2. *Suppose* $U, V \subset \mathbb{C}^n$ *are bounded circular domains with* $0 \in U$*,* $0 \in V$*, and* $f \colon U \to V$ *is a biholomorphic map such that* $f(0) = 0$*. Then* f *is linear.*

For example, $\mathbb{B}_n$ is circular and bounded, so a biholomorphism of $\mathbb{B}_n$ (an automorphism) that fixes the origin is linear. Similarly a polydisc centered at zero is also circular and bounded.

Proof. The map $g(z) = f^{-1}\big(e^{-i\theta} f(e^{i\theta} z)\big)$ is an automorphism of U and via the chain-rule, $g'(0) = I$. Therefore, $f^{-1}\big(e^{-i\theta} f(e^{i\theta} z)\big) = z$, or in other words

$$f(e^{i\theta} z) = e^{i\theta} f(z).$$

Write f near zero as $f(z) = \sum_{j=1}^{\infty} f_j(z)$ where f_j are homogeneous polynomials of degree j (notice $f_0 = 0$). Then

$$\sum_{j=1}^{\infty} e^{i\theta} f_j(z) = e^{i\theta} \sum_{j=1}^{\infty} f_j(z) = \sum_{j=1}^{\infty} f_j(e^{i\theta} z) = \sum_{j=1}^{\infty} e^{ij\theta} f_j(z).$$

By the uniqueness of the Taylor expansion, $e^{i\theta} f_j(z) = e^{ij\theta} f_j(z)$, or $f_j(z) = e^{i(j-1)\theta} f_j(z)$, for all j, all z, and all θ. If $j \neq 1$, we obtain that $f_j \equiv 0$, which proves the claim. ☐

***Exercise* 1.5.3:** *Show that every automorphism f of $\mathbb{D}^n$ (that is a biholomorphism $f\colon \mathbb{D}^n \to \mathbb{D}^n$) is given as*

$$f(z) = P\left(e^{i\theta_1}\frac{z_1 - a_1}{1 - \bar{a}_1 z_1}, e^{i\theta_2}\frac{z_2 - a_2}{1 - \bar{a}_2 z_2}, \ldots, e^{i\theta_n}\frac{z_n - a_n}{1 - \bar{a}_n z_n}\right)$$

for $\theta \in \mathbb{R}^n$, $a \in \mathbb{D}^n$, and a permutation matrix P.

***Exercise* 1.5.4:** *Given $a \in \mathbb{B}_n$, define the linear map $P_a z = \frac{\langle z,a\rangle}{\langle a,a\rangle} a$ if $a \neq 0$ and $P_0 z = 0$. Let $s_a = \sqrt{1 - \|a\|^2}$. Show that every automorphism f of $\mathbb{B}_n$ (that is a biholomorphism $f\colon \mathbb{B}_n \to \mathbb{B}_n$) can be written as*

$$f(z) = U\frac{a - P_a z - s_a (I - P_a) z}{1 - \langle z, a\rangle}$$

for a unitary matrix U and some $a \in \mathbb{B}_n$.

***Exercise* 1.5.5:** *Using the previous two exercises, show that $\mathbb{D}^n$ and $\mathbb{B}_n$, $n \geq 2$, are not biholomorphic via a method more in the spirit of what Poincaré used: Show that the groups of automorphisms of the two domains are different groups when $n \geq 2$.*

***Exercise* 1.5.6:** *Suppose $U \subset \mathbb{C}^n$ is a bounded open set, $a \in U$, and $f\colon U \to U$ is a holomorphic mapping such that $f(a) = a$. Show that every eigenvalue λ of the matrix $Df(a)$ satisfies $|\lambda| \leq 1$.*

***Exercise* 1.5.7 (Tricky):** *Find a domain $U \subset \mathbb{C}^n$ such that the only biholomorphism $f\colon U \to U$ is the identity $f(z) = z$. Hint: Take the polydisc (or the ball) and remove some number of points (be careful in how you choose them). Then show that f extends to a biholomorphism of the polydisc. Then see what happens to those points you took out.*

***Exercise* 1.5.8:**

a) Show that Cartan's uniqueness theorem is not true in the real case, even for rational functions. That is, find a rational function $R(t)$ of a real variable t, such that R that takes $(-1,1)$ to $(-1,1)$, $R'(0) = 1$, and $R(t)$ is not the identity. You can even make R bijective.

b) Show that also Exercise 1.5.6 is not true in the real case. For any $\alpha \in \mathbb{R}$ find a rational function $R(t)$ of a real variable t, such that R takes $(-1,1)$ to $(-1,1)$ and $R'(0) = \alpha$.

***Exercise* 1.5.9:** *Suppose $U \subset \mathbb{C}^n$ is an open set, $a \in U$. Suppose $f\colon U \to U$ is a holomorphic mapping, $f(a) = a$, and suppose that $|\lambda| < 1$ for any eigenvalue λ of $Df(a)$. Prove that there exists a neighborhood W of a, such that $\lim_{\ell\to\infty} f^{\ell}(z) = a$ for all $z \in W$.*

1.6 Riemann extension theorem, zero sets, and injective maps

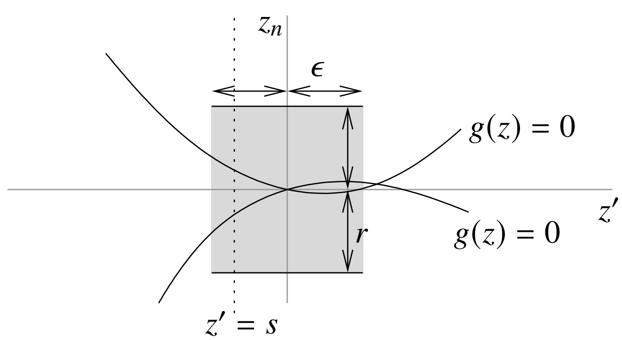
In one dimension if a function is holomorphic in $U \setminus \{p\}$ and locally bounded* in U, in particular bounded near p, then the function extends holomorphically to U (see Proposition B.22 (i)). In several variables the same theorem holds, and the analogue of a single point is the zero set of a holomorphic function.

Theorem 1.6.1 (Riemann extension theorem)**.** *Let $U \subset \mathbb{C}^n$ be a domain and let $g \in \mathcal{O}(U)$ that is not identically zero. Let $N = g^{-1}(0)$ be the zero set of g. Suppose $f \in \mathcal{O}(U \setminus N)$ is locally bounded in U. Then there exists a unique $F \in \mathcal{O}(U)$ such that $F|_{U\setminus N} = f$.*

The proof is an application of the Riemann extension theorem from one dimension.

Proof. Take any $p \in N$, and let L be a complex line through p. That is, L is an image of an affine mapping $\varphi\colon \mathbb{C} \to \mathbb{C}^n$ defined by $\varphi(\xi) = a\xi + p$, for a vector $a \in \mathbb{C}^n$. The composition $g \circ \varphi$ is a holomorphic function of one variable, and it is either identically zero, or the zero at $\xi = 0$ is isolated. The function g is not identically zero in any neighborhood of p. So there is some line L such that $g \circ \varphi$ is not identically zero, or in other words, p is an isolated point of $L \cap N$.

Write $z' = (z_1, \ldots, z_{n-1})$ and $z = (z', z_n)$. Without loss of generality $p = 0$, and L is the line obtained by $z' = 0$. So $g \circ \varphi$ is $\xi \mapsto g(0, \xi)$. There is some small $r > 0$ such that g is nonzero on the set given by $|z_n| = r$ and $z' = 0$. By continuity, g is nonzero on the set given by $|z_n| = r$ and $\|z'\| < \epsilon$ for some $\epsilon > 0$. In particular, for any fixed small $s \in \mathbb{C}^{n-1}$, with $\|s\| < \epsilon$, setting $z' = s$, the zeros of $\xi \mapsto g(s, \xi)$ are isolated.



For $\|z'\| < \epsilon$ and $|z_n| < r$, write

$$F(z', z_n) = \frac{1}{2\pi i} \int_{|\xi|=r} \frac{f(z', \xi)}{\xi - z_n}\, d\xi.$$

The function $\xi \to f(z', \xi)$ extends holomorphically to the entire disc of radius r by the Riemann extension from one dimension. By Cauchy integral formula, F is equal to f at the points where they are both defined. By differentiating under the integral, the function F is holomorphic in all variables.

In a neighborhood of each point of N, f extends to a continuous (holomorphic in fact) function. A continuous extension of f must be unique on the closure of $U \setminus N$ in the subspace topology,

*$f\colon U \setminus X \to \mathbb{C}$ is locally bounded in U if for every $p \in U$, there is a neighborhood W of p such that f is bounded on $W \cap (U \setminus X)$.

$\overline{(U \setminus N)} \cap U$. The set N has empty interior, so $\overline{(U \setminus N)} \cap U = U$. Hence, F is the unique continuous extension of f to U. □

The set of zeros of a holomorphic function has a nice structure at most points.

Theorem 1.6.2. *Let $U \subset \mathbb{C}^n$ be a domain and $f \in \mathcal{O}(U)$ and f is not identically zero. Let $N = f^{-1}(0)$. Then there exists a open and dense (subspace topology) subset $N_{reg} \subset N$ such that near each $p \in N_{reg}$, after possibly reordering variables, N can be locally written as*

$$z_n = g(z_1, \dots, z_{n-1})$$

for a holomorphic function g.

Proof. If N is locally a graph at p, then it is a graph for every point of N near p. So N_{reg} is open. Hence, once we show that N_{reg} is nonempty, we will be done as we can repeat the procedure on any open subset of N.

Since f is not identically zero, then not all derivatives (of arbitrary order) of f vanish identically on N. If some first order derivative of f does not vanish identically on N, let $h = f$. Otherwise, suppose k is such that a derivative of f of order k does not vanish identically on N, and all derivatives of f order less than k vanish identically on N. Let h be one of the derivatives of order $k-1$. We obtain a function $h \colon U \to \mathbb{C}$, holomorphic, vanishing on N, and such that without loss of generality the z_n derivative does not vanish identically on N. Then there is some point $p \in N$ such that $\frac{\partial h}{\partial z_n}(p) \neq 0$. We apply the implicit function theorem at p to find g such that

$$h\bigl(z_1, \dots, z_{n-1}, g(z_1, \dots, z_{n-1})\bigr) = 0,$$

and $z_n = g(z_1, \dots, z_{n-1})$ is the unique solution to $h = 0$ near p.

Near p the zero set of h contains the zero set of f, and we need to show equality. Write $p = (p', p_n)$. Then the function

$$\xi \mapsto f(p', \xi)$$

has an isolated zero in a small disc Δ around p_n and is nonzero on the circle $\partial\Delta$. By Rouchè's theorem, $\xi \mapsto f(z', \xi)$ must have a zero for all z' sufficiently close to p' (close enough to make $|f(q', \xi) - f(z', \xi)| < |f(q', \xi)|$ for all $\xi \in \partial\Delta$). Since $g(z')$ is the unique solution z_n to $h(z', z_n) = 0$ near p and the zeros of f are contained in the zeros of h, we are done. □

The zero set N of a holomorphic function is a so-called *subvariety* or an *analytic set* although the general definition of a subvariety is a little more complicated, and includes more sets. See chapter 6. Points where N is written as a graph of a holomorphic mapping are called *regular points*, and we write them as N_{reg} as above. In particular, since N is a graph of a single holomorphic function, they are called regular points of (complex) dimension $n-1$, or (complex) codimension 1. The set of regular points is what is called an $(n-1)$-dimensional *complex submanifold*. It is also a real submanifold of real dimension $2n-2$. The points on a subvariety that are not regular are called *singular points*.

Example 1.6.3: For $U = \mathbb{C}^2$, let $f(z) = z_1^2 - z_2^2$ and consider $X = f^{-1}(0)$. As $\nabla f = (2z_1, 2z_2)$, outside of the origin, we can solve for z_1 or z_2 and so all points of $X \setminus 0$ are regular. In fact, $z_1 = z_2$ and $z_1 = -z_2$ are the two possibilities. In any neighborhood of the origin, however, there is no way to solve for either z_1 or z_2, since you always get two possible solutions: If you could solve $z_1 = g(z_2)$, then both $z_2 = g(z_2)$ and $-z_2 = g(z_2)$ must be true, a contradiction for any nonzero z_2. Similarly we cannot solve for z_1. So the origin is a singular point.

To see that you may have need to use derivatives of the function, notice that the function $\varphi(z) = (z_1^2 - z_2^2)^2$ has the same zero set X, but both $\frac{\partial \varphi}{\partial z_1}$ and $\frac{\partial \varphi}{\partial z_2}$ vanish on X. Using $h = \frac{\partial \varphi}{\partial z_1}$ or $h = \frac{\partial \varphi}{\partial z_2}$ in the proof will work.

Example 1.6.4: The theorem is not true in the non-holomorphic setting. For example, $x_1^2 + x_2^2 = 0$ in $\mathbb{R}^2$ is only the origin, clearly not a graph of any function of one variable. The first part of the theorem works, but the h you find is either $2x_1$ or $2x_2$, and its zero set is too big.

Exercise* 1.6.1*: *Find all the regular points of the subvariety* $X = \{z \in \mathbb{C}^2 : z_1^2 = z_2^3\}$. *Hint: The trick is showing that you've found all of them.*

Exercise* 1.6.2*: *Suppose* $U \subset \mathbb{C}^n$ *is a domain and* $f \in \mathcal{O}(U)$. *Show that the complement of the zero set,* $U \setminus f^{-1}(0)$*, is connected.*

Let us now prove that a one-to-one holomorphic mapping is biholomorphic, a result definitely not true in the smooth setting: $x \mapsto x^3$ is smooth, one-to-one, onto map of $\mathbb{R}$ to $\mathbb{R}$, but the inverse is not differentiable.

Theorem 1.6.5. *Suppose* $U \subset \mathbb{C}^n$ *is an open set and* $f : U \to \mathbb{C}^n$ *is holomorphic and one-to-one. Then the Jacobian determinant is never equal to zero on* U.

In particular if a holomorphic map $f : U \to V$ *is one-to-one and onto for two open sets* $U, V \subset \mathbb{C}^n$*, then* f *is biholomorphic.*

The function f is locally biholomorphic, in particular f^{-1} is holomorphic, on the set where the Jacobian determinant J_f, that is the determinant

$$J_f(z) = \det Df(z) = \det \left[\frac{\partial f_j}{\partial z_k}(z) \right]_{jk},$$

is not zero. This follows from the inverse function theorem, which is just a special case of the implicit function theorem. The trick is to show that J_f happens to be nonzero everywhere.

In one complex dimension, every holomorphic function f can, in the proper local holomorphic coordinates (and up to adding a constant), be written as z^d for $d = 0, 1, 2, \ldots$: Near a $z_0 \in \mathbb{C}$, there exists a constant c and a local biholomorphic g with $g(z_0) = 0$ such that $f(z) = c + \big(g(z)\big)^d$. Such a simple result does not hold in several variables in general, but if the mapping is locally one-to-one then the present theorem says that such a mapping can be locally written as the identity.

Proof of the theorem. We proceed by induction. We know the theorem for $n = 1$. Suppose $n > 1$ and suppose we know the theorem is true for dimension $n - 1$.

Suppose for contradiction that $J_f = 0$ somewhere. First suppose that J_f is not identically zero. Find a regular point q on the zero set of J_f. Write the zero set of J_f near q as

$$z_n = g(z_1, \ldots, z_{n-1})$$

for some holomorphic g. If we prove the theorem near q, we are done. Without loss of generality assume $q = 0$. The biholomorphic (near the origin) map

$$\Psi(z_1, \ldots, z_n) = \bigl(z_1, z_2, \ldots, z_{n-1}, z_n - g(z_1, \ldots, z_{n-1})\bigr)$$

takes the zero set of J_f to the set given by $z_n = 0$. By considering $f \circ \Psi^{-1}$ instead of f, we may assume that $J_f = 0$ on the set given by $z_n = 0$. We may also assume that $f(0) = 0$.

If J_f vanishes identically then there is no need to do anything other than a translation. In either case, we may assume that $0 \in U$, $f(0) = 0$, and $J_f = 0$ when $z_n = 0$.

We wish to show that all the derivatives of f in the $z_1, \ldots, z_{n-1}$ variables vanish whenever $z_n = 0$. This would clearly contradict f being one-to-one, as $f(z_1, \ldots, z_{n-1}, 0)$ would be constant. So for any point on $z_n = 0$ we consider one of the components of f and one of the derivatives of that component. Without loss of generality, suppose the point is 0, and for contradiction suppose $\frac{\partial f_1}{\partial z_1}(0) \neq 0$. The map

$$G(z_1, \ldots, z_n) = \bigl(f_1(z), z_2, \ldots, z_n\bigr)$$

is biholomorphic on a small neighborhood of the origin. The function $f \circ G^{-1}$ is holomorphic and one-to-one on a small neighborhood. By the definition of G,

$$f \circ G^{-1}(w_1, \ldots, w_n) = \bigl(w_1, h(w)\bigr),$$

where h is a holomorphic mapping taking a neighborhood of the origin in $\mathbb{C}^n$ to $\mathbb{C}^{n-1}$. The mapping

$$\varphi(w_2, \ldots, w_n) = h(0, w_2, \ldots, w_n)$$

is a one-to-one holomorphic mapping of a neighborhood of the origin in $\mathbb{C}^{n-1}$ to $\mathbb{C}^{n-1}$. By the induction hypothesis, the Jacobian determinant of φ is nowhere zero.

If we differentiate $f \circ G^{-1}$, we notice $D(f \circ G^{-1}) = Df \circ D(G^{-1})$. So at the origin

$$\det D(f \circ G^{-1}) = \bigl(\det Df\bigr)\bigl(\det D(G^{-1})\bigr) = 0.$$

We obtain a contradiction, as at the origin

$$\det D(f \circ G^{-1}) = \det D\varphi \neq 0. \qquad \square$$

The theorem is no longer true if the domain and range dimensions of the mapping are not equal.

***Exercise* 1.6.3:** *Take the subvariety $X = \bigl\{z \in \mathbb{C}^2 : z_1^2 = z_2^3\bigr\}$. Find a one-to-one holomorphic mapping $f\colon \mathbb{C} \to X$. Then note that the derivative of f vanishes at a certain point. So Theorem 1.6.5 has no analogue when the domain and range have different dimension.*

***Exercise* 1.6.4:** *Find a continuous function $f\colon \mathbb{R} \to \mathbb{R}^2$ that is one-to-one but such that the inverse $f^{-1}\colon f(\mathbb{R}) \to \mathbb{R}$ is not continuous.*

This is an appropriate place to state a well-known and as yet unsolved conjecture (and most likely ridiculously hard to solve): the *Jacobian conjecture*. This conjecture is a converse to the above theorem in a special case: *Suppose* $F : \mathbb{C}^n \to \mathbb{C}^n$ *is a polynomial map (each component is a polynomial) and the Jacobian derivative* J_F *is never zero, then* F *is invertible with a polynomial inverse.* Clearly F would be locally one-to-one, but proving (or disproving) the existence of a global polynomial inverse is the content of the conjecture.

Exercise* 1.6.5*: *Prove the Jacobian conjecture for* $n = 1$. *That is, prove that if* $F : \mathbb{C} \to \mathbb{C}$ *is a polynomial such that* F' *is never zero, then* F *has an inverse, which is a polynomial.*

Exercise* 1.6.6*: *Let* $F : \mathbb{C}^n \to \mathbb{C}^n$ *be an injective polynomial map. Prove* J_F *is a nonzero constant.*

Exercise* 1.6.7*: *Prove that the Jacobian conjecture is false if "polynomial" is replaced with "entire holomorphic," even for* $n = 1$.

Exercise* 1.6.8*: *Prove that if a holomorphic* $f : \mathbb{C} \to \mathbb{C}$ *is injective then it is onto, and therefore* $f(z) = az + b$ *for* $a \neq 0$.

Let us also remark that while every injective holomorphic map of $f : \mathbb{C} \to \mathbb{C}$ is onto, the same is not true in higher dimensions. In $\mathbb{C}^n$, $n \geq 2$, there exist so-called *Fatou–Bieberbach domains*, that is proper subsets of $\mathbb{C}^n$ that are biholomorphic to $\mathbb{C}^n$.

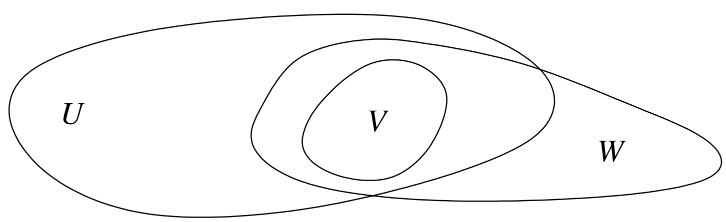
Chapter 2

Convexity and pseudoconvexity

2.1 Domains of holomorphy and holomorphic extensions

It turns out that not every domain in $\mathbb{C}^n$ is a natural domain for holomorphic functions.

Definition 2.1.1. Let $U \subset \mathbb{C}^n$ be a domain[*] (connected open set). The set U is a *domain of holomorphy* if there do not exist nonempty open sets V and W, with $V \subset U \cap W$, $W \not\subset U$, and W connected, such that for every $f \in \mathcal{O}(U)$ there exists an $F \in \mathcal{O}(W)$ with $f(z) = F(z)$ for all $z \in V$.



The idea is that if a domain U is not a domain of holomorphy and V, W exist as in the definition, then f "extends across the boundary" somewhere.

Example 2.1.2: The unit ball $\mathbb{B}_n \subset \mathbb{C}^n$ is a domain of holomorphy. Proof: Consider $U = \mathbb{B}_n$, and suppose V, W as in the definition exist. As W is connected and open, it is path connected. There exist points in W that are not in $\mathbb{B}_n$, so there is a path γ in W that goes from a point $q \in V$ to some $p \in \partial\mathbb{B}_n \cap W$. Without loss of generality (after composing with rotations, that is unitary matrices), assume $p = (1, 0, 0, \ldots, 0)$. Take the function $f(z) = \frac{1}{1-z_1}$. The function F must agree with f on the component of $\mathbb{B}_n \cap W$ that contains q. But that component also contains p and so F must blow up (in particular it cannot be holomorphic) at p. The contradiction shows that no V and W exist.

In one dimension this notion has no real content: Every domain is a domain of holomorphy.

Exercise* 2.1.1** (Easy):*** *In $\mathbb{C}$, every domain is a domain of holomorphy.*

***Exercise* 2.1.2:** *If $U_j \subset \mathbb{C}^n$ are domains of holomorphy (possibly an infinite set of domains), then the interior of $\bigcap_j U_j$ is either empty or every connected component is a domain of holomorphy.*

[*]*Domain of holomorphy* can make sense for disconnected sets (non-domains), and some authors do define it so.

Exercise* 2.1.3** (Easy):*** *Show that a polydisc in $\mathbb{C}^n$ is a domain of holomorphy.*

Exercise* 2.1.4*:

a) *Given $p \in \partial\mathbb{B}_n$, find a function f holomorphic on $\mathbb{B}_n$, C^∞-smooth on $\overline{\mathbb{B}_n}$ (all real partial derivatives of all orders extend continuously to $\overline{\mathbb{B}_n}$), that does not extend past p as a holomorphic function. Hint: For the principal branch of $\sqrt{\cdot}$ the function $\xi \mapsto e^{-1/\sqrt{\xi}}$ is holomorphic for* $\operatorname{Re}\xi > 0$ *and extends to be continuous (even smooth) on all of* $\operatorname{Re}\xi \geq 0$.

b) *Find a function f holomorphic on $\mathbb{B}_n$ that does not extend past any point of $\partial\mathbb{B}_n$.*

Various notions of convexity will play a big role later on. A set S is *geometrically convex* if $tx + (1-t)y \in S$ for all $x, y \in S$ and $t \in [0,1]$. The exercise below says that any geometrically convex domain is a domain of holomorphy. There are other, nonconvex domains of holomorphy (any domain in $\mathbb{C}$), so classical convexity is not the correct notion, but it is in the right direction.

Exercise* 2.1.5*: *Show that a geometrically convex domain in $\mathbb{C}^n$ is a domain of holomorphy.*

In the following when we say $f \in \mathcal{O}(U)$ extends holomorphically to V where $U \subset V$, we mean that there exists a function $F \in \mathcal{O}(V)$ such that $f = F$ on U.

Remark 2.1.3. The subtlety of the definition of a domain of holomorphy is that it does not necessarily talk about functions extending to a larger set, since we must take into account single-valuedness. For example, let f be the principal branch of the logarithm defined on the slit plane $U = \mathbb{C} \setminus \{z \in \mathbb{C} : \operatorname{Im} z = 0, \operatorname{Re} z \leq 0\}$. We can locally define an extension from one side through the boundary of the domain, but we cannot define an extension on a open set that contains U. This example should be motivation for why we let V be a proper subset of $U \cap W$, and why W need not include all of U.

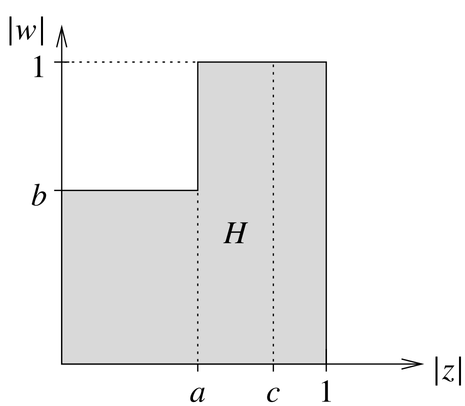
In dimension two or more, not every domain is a domain of holomorphy. We have the following theorem. The domain H in the theorem is called the *Hartogs figure*.

Theorem 2.1.4. *Let $(z,w) = (z_1,\ldots,z_m,w_1,\ldots,w_k) \in \mathbb{C}^m \times \mathbb{C}^k$ be the coordinates. For two numbers $0 < a, b < 1$, let the set $H \subset \mathbb{D}^{m+k}$ be defined by*

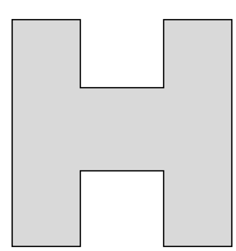
$$H = \left\{(z,w) \in \mathbb{D}^{m+k} : |z_j| > a \text{ for } j = 1,\ldots,m\right\} \cup \left\{(z,w) \in \mathbb{D}^{m+k} : |w_j| < b \text{ for } j = 1,\ldots,k\right\}.$$

If $f \in \mathcal{O}(H)$, then f extends holomorphically to $\mathbb{D}^{m+k}$.

In $\mathbb{C}^2$ if $m = 1$ and $k = 1$, the figure looks like (the c will come up in the proof):



In diagrams, the Hartogs figure is often drawn as:



Proof. Pick a $c \in (a, 1)$. Let

$$\Gamma = \left\{ z \in \mathbb{D}^m : |z_j| = c \text{ for } j = 1, \ldots, m \right\}.$$

That is, Γ is the distinguished boundary of $c\mathbb{D}^m$, a polydisc centered at 0 of radius c in $\mathbb{C}^m$. Define the function

$$F(z, w) = \frac{1}{(2\pi i)^m} \int_\Gamma \frac{f(\xi, w)}{\xi - z} \, d\xi.$$

Clearly F is well-defined on

$$c\mathbb{D}^m \times \mathbb{D}^k$$

as ξ only ranges through Γ and so as long as $w \in \mathbb{D}^k$ then $(\xi, w) \in H$.

The function F is holomorphic in w as we can differentiate underneath the integral and f is holomorphic in w on H. Furthermore, F is holomorphic in z as the kernel $\frac{1}{\xi - z}$ is holomorphic in z as long as $z \in c\mathbb{D}^m$.

For any fixed w with $|w_j| < b$ for all j, the Cauchy integral formula says $F(z, w) = f(z, w)$ for all $z \in c\mathbb{D}^m$. Hence, $F = f$ on the open set $c\mathbb{D}^m \times b\mathbb{D}^k$, and so they are equal on $(c\mathbb{D}^m \times \mathbb{D}^k) \cap H$. Combining F and f we obtain a holomorphic function on $\mathbb{D}^{m+k}$ that extends f. □

The theorem is used in many situations to extend holomorphic functions. We usually need to translate, scale, rotate (apply a unitary matrix), and even take more general biholomorphic mappings of H, to place it wherever we need it. The corresponding polydisc—or the image of the polydisc under the appropriate biholomorphic mapping if one was used—to which all holomorphic functions on H extend is denoted by $\widehat{H}$ and is called the *hull* of H.

Let us state a simple but useful case of the so-called *Hartogs phenomenon*.

Corollary 2.1.5. *Let $U \subset \mathbb{C}^n$, $n \geq 2$, be an open set and $p \in U$. Then every $f \in \mathcal{O}(U \setminus \{p\})$ extends holomorphically to U.*

Proof. Without loss of generality, by translating and scaling (those operations are after all holomorphic), we assume that $p = (\frac{3}{4}, 0, \ldots, 0)$ and the unit polydisc $\mathbb{D}^n$ is contained in U. We fit a Hartogs figure H in U by letting $m = 1$ and $k = n - 1$, writing $\mathbb{C}^n = \mathbb{C}^1 \times \mathbb{C}^{n-1}$, and taking $a = b = \frac{1}{2}$. Then $H \subset U$, and $p \in \mathbb{D}^n \setminus H$. Theorem 2.1.4 says that f extends to be holomorphic at p. □

This result provides another reason why holomorphic functions in several variables have no isolated zeros. That is, suppose $U \subset \mathbb{C}^n$, $n \geq 2$, and $f \in \mathcal{O}(U)$ with f being zero only at p, that is $f^{-1}(0) = \{p\}$. Then $\frac{1}{f}$ would be holomorphic in $U \setminus \{p\}$. It would not be possible to extend f through p (not even continuously let alone holomorphically), and we obtain a contradiction.

The extension works in an even more surprising fashion. We could take out a very large set, for example, any geometrically convex subset:

***Exercise* 2.1.6:** *Suppose $U \subset \mathbb{C}^n$, $n \geq 2$, be an open set and $K \subset\subset U$ is a compact geometrically convex subset. If $f \in \mathcal{O}(U \setminus K)$, then f extends to be holomorphic in U. Hint: Find a nice point on ∂K and try extending a little bit. Then make sure your extension is single-valued.*

Convexity of K is not needed; we only need that $U \setminus K$ is connected, however, the proof is much harder. The single-valuedness of the extension is the key point that makes the general proof harder.

Notice the surprising fact that any holomorphic function on the shell

$$\mathbb{B}_n \setminus \overline{B_{1-\epsilon}(0)} = \left\{ z \in \mathbb{C}^n : 1 - \epsilon < \|z\| < 1 \right\}$$

for any $\epsilon > 0$ automatically extends to a holomorphic function of $\mathbb{B}_n$. We need $n > 1$. The extension result decisively does not work in one dimension; for example take $1/z$. Notice that if $n \geq 2$, then if $f \in \mathcal{O}(\mathbb{B}_n)$ the set of its zeros must "touch the boundary" (in other words $f^{-1}(0)$ is not compact) or be empty: If the set of zeros was compact in $\mathbb{B}_n$, then we could try to extend the function $1/f$.

Exercise* 2.1.7** (Hartogs triangle):*** *Let*

$$T = \left\{ (z_1, z_2) \in \mathbb{D}^2 : |z_2| < |z_1| \right\}.$$

Show that T is a domain of holomorphy. Then show that if

$$\widetilde{T} = T \cup B_\epsilon(0)$$

for an arbitrarily small $\epsilon > 0$, then $\widetilde{T}$ is not a domain of holomorphy. In fact, every function holomorphic on $\widetilde{T}$ extends to a holomorphic function of $\mathbb{D}^2$.

Exercise* 2.1.8*: *Take the natural embedding of $\mathbb{R}^2 \subset \mathbb{C}^2$. Suppose $f \in \mathcal{O}(\mathbb{C}^2 \setminus \mathbb{R}^2)$. Show that f extends to be holomorphic in all of $\mathbb{C}^2$. Hint: Change coordinates before using Hartogs.*

Exercise* 2.1.9*: *Suppose*

$$U = \left\{ (z, w) \in \mathbb{D}^2 : 1/2 < |z| \right\}.$$

Draw U. Let $\gamma = \left\{ z \in \mathbb{C} : |z| = 3/4 \right\}$ oriented positively. If $f \in \mathcal{O}(U)$, then show that the function

$$F(z, w) = \frac{1}{2\pi i} \int_\gamma \frac{f(\xi, w)}{\xi - z}\, d\xi$$

is well-defined in $\left((3/4)\mathbb{D}\right) \times \mathbb{D}$, holomorphic where defined, yet it is not necessarily true that $F = f$ on the intersections of their domains.

Exercise* 2.1.10*: *Suppose $U \subset \mathbb{C}^n$ is an open set such that for every $z \in \mathbb{C}^n \setminus \{0\}$, there is a $\lambda \in \mathbb{C}$ such that $\lambda z \in U$. Let $f \colon U \to \mathbb{C}$ be holomorphic with $f(\lambda z) = f(z)$ whenever $z \in U$, $\lambda \in \mathbb{C}$ and $\lambda z \in U$.*

a) (easy) Prove that f is constant.

b) (hard) Relax the requirement on f to being meromorphic, that is $f = g/h$ for holomorphic g and h, find a nonconstant example and prove that such an f must be rational (that is g and h must be polynomials).

Exercise* 2.1.11*: *Suppose*

$$U = \left\{ z \in \mathbb{D}^3 : 1/2 < |z_1| \quad \text{or} \quad 1/2 < |z_2| \right\}.$$

Prove that every function $f \in \mathcal{O}(U)$ extends to $\mathbb{D}^3$. Compare to Exercise 2.1.9.

Exercise* 2.1.12*: *Suppose $U = \mathbb{C}^n \setminus \{z \in \mathbb{C}^n : z_1 = z_2 = 0\}$, $n \geq 2$. Show that every $f \in \mathcal{O}(U)$ extends holomorphically to $\mathbb{C}^n$.*

Example 2.1.6: By Exercise 2.1.8, $U_1 = \mathbb{C}^2 \setminus \mathbb{R}^2$ is not a domain of holomorphy. On the other hand, $U_2 = \mathbb{C}^2 \setminus \{z \in \mathbb{C}^2 : z_2 = 0\}$ is a domain of holomorphy; simply use $f(z) = \frac{1}{z_2}$ as the function that cannot extend. Therefore U_1 and U_2 are very different as far as complex variables are concerned, yet they are the same set if we ignore the complex structure. They are both simply a 4-dimensional real vector space minus a 2-dimensional real vector subspace. That is, U_1 is the set where either $\operatorname{Im} z_1 \neq 0$ or $\operatorname{Im} z_2 \neq 0$, while U_2 is the set where either $\operatorname{Re} z_2 \neq 0$ or $\operatorname{Im} z_2 \neq 0$.

The condition of being a domain of holomorphy, requires something more than just some real geometric condition on the set. In particular we have shown that the image of a domain of holomorphy via an orthonormal real-linear mapping (so preserving distances, angles, straight lines, etc.) need not be a domain of holomorphy. Therefore, when we want to "rotate" in complex analysis we need to use a complex linear mapping, so a unitary matrix.

2.2 Tangent vectors, the Hessian, and convexity

An exercise in the previous section showed that any convex domain is a domain of holomorphy. However, classical convexity is too strong.

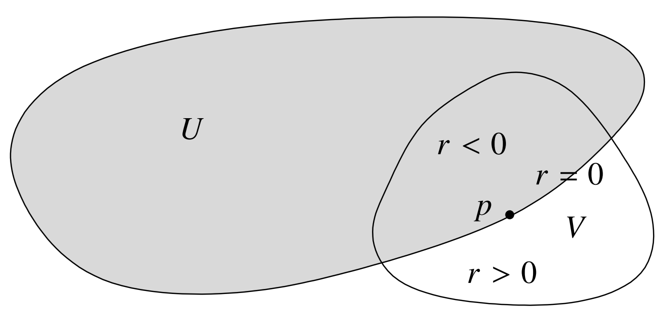
Exercise 2.2.1: *Show that if $U \subset \mathbb{C}^m$ and $V \subset \mathbb{C}^k$ are both domains of holomorphy, then $U \times V$ is a domain of holomorphy.*

In particular, the exercise says that given any domain $U \subset \mathbb{C}$ and any domain $V \subset \mathbb{C}$, the domain $U \times V$ is a domain of holomorphy in $\mathbb{C}^2$. The domains U and V, and therefore $U \times V$ can be spectacularly non-convex. But we should not discard convexity completely, there is a notion of *pseudoconvexity*, which vaguely means "convexity in the complex directions," that is the correct notion to classify which domains are domains of holomorphy. Let us figure out what classical convexity means locally for a smooth boundary.

Definition 2.2.1. A set $M \subset \mathbb{R}^n$ is a real C^k-smooth *hypersurface* if at each point $p \in M$, there exists a k-times continuously differentiable function $r \colon V \to \mathbb{R}$, defined in a neighborhood V of p with nonvanishing derivative such that $M \cap V = \{x \in V : r(x) = 0\}$. The function r is called the *defining function* (at p).

An open set (or domain) U with *C^k-smooth boundary* is a set where ∂U is a C^k-smooth hypersurface, and for every $p \in \partial U$ there is a defining function r such that $r < 0$ for points in U and $r > 0$ for points not in U.

If we say simply *smooth*, we mean C^∞-smooth, that is, the r above is infinitely differentiable.



What we really defined is an *embedded hypersurface*. In particular, in this book the topology on the set M will be the subset topology.

For simplicity, in this book we generally deal with smooth (that is, C^∞) functions and hypersurfaces. Dealing with C^k-smooth functions for finite k introduces technicalities that make certain theorems and arguments unnecessarily difficult.

As the derivative of r is nonvanishing, a hypersurface M is locally the graph of one variable over the rest using the implicit function theorem. That is, M is a smooth hypersurface if it is locally a set defined by $x_j = \varphi(x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n)$ for some j and some smooth function φ.

Notice that the definition of an open set with smooth boundary is not just that the boundary is a smooth hypersurface, that is not enough. It also says that one side of that hypersurface is in U and one side is not in U. That is because if the derivative of r never vanishes, then r must have different signs on different sides of $\{x \in V : r(x) = 0\}$. The verification of this fact is left to the reader (Hint: Look at where the gradient points to).

Same definition works for $\mathbb{C}^n$, where we treat $\mathbb{C}^n$ as $\mathbb{R}^{2n}$. For example, the ball $\mathbb{B}_n$ is a domain with smooth boundary with defining function $r(z, \bar{z}) = \|z\|^2 - 1$. We can, in fact, find a single global defining function for every open set with smooth boundary, but we have no need of this. In $\mathbb{C}^n$ a hypersurface defined as above is a *real hypersurface*, to distinguish it from a complex hypersurface that would be the zero set of a holomorphic function.

Definition 2.2.2. For any $p \in \mathbb{R}^n$, the set of *tangent vectors* $T_p\mathbb{R}^n$ is given by

$$T_p\mathbb{R}^n = \operatorname{span}_{\mathbb{R}} \left\{ \frac{\partial}{\partial x_1}\bigg|_p, \ldots, \frac{\partial}{\partial x_n}\bigg|_p \right\}.$$

That is, a vector $X_p \in T_p\mathbb{R}^n$ is an object of the form

$$X_p = \sum_{j=1}^n a_j \frac{\partial}{\partial x_j}\bigg|_p,$$

for real numbers a_j. So for computations, X_p could be represented by an n-vector $a = (a_1, \ldots, a_n)$. However, if $p \neq q$, then $T_p\mathbb{R}^n$ and $T_q\mathbb{R}^n$ are distinct spaces. An object $\frac{\partial}{\partial x_j}\Big|_p$ is a linear functional on the space of smooth functions: When applied to a smooth function g, it gives $\frac{\partial g}{\partial x_j}\Big|_p$. Therefore X_p is also such a functional. It is the directional derivative from calculus; it is computed as $X_p f = \nabla f|_p \cdot (a_1, \ldots, a_n)$.

Definition 2.2.3. Let $M \subset \mathbb{R}^n$ be a smooth hypersurface, $p \in M$, and r is the defining function at p, then a vector $X_p \in T_p\mathbb{R}^n$ is *tangent* to M at p if

$$X_p r = 0, \qquad \text{or in other words} \qquad \sum_{j=1}^n a_j \frac{\partial r}{\partial x_j}\bigg|_p = 0.$$

The space of tangent vectors to M is denoted by T_pM, and is called the *tangent space* to M at p.

The space T_pM is an $(n-1)$-dimensional real vector space—it is a subspace of an n-dimensional $T_p\mathbb{R}^n$ given by a single linear equation. Recall from calculus that the gradient $\nabla r|_p$ is "normal" to M at p, and the tangent space is given by all the n-vectors a that are orthogonal to the normal, that is $\nabla r|_p \cdot a = 0$.

We cheated in the terminology, and assumed without justification that $T_p M$ depends only on M, not on r. Fortunately, the definition of $T_p M$ is independent of the choice of r by the next two exercises.

***Exercise* 2.2.2:** *Suppose $M \subset \mathbb{R}^n$ is a smooth hypersurface and r is a smooth defining function for M at p.*

a) *Suppose φ is another smooth defining function of M on a neighborhood of p. Show that there exists a smooth nonvanishing function g such that $\varphi = gr$ (in a neighborhood of p).*
b) *Now suppose φ is just any smooth function that vanishes on M (not necessarily a defining function). Again show that $\varphi = gr$, but now g may possibly vanish.*

Hint: First suppose $r = x_n$, then find a g such that $\varphi = x_n g$. Then find a local change of variables to make M into the set given by $x_n = 0$. A useful one variable calculus fact: If $f(0) = 0$ and f is smooth, then $s \int_0^1 f'(ts)\,dt = f(s)$, and $\int_0^1 f'(ts)\,dt$ is a smooth function of s.

***Exercise* 2.2.3:** *Show that $T_p M$ is independent of which defining function we take. That is, prove that if r and $\tilde{r}$ are defining functions for M at p, then $\sum_j a_j \frac{\partial r}{\partial x_j}\Big|_p = 0$ if and only if $\sum_j a_j \frac{\partial \tilde{r}}{\partial x_j}\Big|_p = 0$.*

The tangent space $T_p M$ is the set of derivatives *along* M at p. If r is a defining function of M, and f and h are two smooth functions such that $f = h$ on M, then Exercise 2.2.2 says that

$$f - h = gr, \qquad \text{or} \qquad f = h + gr,$$

for some smooth g. Applying X_p we find

$$X_p f = X_p h + X_p(gr) = X_p h + (X_p g) r + g(X_p r) = X_p h + (X_p g) r.$$

So $X_p f = X_p h$ on M (where $r = 0$). In other words, $X_p f$ only depends on the values of f on M.

Example 2.2.4: If $M \subset \mathbb{R}^n$ is given by $x_n = 0$, then $T_p M$ is given by derivatives of the form

$$X_p = \sum_{j=1}^{n-1} a_j \frac{\partial}{\partial x_j}\Big|_p .$$

That is, derivatives along the first $n-1$ variables only.

Definition 2.2.5. The disjoint union

$$T\mathbb{R}^n = \bigcup_{p \in \mathbb{R}^n} T_p \mathbb{R}^n$$

is called the *tangent bundle*. There is a natural identification $\mathbb{R}^n \times \mathbb{R}^n \cong T\mathbb{R}^n$, that is,

$$(p, a) \in \mathbb{R}^n \times \mathbb{R}^n \quad \mapsto \quad \sum_{j=1}^{n} a_j \frac{\partial}{\partial x_j}\Big|_p \in T\mathbb{R}^n .$$

The topology and smooth structure on $T\mathbb{R}^n$ comes from this identification. The wording "bundle" (a bundle of fibers) comes from the natural projection $\pi\colon T\mathbb{R}^n \to \mathbb{R}^n$, where fibers are $\pi^{-1}(p) = T_p\mathbb{R}^n$.

A smooth *vector field* in $T\mathbb{R}^n$ is an object of the form

$$X = \sum_{j=1}^{n} a_j \frac{\partial}{\partial x_j},$$

where a_j are smooth functions. That is, X is a smooth function $X\colon V \subset \mathbb{R}^n \to T\mathbb{R}^n$ such that $X(p) \in T_p\mathbb{R}^n$. Usually we write X_p rather than $X(p)$. To be more fancy, say X is a *section* of $T\mathbb{R}^n$.

Similarly, the tangent bundle of M is

$$TM = \bigcup_{p\in M} T_pM.$$

A vector field X in TM is a vector field such that $X_p \in T_pM$ for all $p \in M$.

Before we move on, let us note how smooth maps transform tangent spaces. Given a smooth mapping $f\colon U \subset \mathbb{R}^n \to \mathbb{R}^m$, the derivative at p is a linear mapping of the tangent spaces: $Df(p)\colon T_p\mathbb{R}^n \to T_{f(p)}\mathbb{R}^m$. That is, given $X_p \in T_p\mathbb{R}^n$, then $Df(p)X_p$ should be in $T_{f(p)}\mathbb{R}^m$. So if φ is a smooth function of a neighborhood of $f(p)$ in $\mathbb{R}^m$, then $Df(p)X_p$ acts on φ as

$$Df(p)X_p\varphi = X_p(\varphi \circ f).$$

It is the only reasonable way to put those three objects together. When the spaces are $\mathbb{C}^n$ and $\mathbb{C}^m$, we denote this derivative as $D_{\mathbb{R}} f$ to distinguish it from the holomorphic derivative. As far as calculus computations are concerned, the linear mapping $Df(p)$ is the Jacobian matrix acting on vectors in the standard basis of the tangent space as given above. This is why we use the same notation for the Jacobian matrix and the derivative acting on tangent spaces. To see that this is so, it is enough to see where the basis element $\frac{\partial}{\partial x_j}\big|_p$ goes, and the form of $Df(p)$ as a matrix follows by the chain rule. For example, the derivative of the mapping $f(x_1, x_2) = (x_1 + 2x_2 + x_1^2, 3x_1 + 4x_2 + x_1x_2)$ at the origin is given by the matrix $\left[\begin{smallmatrix} 1 & 2 \\ 3 & 4 \end{smallmatrix}\right]$, and so the vector $X_p = a\frac{\partial}{\partial x_1}\big|_0 + b\frac{\partial}{\partial x_2}\big|_0$ gets taken to $Df(0)X_0 = (a+2b)\frac{\partial}{\partial y_1}\big|_0 + (3a+4b)\frac{\partial}{\partial y_2}\big|_0$, where we let (y_1, y_2) be the coordinates on the target. You should check on some test function, such as $\varphi(y_1, y_2) = \alpha y_1 + \beta y_2$, that this satisfies the definition.

Now that we know what tangent vectors are and how they transform, let us define convexity for domains with smooth boundary.

Definition 2.2.6. Suppose $U \subset \mathbb{R}^n$ is an open set with smooth boundary, and r is a defining function for ∂U at $p \in \partial U$ such that $r < 0$ on U. If

$$\sum_{j=1,\ell=1}^{n} a_j a_\ell \frac{\partial^2 r}{\partial x_j \partial x_\ell}\bigg|_p \geq 0, \qquad \text{for all} \qquad X_p = \sum_{j=1}^{n} a_j \frac{\partial}{\partial x_j}\bigg|_p \quad \in \quad T_p\partial U,$$

then U is said to be *convex* at p. If the inequality above is strict for all nonzero $X_p \in T_p\partial U$, then U is said to be *strongly convex* at p.

A domain U is *convex* if it is convex at all $p \in \partial U$. If U is bounded*, we say U is *strongly convex* if it is strongly convex at all $p \in \partial U$.

*Matters are a little more complicated with the "strong" terminology if U is unbounded.

The matrix

$$\left[\frac{\partial^2 r}{\partial x_j \partial x_\ell}\Big|_p \right]_{j\ell}$$

is the *Hessian* of r at p. So, U is convex at $p \in \partial U$ if the Hessian of r at p as a bilinear form is positive semidefinite when restricted to $T_p \partial U$. In the language of vector calculus, let H be the Hessian of r at p, and treat $a \in \mathbb{R}^n$ as a column vector. Then ∂U is convex at p whenever

$$a^t H a \geq 0, \qquad \text{for all } a \in \mathbb{R}^n \text{ such that } \quad \nabla r|_p \cdot a = 0.$$

This bilinear form given by the Hessian is the second fundamental form from Riemannian geometry in mild disguise (or perhaps it is the other way around).

We cheated a little bit, since we have not proved that the notion of convexity is well-defined. In particular, there are many possible defining functions.

Exercise* 2.2.4*: *Show that the definition of convexity is independent of the defining function. Hint: If $\tilde{r}$ is another defining function near p, then there is a smooth function $g > 0$ such that $\tilde{r} = gr$.*

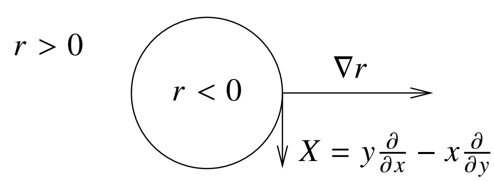
Exercise* 2.2.5*: *Show that if an open set with smooth boundary is strongly convex at a point, then it is strongly convex at all nearby points. On the other hand find an example of an open set with smooth boundary that is convex at one point p, but not convex at points arbitrarily near p.*

Example 2.2.7: Let us prove that the unit disc in $\mathbb{R}^2$ is convex (actually strongly convex). Let (x, y) be the coordinates and let $r(x, y) = x^2 + y^2 - 1$ be the defining function.

The tangent space of the circle is one-dimensional, so we simply need to find a single nonzero tangent vector at each point. Consider the gradient $\nabla r = (2x, 2y)$ to check that

$$X = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}$$

is tangent to the circle, that is, $Xr = X(x^2 + y^2 - 1) = (2x, 2y) \cdot (y, -x) = 0$ on the circle (by chance $Xr = 0$ everywhere). The vector field X is nonzero on the circle, so at each point it gives a basis of the tangent space.



The Hessian matrix of r is

$$\begin{bmatrix} \frac{\partial^2 r}{\partial x^2} & \frac{\partial^2 r}{\partial x \partial y} \\ \frac{\partial^2 r}{\partial y \partial x} & \frac{\partial^2 r}{\partial y^2} \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}.$$

Applying the vector $(y, -x)$ gets us

$$\begin{bmatrix} y & -x \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} y \\ -x \end{bmatrix} = 2y^2 + 2x^2 = 2 > 0.$$

So the domain given by $r < 0$ is strongly convex at all points.

In general, to construct a tangent vector field for a curve in $\mathbb{R}^2$, consider $r_y \frac{\partial}{\partial x} - r_x \frac{\partial}{\partial y}$. In higher dimensions we can just run through enough pairs of variables to get a basis of TM.

Exercise 2.2.6: *Show that the domain in $\mathbb{R}^2$ defined by $x^4 + y^4 < 1$ is convex, but not strongly convex. Find all the points where the domain is not strongly convex.*

Exercise 2.2.7: *Show that the domain in $\mathbb{R}^3$ defined by $(x_1^2 + x_2^2)^2 < x_3$ is strongly convex at all points except the origin, where it is just convex (but not strongly).*

In the following, we use the *big-oh notation*, although we use a perhaps less standard short hand*. A smooth function is $O(\ell)$ at a point p (usually the origin), if all its derivatives of order $0, 1, \ldots, \ell - 1$ vanish at p. For example, if f is $O(3)$ at the origin then $f(0) = 0$, and its first and second derivatives vanish at the origin.

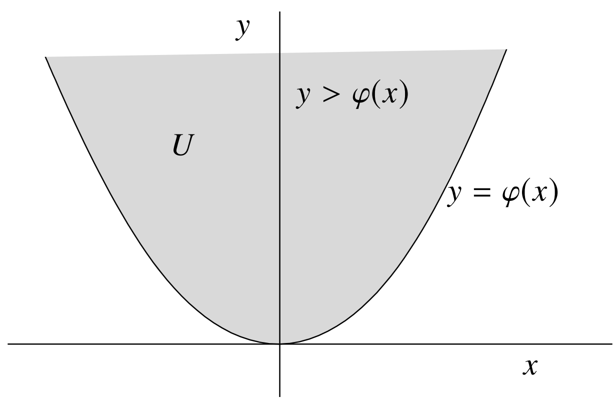
For computations it is often useful to use a more convenient defining function, that is, it is convenient to write M as a graph.

Lemma 2.2.8. *Suppose $M \subset \mathbb{R}^n$ is a smooth hypersurface, and $p \in M$. Then after a rotation and translation, p is the origin and near the origin M is defined by*

$$y = \varphi(x),$$

where $(x, y) \in \mathbb{R}^{n-1} \times \mathbb{R}$ are our coordinates and φ is a smooth function that is $O(2)$ at the origin, that is $\varphi(0) = 0$ and $d\varphi(0) = 0$.

If M is the boundary of an open set U with smooth boundary and $r < 0$ on U, then the rotation can be chosen such that for points in U we have $y > \varphi(x)$.



Proof. Let r be a defining function at p. Take $v = \nabla r|_p$. By translating p to zero, and applying a rotation (an orthogonal matrix), we assume $v = (0, 0, \ldots, 0, v_n)$, where $v_n < 0$. Denote our coordinates by $(x, y) \in \mathbb{R}^{n-1} \times \mathbb{R}$. As $\nabla r|_0 = v$, then $\frac{\partial r}{\partial y}(0) \neq 0$. We apply the implicit function theorem to find a smooth function φ such that $r\bigl(x, \varphi(x)\bigr) = 0$ for all x in a neighborhood of the origin, and $\bigl\{(x, y) : y = \varphi(x)\bigr\}$ are all the solutions to $r = 0$ near the origin.

What is left is to show that the derivative at 0 of φ vanishes. As $r\bigl(x, \varphi(x)\bigr) = 0$ for all x in a neighborhood of the origin, we differentiate. For any $j = 1, \ldots, n - 1$,

$$0 = \frac{\partial}{\partial x_j}\Bigl[r\bigl(x, \varphi(x)\bigr)\Bigr] = \left(\sum_{\ell=1}^{n-1} \frac{\partial r}{\partial x_\ell}\frac{\partial x_\ell}{\partial x_j}\right) + \frac{\partial r}{\partial y}\frac{\partial \varphi}{\partial x_j} = \frac{\partial r}{\partial x_j} + \frac{\partial r}{\partial y}\frac{\partial \varphi}{\partial x_j}.$$

*The standard notation for $O(\ell)$ is $O(\lVert x \rVert^\ell)$ and is usually defined to mean that $\left|\frac{f(x)}{\lVert x \rVert^\ell}\right|$ is bounded as $x \to p$.

At the origin, $\frac{\partial r}{\partial x_j}(0,0) = 0$ and $\frac{\partial r}{\partial y}(0,0) = v_n \neq 0$, and therefore $\frac{\partial \varphi}{\partial x_j}(0) = 0$.

To prove the final statement, note that $r < 0$ on U. It is enough to check that r is negative for $(0, y)$ if $y > 0$ is small, which follows as $\frac{\partial r}{\partial y}(0,0) = v_n < 0$. □

The advantage of this representation is that the tangent space at p can be identified with the x coordinates for the purposes of computation. Considering x as a column vector, the Taylor expansion of any smooth function φ at the origin is

$$\varphi(x) = \varphi(0) + \nabla\varphi|_0 \cdot x + \frac{1}{2}\, x^t H x + E(x),$$

where H is the Hessian matrix of φ at the origin, that is $H = \left[\frac{\partial^2 \varphi}{\partial x_j \partial x_k}\Big|_0 \right]_{jk}$, and E is $O(3)$. That is, $E(0) = 0$, and all first and second derivatives of E vanish at 0. For the situation from the lemma above, the φ is $O(2)$ at the origin, i.e. $\varphi(0) = 0$ and $\nabla\varphi|_0 = 0$. So we write the hypersurface M as

$$y = \frac{1}{2}\, x^t H x + E(x).$$

If M is the boundary ∂U of an open set, then we pick the rotation so that $y > \frac{1}{2}\, x^t H x + E(x)$ on U. It is an easy exercise to show that U is convex at p if H positive semidefinite, and U is strongly convex at p if H is positive definite.

Exercise* 2.2.8*: *Prove the above statement about H and convexity at p.*

Exercise* 2.2.9*: *M is convex from both sides at p if and only if for a defining function r for M at p, both the set given by $r > 0$ and the set given by $r < 0$ are convex at p. Prove that if a hypersurface $M \subset \mathbb{R}^n$ is convex from both sides at all points then it is locally just a hyperplane (the zero set of a real affine function).*

Recall that U is *geometrically convex* if for every $p, q \in U$ the line between p and q is in U, or in other words $tp + (1-t)q \in U$ for all $t \in [0,1]$. In particular note that geometric convexity is a *global* condition. You need to know all of U. On the other hand, the notion of convex for a smooth boundary is *local* in that you only need to know ∂U in a small neighborhood. It turns out for domains with smooth boundaries the two notions are equivalent. One direction is easy.

Exercise* 2.2.10*: *Suppose a domain $U \subset \mathbb{R}^n$ with smooth boundary is geometrically convex. Show that it is convex.*

The other direction is considerably more complicated, and we will not worry about it here. Proving a global condition from a local one is often trickier, but also often more interesting. Similar difficulties will be present once we move back to several complex variables and try to relate pseudoconvexity with domains of holomorphy.

2.3 Holomorphic vectors, the Levi form, and pseudoconvexity

As $\mathbb{C}^n$ is identified with $\mathbb{R}^{2n}$ using $z = x + iy$, we have $T_p\mathbb{C}^n = T_p\mathbb{R}^{2n}$. If we take the complex span instead of the real span we get the *complexified tangent space*

$$\mathbb{C} \otimes T_p\mathbb{C}^n = \operatorname{span}_{\mathbb{C}} \left\{ \frac{\partial}{\partial x_1}\Big|_p, \frac{\partial}{\partial y_1}\Big|_p, \ldots, \frac{\partial}{\partial x_n}\Big|_p, \frac{\partial}{\partial y_n}\Big|_p \right\}.$$

We simply replace all the real coefficients with complex ones. The space $\mathbb{C} \otimes T_p\mathbb{C}^n$ is a $2n$-dimensional complex vector space. Both $\frac{\partial}{\partial z_j}\big|_p$ and $\frac{\partial}{\partial \bar{z}_j}\big|_p$ are in $\mathbb{C} \otimes T_p\mathbb{C}^n$, and in fact:

$$\mathbb{C} \otimes T_p\mathbb{C}^n = \operatorname{span}_{\mathbb{C}} \left\{ \frac{\partial}{\partial z_1}\Big|_p, \frac{\partial}{\partial \bar{z}_1}\Big|_p, \ldots, \frac{\partial}{\partial z_n}\Big|_p, \frac{\partial}{\partial \bar{z}_n}\Big|_p \right\}.$$

Define

$$T_p^{(1,0)}\mathbb{C}^n \overset{\text{def}}{=} \operatorname{span}_{\mathbb{C}} \left\{ \frac{\partial}{\partial z_1}\Big|_p, \ldots, \frac{\partial}{\partial z_n}\Big|_p \right\} \qquad \text{and} \qquad T_p^{(0,1)}\mathbb{C}^n \overset{\text{def}}{=} \operatorname{span}_{\mathbb{C}} \left\{ \frac{\partial}{\partial \bar{z}_1}\Big|_p, \ldots, \frac{\partial}{\partial \bar{z}_n}\Big|_p \right\}.$$

The vectors in $T_p^{(1,0)}\mathbb{C}^n$ are the *holomorphic vectors* and vectors in $T_p^{(0,1)}\mathbb{C}^n$ are the *antiholomorphic vectors*. We decompose the full tangent space as the direct sum

$$\mathbb{C} \otimes T_p\mathbb{C}^n = T_p^{(1,0)}\mathbb{C}^n \oplus T_p^{(0,1)}\mathbb{C}^n.$$

A holomorphic function is one that vanishes on $T_p^{(0,1)}\mathbb{C}^n$.

Let us note what holomorphic functions do to these spaces. Given a smooth mapping f from $\mathbb{C}^n$ to $\mathbb{C}^m$, its derivative at $p \in \mathbb{C}^n$ is a real-linear mapping $D_{\mathbb{R}}f(p)\colon T_p\mathbb{C}^n \to T_{f(p)}\mathbb{C}^m$. Given the basis above, this mapping is represented by the standard real Jacobian matrix, that is, a real $2m \times 2n$ matrix that we wrote before as $D_{\mathbb{R}}f(p)$.

Proposition 2.3.1. *Let $f\colon U \subset \mathbb{C}^n \to \mathbb{C}^m$ be a holomorphic mapping with $p \in U$. Suppose $D_{\mathbb{R}}f(p)\colon T_p\mathbb{C}^n \to T_{f(p)}\mathbb{C}^m$ is the real derivative of f at p. We naturally extend the derivative to $D_{\mathbb{C}}f(p)\colon \mathbb{C} \otimes T_p\mathbb{C}^n \to \mathbb{C} \otimes T_{f(p)}\mathbb{C}^m$. Then*

$$D_{\mathbb{C}}f(p)\Big(T_p^{(1,0)}\mathbb{C}^n\Big) \subset T_{f(p)}^{(1,0)}\mathbb{C}^m \qquad \textit{and} \qquad D_{\mathbb{C}}f(p)\Big(T_p^{(0,1)}\mathbb{C}^n\Big) \subset T_{f(p)}^{(0,1)}\mathbb{C}^m.$$

If f is a biholomorphism, then $D_{\mathbb{C}}f(p)$ restricted to $T_p^{(1,0)}\mathbb{C}^n$ is a vector space isomorphism. Similarly for $T_p^{(0,1)}\mathbb{C}^n$.

Exercise* 2.3.1*: *Prove the proposition. Hint: Start with $D_{\mathbb{R}}f(p)$ as a real $2m \times 2n$ matrix to show it extends (it is the same matrix if you think of it as a matrix and use the same basis vectors). Think of $\mathbb{C}^n$ and $\mathbb{C}^m$ in terms of the zs and the $\bar{z}$s and think of f as a mapping*

$$(z, \bar{z}) \mapsto \big(f(z), \bar{f}(\bar{z})\big).$$

Write the derivative as a matrix in terms of the zs and the $\bar{z}$s and fs and $\bar{f}$s and the result will follow. That is just changing the basis.

When talking about only holomorphic functions and holomorphic vectors, when we say derivative of f, we mean the holomorphic part of the derivative, which we write as

$$Df(p)\colon T_p^{(1,0)}\mathbb{C}^n \to T_{f(p)}^{(1,0)}\mathbb{C}^m.$$

That is, $Df(p)$ is the restriction of $D_{\mathbb{C}}f(p)$ to $T_p^{(1,0)}\mathbb{C}^n$. In other words, let z be the coordinates on $\mathbb{C}^n$ and w be coordinates on $\mathbb{C}^m$. Using the bases $\left\{ \frac{\partial}{\partial z_1}\Big|_p, \ldots, \frac{\partial}{\partial z_n}\Big|_p \right\}$ on $\mathbb{C}^n$ and $\left\{ \frac{\partial}{\partial w_1}\Big|_{f(p)}, \ldots, \frac{\partial}{\partial w_m}\Big|_{f(p)} \right\}$ on $\mathbb{C}^m$, the holomorphic derivative of $f\colon \mathbb{C}^n \to \mathbb{C}^m$ is represented as the $m \times n$ Jacobian matrix

$$\left[\frac{\partial f_j}{\partial z_k}\Big|_p \right]_{jk},$$

which we have seen before and for which we also used the notation $Df(p)$.

As before, define the tangent bundles

$$\mathbb{C} \otimes T\mathbb{C}^n, \quad T^{(1,0)}\mathbb{C}^n, \quad \text{and} \quad T^{(0,1)}\mathbb{C}^n,$$

by taking the disjoint unions. One can also define vector fields in these bundles.

Let us describe $\mathbb{C} \otimes T_pM$ for a real smooth hypersurface $M \subset \mathbb{C}^n$. Let r be a real-valued defining function of M at p. A vector $X_p \in \mathbb{C} \otimes T_p\mathbb{C}^n$ is in $\mathbb{C} \otimes T_pM$ whenever $X_pr = 0$. That is,

$$X_p = \sum_{j=1}^n \left(a_j \frac{\partial}{\partial z_j}\Big|_p + b_j \frac{\partial}{\partial \bar{z}_j}\Big|_p \right) \in \mathbb{C} \otimes T_pM \qquad \text{whenever} \qquad \sum_{j=1}^n \left(a_j \frac{\partial r}{\partial z_j}\Big|_p + b_j \frac{\partial r}{\partial \bar{z}_j}\Big|_p \right) = 0.$$

Therefore, $\mathbb{C} \otimes T_pM$ is a $(2n-1)$-dimensional complex vector space. We decompose $\mathbb{C} \otimes T_pM$ as

$$\mathbb{C} \otimes T_pM = T_p^{(1,0)}M \oplus T_p^{(0,1)}M \oplus B_p,$$

where

$$T_p^{(1,0)}M \overset{\text{def}}{=} \big(\mathbb{C} \otimes T_pM\big) \cap \big(T_p^{(1,0)}\mathbb{C}^n\big), \qquad \text{and} \qquad T_p^{(0,1)}M \overset{\text{def}}{=} \big(\mathbb{C} \otimes T_pM\big) \cap \big(T_p^{(0,1)}\mathbb{C}^n\big).$$

The B_p is just the "left-over" and must be included, otherwise the dimensions will not work out.

Make sure that you understand what all the objects are. The space T_pM is a real vector space; $\mathbb{C} \otimes T_pM$, $T_p^{(1,0)}M$, $T_p^{(0,1)}M$, and B_p are complex vector spaces. To see that these give vector bundles, we must first show that their dimensions do not vary from point to point. The easiest way to see this fact is to write down convenient local coordinates. First, let us see what a biholomorphic map does to the holomorphic and antiholomorphic vectors. A biholomorphic map f is a diffeomorphism. And if a real hypersurface M is defined by a function r near p, then the image $f(M)$ is also a real hypersurface is given by the defining function $r \circ f^{-1}$ near $f(p)$.

Proposition 2.3.2. *Suppose $M \subset \mathbb{C}^n$ is a smooth real hypersurface, $p \in M$, and $U \subset \mathbb{C}^n$ is an open set such that $M \subset U$. Let $f\colon U \to \mathbb{C}^n$ be a holomorphic mapping such that $Df(p)$ is invertible (a biholomorphism near p). Let $D_{\mathbb{C}}f(p)$ be the complexified real derivative as before. Then*

$$D_{\mathbb{C}}f(p)\Big(T_p^{(1,0)}M\Big) = T_{f(p)}^{(1,0)}f(M), \qquad D_{\mathbb{C}}f(p)\Big(T_p^{(0,1)}M\Big) = T_{f(p)}^{(0,1)}f(M).$$

That is, the spaces are isomorphic as complex vector spaces.

The proposition is local, if U is only a neighborhood of p, replace M with $M \cap U$.

Proof. The proof is an application of Proposition 2.3.1. As the map is a biholomorphism at p,

$$D_{\mathbb{C}} f(p)\Big(T_p^{(1,0)}\mathbb{C}^n\Big) = T_{f(p)}^{(1,0)}\mathbb{C}^n, \quad D_{\mathbb{C}} f(p)\Big(T_p^{(0,1)}\mathbb{C}^n\Big) = T_{f(p)}^{(0,1)}\mathbb{C}^n, \quad \text{and}$$
$$D_{\mathbb{C}} f(p)\big(\mathbb{C} \otimes T_p M\big) = \mathbb{C} \otimes T_{f(p)} f(M).$$

Then it is clear that $D_{\mathbb{C}} f(p)$ must take $T_p^{(1,0)}M$ to $T_{f(p)}^{(1,0)} f(M)$ and $T_p^{(0,1)}M$ to $T_{f(p)}^{(0,1)} f(M)$. □

In the next proposition it is important to note that a translation and applying a unitary matrix are biholomorphic changes of coordinates.

Proposition 2.3.3. *Let $M \subset \mathbb{C}^n$ be a smooth real hypersurface, $p \in M$. After a translation and rotation by a unitary matrix, $p = 0$ and near the origin, M is written in variables $(z, w) \in \mathbb{C}^{n-1} \times \mathbb{C}$ as*

$$\operatorname{Im} w = \varphi(z, \bar{z}, \operatorname{Re} w),$$

with the $\varphi(0)$ and $d\varphi(0) = 0$. Consequently

$$T_0^{(1,0)}M = \operatorname{span}_{\mathbb{C}} \left\{ \frac{\partial}{\partial z_1}\Big|_0, \ldots, \frac{\partial}{\partial z_{n-1}}\Big|_0 \right\}, \qquad T_0^{(0,1)}M = \operatorname{span}_{\mathbb{C}} \left\{ \frac{\partial}{\partial \bar{z}_1}\Big|_0, \ldots, \frac{\partial}{\partial \bar{z}_{n-1}}\Big|_0 \right\},$$
$$B_0 = \operatorname{span}_{\mathbb{C}} \left\{ \frac{\partial}{\partial(\operatorname{Re} w)}\Big|_0 \right\}.$$

In particular, $\dim_{\mathbb{C}} T_p^{(1,0)}M = \dim_{\mathbb{C}} T_p^{(0,1)}M = n - 1$ *and* $\dim_{\mathbb{C}} B_p = 1$.

If M is the boundary of a open set U with smooth boundary, the rotation can be chosen so that $\operatorname{Im} w > \varphi(z, \bar{z}, \operatorname{Re} w)$ *on U.*

Proof. We apply a translation to put $p = 0$ and in the same manner as in Lemma 2.2.8 apply a unitary matrix to make sure that ∇r is in the direction $-\frac{\partial}{\partial(\operatorname{Im} w)}\big|_0$. That $\varphi(0) = 0$ and $d\varphi(0) = 0$ follows as before. As a translation and a unitary matrix are holomorphic and in fact biholomorphic, then via Proposition 2.3.1 we obtain that the tangent spaces are all transformed correctly.

The rest of the proposition follows at once as $\frac{\partial}{\partial(\operatorname{Im} w)}\big|_0$ is the normal vector to M at 0. □

Remark 2.3.4. When M is of smaller dimension than $2n - 1$ (no longer a hypersurface, but a submanifold of higher codimension), then the proposition above does not hold. That is, we would still have $\dim_{\mathbb{C}} T_p^{(1,0)}M = \dim_{\mathbb{C}} T_p^{(0,1)}M$, but this number need not be constant from point to point. Fortunately, when talking about domains with smooth boundaries, the boundaries are hypersurfaces, and this complication does not arise.

Definition 2.3.5. Suppose $U \subset \mathbb{C}^n$ is an open set with smooth boundary, and r is a defining function for ∂U at $p \in \partial U$ such that $r < 0$ on U. If

$$\sum_{j=1,\ell=1}^{n} \bar{a}_j a_\ell \frac{\partial^2 r}{\partial \bar{z}_j \partial z_\ell}\Big|_p \geq 0 \qquad \text{for all} \qquad X_p = \sum_{j=1}^{n} a_j \frac{\partial}{\partial z_j}\Big|_p \quad \in \quad T_p^{(1,0)}\partial U,$$

then U is said to be *pseudoconvex* at p (or *Levi pseudoconvex*). If the inequality above is strict for all nonzero $X_p \in T_p^{(1,0)} \partial U$, then U is said to be *strongly pseudoconvex*. If U is pseudoconvex, but not strongly pseudoconvex at p, then we say that U is *weakly pseudoconvex*.

A domain U is *pseudoconvex* if it is pseudoconvex at all $p \in \partial U$. For a bounded* U, we say U is *strongly pseudoconvex* if it is strongly pseudoconvex at all $p \in \partial U$.

For $X_p \in T_p^{(1,0)} \partial U$, the sesquilinear form

$$\mathcal{L}(X_p, X_p) = \sum_{j=1,\ell=1}^{n} \bar{a}_j a_\ell \frac{\partial^2 r}{\partial \bar{z}_j \partial z_\ell}\Big|_p$$

is called the *Levi form* at p. So U is pseudoconvex (resp. strongly pseudoconvex) at $p \in \partial U$ if the Levi form is positive semidefinite (resp. positive definite) at p. The Levi form can be defined for any real hypersurface M, although one has to decide which side of M is "the inside."

The matrix

$$\left[\frac{\partial^2 r}{\partial \bar{z}_j \partial z_\ell}\Big|_p \right]_{j\ell}$$

is called the *complex Hessian* of r at p. So, U is pseudoconvex at $p \in \partial U$ if the complex Hessian of r at p as a sesquilinear form is positive (semi)definite when restricted to tangent vectors in $T_p^{(1,0)} \partial U$. For example, the unit ball $\mathbb{B}_n$ is strongly pseudoconvex as can be seen by computing the Levi form directly from $r(z, \bar{z}) = \|z\|^2 - 1$, that is, the complex Hessian of r is the identity matrix.

We remark that the complex Hessian is not the full Hessian. Let us write down the full Hessian, using the basis of $\frac{\partial}{\partial z}$s and $\frac{\partial}{\partial \bar{z}}$s. It is the symmetric matrix

$$\begin{bmatrix}
\frac{\partial^2 r}{\partial z_1 \partial z_1} & \cdots & \frac{\partial^2 r}{\partial z_1 \partial z_n} & \frac{\partial^2 r}{\partial z_1 \partial \bar{z}_1} & \cdots & \frac{\partial^2 r}{\partial z_1 \partial \bar{z}_n} \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
\frac{\partial^2 r}{\partial z_n \partial z_1} & \cdots & \frac{\partial^2 r}{\partial z_n \partial z_n} & \frac{\partial^2 r}{\partial z_n \partial \bar{z}_1} & \cdots & \frac{\partial^2 r}{\partial z_n \partial \bar{z}_n} \\
\frac{\partial^2 r}{\partial \bar{z}_1 \partial z_1} & \cdots & \frac{\partial^2 r}{\partial \bar{z}_1 \partial z_n} & \frac{\partial^2 r}{\partial \bar{z}_1 \partial \bar{z}_1} & \cdots & \frac{\partial^2 r}{\partial \bar{z}_1 \partial \bar{z}_n} \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
\frac{\partial^2 r}{\partial \bar{z}_n \partial z_1} & \cdots & \frac{\partial^2 r}{\partial \bar{z}_n \partial z_n} & \frac{\partial^2 r}{\partial \bar{z}_n \partial \bar{z}_1} & \cdots & \frac{\partial^2 r}{\partial \bar{z}_n \partial \bar{z}_n}
\end{bmatrix}.$$

The complex Hessian is the lower left, or the transpose of the upper right, block. In particular, it is a smaller matrix. And we also apply it only to a subspace of the complexified tangent space.

Let us illustrate the change of basis on one dimension. The change of variables is left to student for higher dimensions. Let $z = x + iy$ be in $\mathbb{C}$ and we denote by T the change of basis matrix:

$$T = \begin{bmatrix} 1/2 & 1/2 \\ -i/2 & i/2 \end{bmatrix}, \qquad T^t \begin{bmatrix} \frac{\partial^2 r}{\partial x \partial x} & \frac{\partial^2 r}{\partial x \partial y} \\ \frac{\partial^2 r}{\partial y \partial x} & \frac{\partial^2 r}{\partial y \partial y} \end{bmatrix} T = \begin{bmatrix} \frac{\partial^2 r}{\partial z \partial z} & \frac{\partial^2 r}{\partial z \partial \bar{z}} \\ \frac{\partial^2 r}{\partial \bar{z} \partial z} & \frac{\partial^2 r}{\partial \bar{z} \partial \bar{z}} \end{bmatrix}.$$

Let us also mention how a complex linear change of variables acts on the Hessian matrix. A complex linear change of variables is not an arbitrary $2n \times 2n$ matrix. If the Hessian is in the basis

*The definition for unbounded domains is not consistent in the literature. Sometimes *strictly pseudoconvex* is used.

of $\frac{\partial}{\partial z}$s and $\frac{\partial}{\partial \bar{z}}$s, an $n \times n$ complex linear matrix A acts on the Hessian as $A \oplus \overline{A}$, that is $\left[\begin{smallmatrix} A & 0 \\ 0 & \overline{A} \end{smallmatrix}\right]$. Write the real Hessian as $\left[\begin{smallmatrix} X & L^t \\ L & \overline{X} \end{smallmatrix}\right]$, where L is the complex Hessian. Then the complex linear change of variables A transforms the Hessian as

$$\begin{bmatrix} A & 0 \\ 0 & \overline{A} \end{bmatrix}^t \begin{bmatrix} X & L^t \\ L & \overline{X} \end{bmatrix} \begin{bmatrix} A & 0 \\ 0 & \overline{A} \end{bmatrix} = \begin{bmatrix} A^t X A & (A^* L A)^t \\ A^* L A & \overline{A^t X A} \end{bmatrix},$$

where $A^* = \overline{A}^t$ is the conjugate transpose of A. So A transforms the complex Hessian L as A^*LA, that is, by $*$-congruence. Star congruence preserves the *inertia* (the number of positive, negative, and zero eigenvalues) of a Hermitian matrix by the Sylvester's law of inertia from linear algebra.

The Levi form itself does depend on the defining function, but the signs of the eigenvalues do not. It is common to say "the Levi form" without mentioning a specific defining function even though that is not completely correct. The proof of the following proposition is left as an exercise.

Proposition 2.3.6. *Let $U \subset \mathbb{C}^n$ be an open set with smooth boundary and $p \in \partial U$. The inertia of the Levi form at p does not depend on the defining function at p. In particular, the definition of pseudoconvexity and strong pseudoconvexity is independent of the defining function.*

***Exercise* 2.3.2:** *If r is real-valued, then the complex Hessian of r is Hermitian, that is, the matrix is equal to its conjugate transpose.*

***Exercise* 2.3.3:** *Prove Proposition 2.3.6.*

***Exercise* 2.3.4:** *Show that a convex domain with smooth boundary is pseudoconvex, and show that (a bounded) strongly convex domain with smooth boundary is strongly pseudoconvex.*

***Exercise* 2.3.5:** *Show that if an open set with smooth boundary is strongly pseudoconvex at a point, it is strongly pseudoconvex at all nearby points.*

We are generally interested what happens under a holomorphic change of coordinates, that is, a biholomorphic mapping. And as far as pseudoconvexity is concerned we are interested in local changes of coordinates as pseudoconvexity is a local property. Before proving that pseudoconvexity is a biholomorphic invariant, let us note where the Levi form appears in the graph coordinates from Proposition 2.3.3, that is, if our boundary (the hypersurface) is given by

$$\operatorname{Im} w = \varphi(z, \bar{z}, \operatorname{Re} w).$$

Let $r(z, \bar{z}, w, \bar{w}) = \varphi(z, \bar{z}, \operatorname{Re} w) - \operatorname{Im} w$ be our defining function. Then the complex Hessian of r is of the form

$$\begin{bmatrix} L & 0 \\ 0 & 0 \end{bmatrix} \qquad \text{where} \quad L = \left[\frac{\partial^2 \varphi}{\partial \bar{z}_j \partial z_\ell}\Big|_0 \right]_{j\ell}.$$

Note that L is an $(n-1)\times(n-1)$ matrix. The vectors in $T_0^{(1,0)}\partial U$ are in the span of $\left\{ \frac{\partial}{\partial z_1}\big|_0, \ldots, \frac{\partial}{\partial z_{n-1}}\big|_0 \right\}$. That is, as an n-vector, a vector in $T_0^{(1,0)}\partial U$ is represented by $(a, 0) \in \mathbb{C}^n$ for an arbitrary $a \in \mathbb{C}^{n-1}$. The Levi form is then a^*La, in other words, it is given by the $(n-1) \times (n-1)$ matrix L. If this matrix L is positive semidefinite, then ∂U is pseudoconvex at 0.

Example 2.3.7: Let us change variables to show how we write the ball $\mathbb{B}_n$ in different local holomorphic coordinates where the Levi form is displayed nicely. Suppose the sphere $\partial \mathbb{B}_n$ is defined in the variables $Z = (Z_1, \ldots, Z_n) \in \mathbb{C}^n$ by $\lVert Z \rVert = 1$.

Let us change variables to $(z_1, \ldots, z_{n-1}, w)$ where

$$z_j = \frac{Z_1}{1 - Z_n} \qquad \text{for all } j = 1, \ldots, n-1, \qquad w = i \frac{1 + Z_n}{1 - Z_n}.$$

This change of variables is a biholomorphic mapping from the set where $Z_n \neq 1$ to the set where $w \neq -i$ (exercise). For us it is sufficient to notice that the map is invertible near $(0, \ldots, 0, -1)$, which follows by simply computing the derivative. Notice that the last component is the inverse of the Cayley transform (that takes the disc to the upper half plane).

We claim that the mapping takes the unit sphere given by $\lVert Z \rVert = 1$ (without the point $(0, \ldots, 0, 1)$), to the set defined by

$$\operatorname{Im} w = |z_1|^2 + \cdots + |z_{n-1}|^2,$$

and that it takes $(0, \ldots, 0, -1)$ to the origin (this part is trivial). Let us check:

$$\begin{aligned}
|z_1|^2 + \cdots + |z_{n-1}|^2 - \operatorname{Im} w &= \left| \frac{Z_1}{1 - Z_n} \right|^2 + \cdots + \left| \frac{Z_{n-1}}{1 - Z_n} \right|^2 - \frac{i \frac{1+Z_n}{1-Z_n} - \overline{i \frac{1+Z_n}{1-Z_n}}}{2i} \\
&= \frac{|Z_1|^2}{|1 - Z_n|^2} + \cdots + \frac{|Z_{n-1}|^2}{|1 - Z_n|^2} - \frac{1 + Z_n}{2(1 - Z_n)} - \frac{1 + \bar{Z}_n}{2(1 - \bar{Z}_n)} \\
&= \frac{|Z_1|^2 + \cdots + |Z_{n-1}|^2 + |Z_n|^2 - 1}{|1 - Z_n|^2}.
\end{aligned}$$

Therefore $|Z_1|^2 + \cdots + |Z_n|^2 = 1$ if and only if $\operatorname{Im} w = |z_1|^2 + \cdots + |z_{n-1}|^2$. As the map takes the point $(0, \ldots, 0, -1)$ to the origin, we can think of the set given by

$$\operatorname{Im} w = |z_1|^2 + \cdots + |z_{n-1}|^2$$

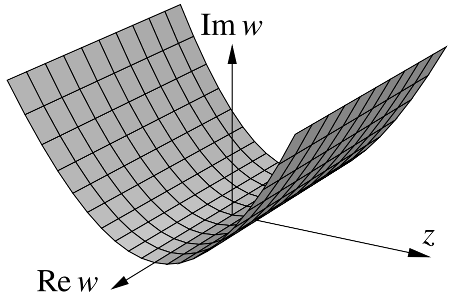
as the sphere in local holomorphic coordinates at $(0, \ldots, 0, -1)$ (by symmetry of the sphere we could have done this at any point by rotation). In the coordinates (z, w), the ball (the inside of the sphere) is the set given by

$$\operatorname{Im} w > |z_1|^2 + \cdots + |z_{n-1}|^2.$$

In these new coordinates, the Levi form is just the identity matrix at the origin. In particular the domain is strongly pseudoconvex. We have not yet proved that pseudoconvexity is a biholomorphic invariant, but when we do, it will also mean that the ball is strongly pseudoconvex.

Of course not the entire sphere gets transformed, the points where $Z_n = 1$ get "sent to infinity." The hypersurface $\operatorname{Im} w = |z_1|^2 + \cdots + |z_{n-1}|^2$ is sometimes called the *Lewy hypersurface*, and in the literature some even say it *is* the sphere[*]. Pretending z is just one real direction, it looks like this:

[*] That is not, in fact, completely incorrect. If you think of the sphere in the complex projective space, we are simply looking at the sphere in a different coordinate patch.



As an aside, the hypersurface $\operatorname{Im} w = |z_1|^2 + \cdots + |z_{n-1}|^2$ is also called the *Heisenberg group*. The group in this case is the group defined on the parameters $(z, \operatorname{Re} w)$ of this hypersurface with the group law defined by $(z, \operatorname{Re} w)(z', \operatorname{Re} w') = (z + z', \operatorname{Re} w + \operatorname{Re} w' + 2 \operatorname{Im} z \cdot z')$.

***Exercise* 2.3.6:** *Prove the assertion in the example about the mapping being biholomorphic on the sets described above.*

Let us see how the Hessian of r changes under a biholomorphic change of coordinates. That is, let $f : V \to V'$ be a biholomorphic map between two domains in $\mathbb{C}^n$, and let $r : V' \to \mathbb{R}$ be a smooth function with nonvanishing derivative. Let us compute the Hessian of $r \circ f : V \to \mathbb{R}$. We first compute what happens to the non-mixed derivatives. As we have to apply chain rule twice, to keep track better track of things, we write the derivatives as functions. Also for clarity, let z be the coordinates in V and ζ be the coordinates in V'. That is, r is a function of ζ and $\bar{\zeta}$, f is a function of z, and $\bar{f}$ is a function of $\bar{z}$. Therefore $r \circ f$ is a function of z and $\bar{z}$.

$$
\begin{aligned}
\frac{\partial^2 (r \circ f)}{\partial z_j \partial z_k}(z, \bar{z}) &= \frac{\partial}{\partial z_j} \sum_{\ell=1}^{n} \left(\frac{\partial r}{\partial \zeta_\ell}\big(f(z), \bar{f}(\bar{z})\big) \frac{\partial f_\ell}{\partial z_k}(z) + \frac{\partial r}{\partial \bar{\zeta}_\ell}\big(f(z), \bar{f}(\bar{z})\big) \cancelto{0}{\frac{\partial \bar{f}_\ell}{\partial z_k}(\bar{z})} \right) \\
&= \sum_{\ell, m=1}^{n} \left(\frac{\partial^2 r}{\partial \zeta_m \partial \zeta_\ell}\big(f(z), \bar{f}(\bar{z})\big) \frac{\partial f_m}{\partial z_j}(z) \frac{\partial f_\ell}{\partial z_k}(z) + \frac{\partial^2 r}{\partial \bar{\zeta}_m \partial \zeta_\ell}\big(f(z), \bar{f}(\bar{z})\big) \cancelto{0}{\frac{\partial \bar{f}_m}{\partial z_j}(\bar{z})} \frac{\partial f_\ell}{\partial z_k}(z) \right) \\
&\qquad + \sum_{\ell=1}^{n} \frac{\partial r}{\partial \zeta_\ell}\big(f(z), \bar{f}(\bar{z})\big) \frac{\partial^2 f_\ell}{\partial z_j \partial z_k}(z) \\
&= \sum_{\ell, m=1}^{n} \frac{\partial^2 r}{\partial \zeta_m \partial \zeta_\ell} \frac{\partial f_m}{\partial z_j} \frac{\partial f_\ell}{\partial z_k} + \sum_{\ell=1}^{n} \frac{\partial r}{\partial \zeta_\ell} \frac{\partial^2 f_\ell}{\partial z_j \partial z_k}.
\end{aligned}
$$

The matrix $\left[\frac{\partial^2 (r \circ f)}{\partial z_j \partial z_k} \right]$ can have different eigenvalues than the matrix $\left[\frac{\partial^2 r}{\partial \zeta_j \partial \zeta_k} \right]$. If r has nonvanishing gradient, then using the second term, we can (locally) choose f in such a way as to make the matrix $\left[\frac{\partial^2 (r \circ f)}{\partial z_j \partial z_k} \right]$ be the zero matrix (or anything else) at a certain point by choosing the second derivatives of f arbitrarily at that point. See the exercise below. Nothing about the matrix $\left[\frac{\partial^2 r}{\partial \zeta_j \partial \zeta_k} \right]$ is preserved under a biholomorphic map. And that is precisely why it does not appear in the definition of pseudoconvexity. The story for $\left[\frac{\partial^2 (r \circ f)}{\partial \bar{z}_j \partial \bar{z}_k} \right]$ and $\left[\frac{\partial^2 r}{\partial \bar{\zeta}_j \partial \bar{\zeta}_k} \right]$ is exactly the same.

***Exercise* 2.3.7:** *Given a real function r with nonvanishing gradient at $p \in \mathbb{C}^n$. Find a local change of coordinates f at p (so f ought to be a holomorphic mapping with an invertible derivative at p) such that $\left[\frac{\partial^2 (r \circ f)}{\partial z_j \partial z_k}\Big|_p \right]$ and $\left[\frac{\partial^2 (r \circ f)}{\partial \bar{z}_j \partial \bar{z}_k}\Big|_p \right]$ are just the zero matrices.*

Let us look at the mixed derivatives:

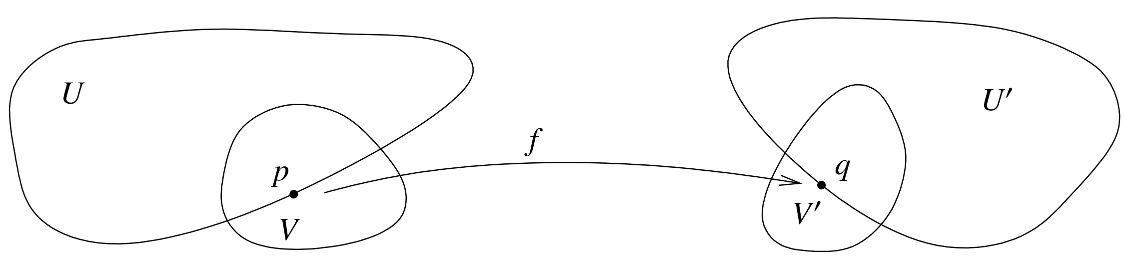
$$\begin{aligned}
\frac{\partial^2 (r \circ f)}{\partial \bar{z}_j \partial z_k}(z, \bar{z}) &= \frac{\partial}{\partial \bar{z}_j} \sum_{\ell=1}^{n} \left(\frac{\partial r}{\partial \zeta_\ell}\big(f(z), \bar{f}(\bar{z})\big) \frac{\partial f_\ell}{\partial z_k}(z) \right) \\
&= \sum_{\ell,m=1}^{n} \frac{\partial^2 r}{\partial \bar{\zeta}_m \partial \zeta_\ell}\big(f(z), \bar{f}(\bar{z})\big) \frac{\partial \bar{f}_m}{\partial \bar{z}_j}(\bar{z}) \frac{\partial f_\ell}{\partial z_k}(z) + \sum_{\ell=1}^{n} \frac{\partial r}{\partial \zeta_\ell}\big(f(z), \bar{f}(\bar{z})\big) \cancelto{0}{\frac{\partial^2 f_\ell}{\partial \bar{z}_j \partial z_k}}(z) \\
&= \sum_{\ell,m=1}^{n} \frac{\partial^2 r}{\partial \bar{\zeta}_m \partial \zeta_\ell} \frac{\partial \bar{f}_m}{\partial \bar{z}_j} \frac{\partial f_\ell}{\partial z_k}.
\end{aligned}$$

The complex Hessian of $r \circ f$ is the complex Hessian H of r conjugated as $D^* H D$, where D is the holomorphic derivative matrix of f at z and D^* is the conjugate transpose. Sylvester's law of inertia says that the number of positive, negative, and zero eigenvalues of $D^* H D$ is the same as that for H. The eigenvalues might have changed, but their sign did not. We are only considering H and $D^* H D$ on a subspace. In linear algebra language, suppose D is invertible and consider a subspace T and its image DT. Then the inertia of H restricted to DT is the same as the inertia of $D^* H D$ restricted to T.

Let M be a smooth hypersurface given by $r = 0$, then $f^{-1}(M)$ is a smooth hypersurface given by $r \circ f = 0$. The holomorphic derivative $D = Df(p)$ takes $T_p^{(1,0)} f^{-1}(M)$ isomorphically to $T_{f(p)}^{(1,0)} M$. So H is positive (semi)definite on $T_{f(p)}^{(1,0)} M$ if and only if $D^* H D$ is positive (semi)definite on $T_p^{(1,0)} f^{-1}(M)$. We have almost proved the following theorem. In short, pseudoconvexity is a biholomorphic invariant.

Theorem 2.3.8. *Suppose $U, U' \subset \mathbb{C}^n$ are open sets with smooth boundary, $p \in \partial U$, $V \subset \mathbb{C}^n$ a neighborhood of p, $q \in \partial U'$, $V' \subset \mathbb{C}^n$ a neighborhood of q, and $f : V \to V'$ a biholomorphic map with $f(p) = q$, such that $f(U \cap V) = U' \cap V'$.*

Then the inertia of the Levi form of U at p is the same as the inertia of the Levi form of U' at q. In particular, U is pseudoconvex at p if and only if U' is pseudoconvex at q. Similarly U is strongly pseudoconvex at p if and only if U' is strongly pseudoconvex at q.



To finish proving the theorem, the only thing left is to observe that if $f(U \cap V) = U' \cap V'$, then $f(\partial U \cap V) = \partial U' \cap V'$, and to note that if r is a defining function for U' at q, then $f \circ r$ is a defining function for U at p.

***Exercise* 2.3.8:** *Find an example of a bounded domain with smooth boundary that is not convex, but that is pseudoconvex.*

While the Levi form is not invariant under holomorphic changes of coordinates, its inertia is. Putting this together with the other observations we made above, we will find the normal form for the quadratic part of the defining equation for a smooth real hypersurface under biholomorphic transformations. It is possible to do better than the following lemma, but it is not possible to always get rid of the dependence on $\operatorname{Re} w$ in the higher order terms.

Recall that $O(k)$ at the origin means a function that together with its derivatives up to order $k-1$ vanish at the origin.

Lemma 2.3.9. *Let M be a smooth real hypersurface in $\mathbb{C}^n$ and $p \in M$. Then there exists a local biholomorphic change of coordinates taking p to the origin and M to the hypersurface given by*

$$\operatorname{Im} w = \sum_{j=1}^{\alpha} |z_j|^2 - \sum_{j=\alpha+1}^{\alpha+\beta} |z_j|^2 + E(z,\bar{z},\operatorname{Re} w),$$

where E is $O(3)$ at the origin. Here α is the number of positive eigenvalues of the Levi form at p, β is the number of negative eigenvalues, and $\alpha+\beta \leq n-1$.

Proof. Change coordinates so that M is given by $\operatorname{Im} w = \varphi(z,\bar{z},\operatorname{Re} w)$, where φ is $O(2)$. Apply Taylor's theorem to φ up to the second order:

$$\varphi(z,\bar{z},\operatorname{Re} w) = q(z,\bar{z}) + (\operatorname{Re} w)(Lz + \overline{Lz}) + a(\operatorname{Re} w)^2 + O(3),$$

where q is quadratic, $L \colon \mathbb{C}^{n-1} \to \mathbb{C}$ is linear, and $a \in \mathbb{R}$. If $L \neq 0$, do a linear change of coordinates in the z only to make $Lz = z_1$. So assume $Lz = \epsilon z_1$ where $\epsilon = 0$ or $\epsilon = 1$.

Change coordinates by leaving z unchanged and letting $w = w' + bw'^2 + cw'z_1$. Ignore $q(z,\bar{z})$ for a moment as this change of coordinates does not affect it. Also we only look up to second order.

$$\begin{aligned}
-\operatorname{Im} w + \epsilon(\operatorname{Re} w)(z_1+\bar{z}_1) + a(\operatorname{Re} w)^2 &= -\frac{w-\bar{w}}{2i} + \epsilon\frac{w+\bar{w}}{2}(z_1+\bar{z}_1) + a\left(\frac{w+\bar{w}}{2}\right)^2 \\
&= -\frac{w' + bw'^2 + cw'z_1 - \bar{w}' - \bar{b}\bar{w}'^2 - \bar{c}\bar{w}'\bar{z}_1}{2i} \\
&\quad + \epsilon\frac{w' + bw'^2 + cw'z_1 + \bar{w}' + \bar{b}\bar{w}'^2 + \bar{c}\bar{w}'\bar{z}_1}{2}(z_1+\bar{z}_1) \\
&\quad + a\frac{(w' + bw'^2 + cw'z_1 + \bar{w}' + \bar{b}\bar{w}'^2 + \bar{c}\bar{w}'\bar{z}_1)^2}{4} \\
&= -\frac{w'-\bar{w}'}{2i} \\
&\quad + \frac{\big((\epsilon i - c)w' + \epsilon i\bar{w}'\big)z_1 + \big((\epsilon i + \bar{c})\bar{w}' + \epsilon i w'\big)\bar{z}_1}{2i} \\
&\quad + \frac{(ia-2b)w'^2 + (ia+2\bar{b})\bar{w}'^2 + 2iaw'\bar{w}'}{4i} + O(3).
\end{aligned}$$

We cannot quite get rid of all the quadratic terms in φ, but we choose b and c to make the second order terms not depend on $\operatorname{Re} w'$. Set $b = ia$ and $c = 2i\epsilon$, and add $q(z, \bar{z}) + O(3)$ into the mix to get

$$\begin{aligned}
-\operatorname{Im} w + \varphi(z, \bar{z}, \operatorname{Re} w) &= -\operatorname{Im} w + q(z, \bar{z}) + \epsilon(\operatorname{Re} w)(z_1 + \bar{z}_1) + a(\operatorname{Re} w)^2 + O(3) \\
&= -\frac{w' - \bar{w}'}{2i} + q(z, \bar{z}) - \epsilon i \frac{w' - \bar{w}'}{2i}(z_1 - \bar{z}_1) + a\left(\frac{w' - \bar{w}'}{2i}\right)^2 + O(3) \\
&= -\operatorname{Im} w' + q(z, \bar{z}) - \epsilon i(\operatorname{Im} w')(z_1 - \bar{z}_1) + a(\operatorname{Im} w')^2 + O(3).
\end{aligned}$$

The right hand side is the defining equation in the (z, w') coordinates. However, it is no longer written as a graph of $\operatorname{Im} w'$ over the rest, so we apply the implicit function theorem to solve for $\operatorname{Im} w'$ and write the hypersurface as a graph again. The expression for $\operatorname{Im} w'$ is $O(2)$, and therefore $-i\epsilon(\operatorname{Im} w')(z_1 - \bar{z}_1) + a(\operatorname{Im} w')^2$ is $O(3)$. So if we write M as a graph,

$$\operatorname{Im} w' = q(z, \bar{z}) + E(z, \bar{z}, \operatorname{Re} w'),$$

then E is $O(3)$.

The quadratic polynomial q can be written as

$$q(z, \bar{z}) = \sum_{j,k=1}^{n-1} a_{jk} z_j z_k + b_{jk} \bar{z}_j \bar{z}_k + c_{jk} \bar{z}_j z_k. \tag{2.1}$$

As q is real-valued, it is left as an exercise to show that $a_{jk} = \overline{b_{jk}}$ and $c_{jk} = \overline{c_{kj}}$. That is, the matrix $[b_{jk}]$ is the complex conjugate of $[a_{jk}]$ and $[c_{jk}]$ is Hermitian.

We make another change of coordinates; we fix the zs again and we set

$$w' = w'' + i \sum_{j,k=1}^{n-1} a_{jk} z_j z_k. \tag{2.2}$$

In particular,

$$\operatorname{Im} w' = \operatorname{Im} w'' + \operatorname{Im}\left(i \sum_{j,k=1}^{n-1} a_{jk} z_j z_k\right) = \operatorname{Im} w'' + \sum_{j,k=1}^{n-1} \left(a_{jk} z_j z_k + b_{jk} \bar{z}_j \bar{z}_k\right),$$

as $\overline{a_{jk}} = b_{jk}$. Plugging (2.2) into $\operatorname{Im} w' = q(z, \bar{z}) + E(z, \bar{z}, \operatorname{Re} w')$ and solving for $\operatorname{Im} w''$ cancels the holomorphic and antiholomorphic terms in q, and leaves E as $O(3)$. After this change of coordinates we may assume

$$q(z, \bar{z}) = \sum_{j,k=1}^{n-1} c_{jk} z_j \bar{z}_k.$$

That is, q is a sesquilinear form. Since q is real-valued the matrix $C = [c_{jk}]$ is Hermitian. In linear algebra notation, $q(z, \bar{z}) = z^* C z$, where we think of z as a column vector. If T is a linear transformation on the z variables, say $z' = Tz$, we obtain $z'^* C z' = (Tz)^* C T z = z^*(T^* C T) z$. Thus, we normalize C up to $*$-congruence. A Hermitian matrix is $*$-congruent to a diagonal matrix with only 1s, -1s, and 0s on the diagonal, again by Sylvester's law of inertia. Writing out what that means is precisely the conclusion of the proposition. $\square$

***Exercise* 2.3.9:** *Prove the assertion in the proof, that is, if q is a quadratic as in (2.1) that is real-valued, then $a_{jk} = \overline{b_{jk}}$ and $c_{jk} = \overline{c_{kj}}$.*

Lemma 2.3.10 (Narasimhan's lemma*)**.** *Let $U \subset \mathbb{C}^n$ be a an open set with smooth boundary that is strongly pseudoconvex at $p \in \partial U$. Then there exists a local biholomorphic change of coordinates fixing p such that in these new coordinates, U is strongly convex at p and hence strongly convex at all points near p.*

***Exercise* 2.3.10:** *Prove the above lemma. Hint: See the proof of Lemma 2.3.9.*

***Exercise* 2.3.11:** *Prove that an open $U \subset \mathbb{C}^n$ with smooth boundary is pseudoconvex at p if and only if there exist local holomorphic coordinates at p such that U is convex at p.*

Narasimhan's lemma only works at points of strong pseudoconvexity. For weakly pseudoconvex points the situation is far more complicated. The difficulty is that we cannot make a U that weakly pseudoconvex at all points near p to be convex at all points near p.

Let us prove the easy direction of the famous *Levi problem*. The Levi problem was a long-standing problem† in several complex variables to classify domains of holomorphy in $\mathbb{C}^n$. The answer is that a domain is a domain of holomorphy if and only if it is pseudoconvex. Just as the problem of trying to show that the classical geometric convexity is the same as convexity as we have defined it, the Levi problem has an easier direction and a harder direction. The easier direction is to show that a domain of holomorphy is pseudoconvex, and the harder direction is to show that a pseudoconvex domain is a domain of holomorphy. See Hörmander's book [H] for the proof of the hard direction.

Theorem 2.3.11 (Tomato can principle)**.** *Suppose $U \subset \mathbb{C}^n$ is an open set with smooth boundary and at some point $p \in \partial U$ the Levi form has a negative eigenvalue. Then every holomorphic function on U extends to a neighborhood of p. In particular, U is not a domain of holomorphy.*

Pseudoconvex at p means that all eigenvalues of the Levi form are nonnegative. The theorem says that a domain of holomorphy must be pseudoconvex. The theorem's name comes from the proof, and sometimes other theorems using a similar proof of a "tomato can" of analytic discs are called tomato can principles. The general statement of proof of the principle is that "an analytic function holomorphic in a neighborhood of the sides and the bottom of a tomato can extends to the inside." And the theorem we named after the principle states that "if the Levi form at p has a negative eigenvalue, we can fit a tomato can from inside the domain over p."

Proof. We change variables so that $p = 0$, and near p, U is given by

$$\operatorname{Im} w > -|z_1|^2 + \sum_{j=2}^{n-1} \epsilon_j |z_j|^2 + E(z_1, z', \bar{z}_1, \bar{z}', \operatorname{Re} w),$$

*A statement essentially of Narasimhan's lemma was already used by Helmut Knesser in 1936.

†E. E. Levi stated the problem in 1911, but it was not completely solved until the 1950s, by Oka and others.

where $z' = (z_2, \ldots, z_{n-1})$, $\epsilon_j = -1, 0, 1$, and E is $O(3)$. We embed an analytic disc via the map $\xi \overset{\varphi}{\mapsto} (\lambda\xi, 0, 0, \ldots, 0)$ for some small $\lambda > 0$. Clearly $\varphi(0) = 0 \in \partial U$. For $\xi \neq 0$ near the origin

$$-\lambda^2|\xi|^2 + \sum_{j=2}^{n-1} \epsilon_j |0|^2 + E(\lambda\xi, 0, \lambda\bar{\xi}, 0, 0) = -\lambda^2|\xi|^2 + E(\lambda\xi, 0, \lambda\bar{\xi}, 0, 0) < 0.$$

That is because by second derivative test the function above has a strict maximum at $\xi = 0$. Therefore for $\xi \neq 0$ near the origin, $\varphi(\xi) \in U$. By picking λ small enough, $\varphi(\overline{\mathbb{D}} \setminus \{0\}) \subset U$.

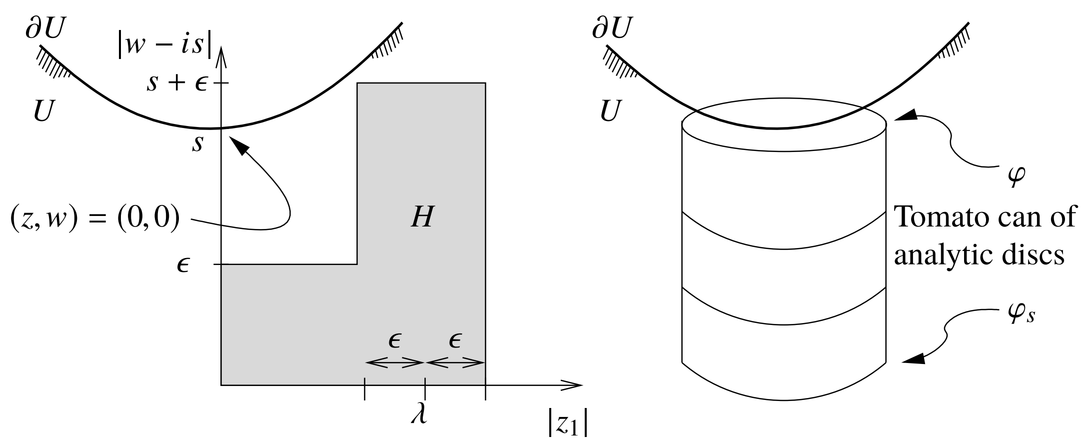
As $\varphi(\partial\mathbb{D})$ is compact we can "wiggle it a little" and find discs in U. In particular, for all small enough $s > 0$, the closed disc given by

$$\xi \overset{\varphi_s}{\mapsto} (\lambda\xi, 0, 0, \ldots, 0, is)$$

is entirely inside U (that is, for slightly positive $\operatorname{Im} w$). Fix such a small $s > 0$. Suppose $\epsilon > 0$ is small and $\epsilon < s$. Define the Hartogs figure

$$\begin{aligned} H = &\bigl\{(z, w) : \lambda - \epsilon < |z_1| < \lambda + \epsilon \text{ and } |z_j| < \epsilon \text{ for } j = 2, \ldots, n-1, \text{ and } |w - is| < s + \epsilon\bigr\} \\ &\cup \bigl\{(z, w) : |z_1| < \lambda + \epsilon, \text{ and } |z_j| < \epsilon \text{ for } j = 2, \ldots, n-1, \text{ and } |w - is| < \epsilon\bigr\}. \end{aligned}$$

For small enough $\epsilon > 0$, $H \subset U$. That is because the set where $|z_1| = \lambda$, $z' = 0$ and $|w| \leq s$, is inside U for all small enough s. So we can take an ϵ-neighborhood of that. Further for $w = is$ the whole disc where $|z_1| \leq \lambda$ is in U. So we can take an ϵ-neighborhood of that. We are really just taking a Hartogs figure in the z_1, w variables, and then "fattening it up" to the z' variables. We picture the Hartogs figure in the $|z_1|$ and $|w - is|$ variables. The boundary ∂U and U are only pictured diagrammatically. Also we make a "picture" the analytic discs giving the "tomato can." In the picture the U is below its boundary ∂U, unlike usually.



The origin is in the hull of H, and so every function holomorphic in U, and so in H, extends through the origin. Hence U is not a domain of holomorphy. ◻

Exercise* 2.3.12*: *For the following domains in $U \subset \mathbb{C}^2$, find all the points in ∂U where U is weakly pseudoconvex, all the points where it is strongly pseudoconvex, and all the points where it is not pseudoconvex. Is U pseudoconvex?*

a) $\operatorname{Im} w > |z|^4$
b) $\operatorname{Im} w > |z|^2(\operatorname{Re} w)$
c) $\operatorname{Im} w > (\operatorname{Re} z)(\operatorname{Re} w)$

Exercise* 2.3.13*: *Let $U \subset \mathbb{C}^n$ be an open set with smooth boundary that is strongly pseudoconvex at $p \in \partial U$. Show that there exists a neighborhood W of p and a smooth function $f : \overline{W \cap U} \to \mathbb{C}$ that is holomorphic on $W \cap U$ such that $f(p) = 1$ and $|f(z)| < 1$ for all $z \in \overline{W \cap U} \setminus \{p\}$.*

Exercise* 2.3.14*: *Suppose $U \subset \mathbb{C}^n$ is an open set with smooth boundary. Suppose for $p \in \partial U$, there is a neighborhood W of p and a holomorphic function $f : W \to \mathbb{C}$ such that $df(p) \neq 0$, $f(p) = 0$, but f is never zero on $W \cap U$. Show that U is pseudoconvex at p. Hint: You may need the holomorphic implicit function theorem, see Theorem 1.3.8. Note: The result does not require the derivative of f to not vanish, but is much harder to prove without that hypothesis.*

A flat hyperplane is the "degenerate" case of normal convexity. There is similarly a flat case of pseudoconvexity. A smooth hypersurface $M \subset \mathbb{C}^n$ is *Levi-flat* if the Levi form vanishes at every point of M. A zero matrix is both positive semidefinite and negative semidefinite, so both sides of M are pseudoconvex. Conversely the only hypersurface pseudoconvex from both sides is a Levi-flat.

Exercise* 2.3.15*: *Suppose $U = V \times \mathbb{C}^{n-1} \subset \mathbb{C}^n$, where $V \subset \mathbb{C}$ is an open set with smooth boundary. Show that U is has a smooth Levi-flat boundary.*

Exercise* 2.3.16*: *Prove that a real hyperplane is Levi-flat.*

Exercise* 2.3.17*: *Let $f \in \mathcal{O}(U)$ for some domain $U \subset \mathbb{C}^n$, $f \not\equiv 0$. Let $M = \bigl\{ z \in U : \operatorname{Im} f(z) = 0 \bigr\}$. Show that if $df(p) \neq 0$ for some $p \in M$, then near p, M is a Levi-flat hypersurface.*

Exercise* 2.3.18*: *Suppose $M \subset \mathbb{C}^n$ is a smooth Levi-flat hypersurface and $p \in M$. Suppose a complex line L is tangent to M at p. Prove that p is not an isolated point of $L \cap M$.*

Exercise* 2.3.19*: *Suppose $U \subset \mathbb{C}^n$ is an open set with smooth boundary and ∂U is Levi-flat. Show that U is unbounded. Hint: If U were bounded, consider the point on ∂U farthest from the origin.*

2.4 Harmonic, subharmonic, and plurisubharmonic functions

Definition 2.4.1. Let $U \subset \mathbb{R}^n$ be an open set. A C^2-smooth function $f : U \to \mathbb{R}$ is *harmonic* if*

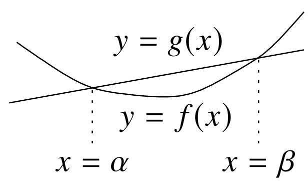
$$\nabla^2 f = \frac{\partial^2 f}{\partial x_1^2} + \cdots + \frac{\partial^2 f}{\partial x_n^2} = 0 \quad \text{on } U.$$

*Recall the operator ∇^2, sometimes also written Δ, is the *Laplacian*. It is the trace of the Hessian matrix.

A function $f: U \to \mathbb{R} \cup \{-\infty\}$ is *subharmonic* if it is upper-semicontinuous[*] and for every ball $B_r(a)$ with $\overline{B_r(a)} \subset U$, and every function g continuous on $\overline{B_r(a)}$ and harmonic on $B_r(a)$, such that $f(x) \leq g(x)$ for $x \in \partial B_r(a)$, we have

$$f(x) \leq g(x), \qquad \text{for all } x \in B_r(a).$$

In other words, a subharmonic function is a function that is less than any harmonic function on every ball. We remark that when $n = 1$ in the definition of a subharmonic function, it is the same as the standard definition of a convex function of one real variable, where affine linear functions play the role of harmonic functions: A function of one real variable is *convex* if for every interval it is less than the affine linear function with the same end points. After all, a function of one real variable is harmonic if the second derivative vanishes, and it is therefore affine linear. In one real dimension it is also easier to picture. The function f is convex if on every interval $[\alpha, \beta]$, $f \leq g$ for every affine linear g bigger than f at the endpoints α and β. In particular we can take the g that is equal to f at the endpoints. The picture is analogous for subharmonic functions for $n > 1$, but it is harder to draw.



We will consider harmonic and subharmonic functions in $\mathbb{C} \cong \mathbb{R}^2$. Let us go through some basic results on harmonic and subharmonic functions in $\mathbb{C}$ that you have seen in detail in your one-variable class. Consequently we leave some of these results as exercises. In this section (and not just here) we often write $f(z)$ for a function even if it is not holomorphic.

Exercise* 2.4.1*: *An upper-semicontinuous function achieves a maximum on compact sets.*

Exercise* 2.4.2*: *Let $U \subset \mathbb{C}$ be open. Show that for a C^2 function $f: U \to \mathbb{R}$,*

$$\frac{\partial^2}{\partial \bar{z} \partial z} f = \frac{1}{4} \nabla^2 f.$$

Use this to show that f is harmonic if and only if it is (locally) the real or imaginary part of a holomorphic function. Hint: The key is finding an antiderivative of a holomorphic function.

Exercise* 2.4.3*: *Prove the identity theorem. Let $U \subset \mathbb{C}$ be a domain and $f: U \to \mathbb{R}$ harmonic such that $f = 0$ on a nonempty open subset of U. Then $f \equiv 0$.*

It follows from the exercise that a harmonic function is infinitely differentiable. It is useful to find a harmonic function given boundary values. This problem is called the *Dirichlet problem*, and

[*]Recall f is *upper-semicontinuous* if $\limsup_{t \to x} f(t) \leq f(x)$ for all x.

it is solvable for many (though not all) domains. The proof of the following special case is contained in the exercises following the theorem. The *Poisson kernel* for the unit disc $\mathbb{D} \subset \mathbb{C}$ is

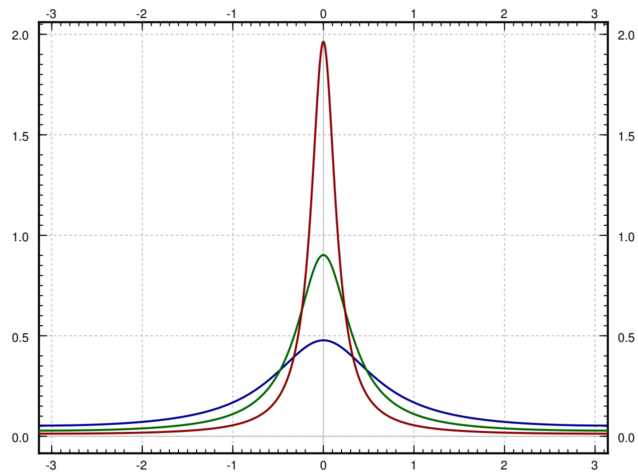
$$P_r(\theta) = \frac{1}{2\pi} \frac{1-r^2}{1+r^2-2r\cos\theta} = \frac{1}{2\pi} \operatorname{Re}\left(\frac{1+re^{i\theta}}{1-re^{i\theta}}\right), \qquad \text{for } 0 \leq r < 1.$$

Theorem 2.4.2. *Given any continuous function $u\colon \partial\mathbb{D} \to \mathbb{C}$, the function $Pu\colon \overline{\mathbb{D}} \to \mathbb{C}$, defined by*

$$Pu(re^{i\theta}) = \int_{-\pi}^{\pi} u(e^{i\varphi}) P_r(\theta-\varphi)\, d\varphi \quad \text{if } r < 1 \qquad \text{and} \qquad Pu(e^{i\theta}) = u(e^{i\theta}),$$

is harmonic in $\mathbb{D}$ and continuous on $\overline{\mathbb{D}}$.

In the proof it is useful to consider how does the graph of P_r as a function of θ look for a fixed r. The following are the graphs of P_r for $r = 0.5$, $r = 0.7$, and $r = 0.85$ on $[-\pi, \pi]$:



Exercise* 2.4.4*:
1. *a) Prove $P_r(\theta) > 0$ for all $0 \leq r < 1$ and all θ.*
2. *b) Prove $\int_{-\pi}^{\pi} P_r(\theta)\, d\theta = 1$.*
3. *c) Prove for any given $\delta > 0$, $\sup\{P_r(\theta) : \delta \leq |\theta| \leq \pi\} \to 0$ as $r \to 1$.*

Exercise* 2.4.5*: *Prove Theorem 2.4.2 using the following guideline:*
- *a) Poisson kernel is harmonic as a function of $z = re^{i\theta} \in \mathbb{D}$, and hence Pu is harmonic.*
- *b) P acts like an approximate identity: prove that $Pu(re^{i\theta}) \to u(e^{i\theta})$ uniformly as $r \to 1$ (Hint: Split the integral to $[-\delta, \delta]$ and the rest and use the previous exercise).*
- *c) Prove that $Pu(z)$ tends to $u(z_0)$ as $z \in \mathbb{D} \to z_0 \in \partial\mathbb{D}$.*

Exercise* 2.4.6*: *State and prove a version of Theorem 2.4.2 for an arbitrary disc $\Delta_r(a)$.*

Exercise* 2.4.7*: *Prove that the Dirichlet problem is not solvable in the punctured disc $\mathbb{D} \setminus \{0\}$.*

The Poisson kernel is also used as a reproducing kernel for holomorphic functions, as holomorphic functions are harmonic (their real and imaginary parts are). A Poisson kernel also exists for higher dimensions, and has analogous properties. The solution to the Dirichlet problem using the Poisson kernel leads to the following proposition.

Proposition 2.4.3 (Mean-value property and sub-mean-value property)**.** *Let $U \subset \mathbb{C}$ be an open set.*

(i) A continuous function $f: U \to \mathbb{R}$ is harmonic if and only if

$$f(a) = \frac{1}{2\pi} \int_0^{2\pi} f(a + re^{i\theta})\, d\theta \qquad \text{whenever} \quad \overline{\Delta_r(a)} \subset U.$$

(ii) An upper-semicontinuous function $f: U \to \mathbb{R} \cup \{-\infty\}$ is subharmonic if and only if

$$f(a) \leq \frac{1}{2\pi} \int_0^{2\pi} f(a + re^{i\theta})\, d\theta \qquad \text{whenever} \quad \overline{\Delta_r(a)} \subset U.$$

For the sub-mean-value property you may have to use the Lebesgue integral to integrate an upper-semicontinuous function, and to use the version of the Poisson integral above, you need to approximate by continuous functions on the boundary in the right way. On first reading, feel free to think of continuous subharmonic functions and not too much will be lost.

Exercise* 2.4.8*: *Fill in the details of the proof of Proposition 2.4.3.*

Exercise* 2.4.9*: *Let $U \subset \mathbb{C}$ be open. Show that if $f: U \to \mathbb{R} \cup \{-\infty\}$ is subharmonic then*

$$\limsup_{w \to z} f(w) = f(z) \qquad \text{for all } z \in U.$$

Exercise* 2.4.10*: *Suppose $U \subset \mathbb{C}$ is open and $g: U \to \mathbb{R}$ is harmonic. Then $f: U \to \mathbb{R} \cup \{-\infty\}$ is subharmonic if and only if $f - g$ is subharmonic.*

Proposition 2.4.4 (Maximum principle)**.** *Suppose $U \subset \mathbb{C}$ is a domain and $f: U \to \mathbb{R} \cup \{-\infty\}$ is subharmonic. If f attains a maximum in U, then f is constant.*

Proof. Suppose f attains a maximum at $a \in U$. If $\overline{\Delta_r(a)} \subset U$ then

$$f(a) \leq \frac{1}{2\pi} \int_0^{2\pi} f(a + re^{i\theta})\, d\theta \leq f(a).$$

Hence, $f = f(a)$ almost everywhere on $\partial\Delta_r(a)$. By upper-semicontinuity, $f = f(a)$ everywhere on $\partial\Delta_r(a)$. This was true for all r with $\overline{\Delta_r(a)} \subset U$, so $f = f(a)$ on $\Delta_r(a)$, and so the set where $f = f(a)$ is open. The set where an upper-semicontinuous function attains a maximum is closed. So $f = f(a)$ on U as U is connected. □

Exercise* 2.4.11*: *Prove that subharmonicity is a local property. That is, given an open set $U \subset \mathbb{C}$, a function $f: U \to \mathbb{R} \cup \{-\infty\}$ is subharmonic if and only if for every $p \in U$ there exists a neighborhood W of p, $W \subset U$, such that $f|_W$ is subharmonic. Hint: Perhaps try to use the maximum principle and Exercise 2.4.10.*

Exercise* 2.4.12*: *Suppose $U \subset \mathbb{C}$ is a bounded open set, $f : \overline{U} \to \mathbb{R} \setminus \{-\infty\}$ is an upper-semicontinuous function, such that $f|_U$ is subharmonic, $g : \overline{U} \to \mathbb{R}$ is a continuous function such that $g|_U$ is harmonic and $f(z) \leq g(z)$ for all $z \in \partial U$. Prove that $f(z) \leq g(z)$ for all $z \in U$.*

Exercise* 2.4.13*: *Let g be a function harmonic on a disc $\Delta \subset \mathbb{C}$ and continuous on $\overline{\Delta}$. Prove that for any $\epsilon > 0$ there exists a function g_ϵ, harmonic in a neighborhood of $\overline{\Delta}$, such that $g(z) \leq g_\epsilon(z) \leq g(z) + \epsilon$ for all $z \in \overline{\Delta}$. In particular, to test subharmonicity, we only need to consider those g that are harmonic a bit past the boundary of the disc.*

Proposition 2.4.5. *Suppose $U \subset \mathbb{C}$ is an open set and $f : U \to \mathbb{R}$ is a C^2 function. The function f is subharmonic if and only if $\nabla^2 f \geq 0$.*

Proof. We have a C^2-smooth function on a subset of $\mathbb{C} \cong \mathbb{R}^2$ with $\nabla^2 f \geq 0$ and we wish to show that it is subharmonic. Take a disc $\Delta_r(a)$ such that $\overline{\Delta_r(a)} \subset U$. Consider a function g continuous on $\overline{\Delta_r(a)}$, harmonic on $\Delta_r(a)$, and such that $f \leq g$ on the boundary $\partial\Delta_r(a)$. Because $\nabla^2(f - g) = \nabla^2 f \geq 0$, we assume $g = 0$ and $f \leq 0$ on the boundary $\partial\Delta_r(a)$.

Suppose $\nabla^2 f > 0$ at all points on $\Delta_r(a)$. Suppose f attains a maximum in $\Delta_r(a)$, call this point p. The Laplacian $\nabla^2 f$ is the trace of the Hessian matrix, but for f to have a maximum, the Hessian must have only nonpositive eigenvalues at the critical points, which is a contradiction as the trace is the sum of the eigenvalues. So f has no maximum inside, and therefore $f \leq 0$ on all of $\overline{\Delta_r(a)}$.

Next suppose $\nabla^2 f \geq 0$. Let M be the maximum of $x^2 + y^2$ on $\overline{\Delta_r(a)}$. Take $f_n(x, y) = f(x, y) + \frac{1}{n}(x^2 + y^2) - \frac{1}{n}M$. Clearly $\nabla^2 f_n > 0$ everywhere on $\Delta_r(a)$ and $f_n \leq 0$ on the boundary, so $f_n \leq 0$ on all of $\overline{\Delta_r(a)}$. As $f_n \to f$ we obtain that $f \leq 0$ on all of $\overline{\Delta_r(a)}$.

The other direction is left as an exercise. □

Exercise* 2.4.14*: *Finish the proof of the above proposition.*

In analogy to convex functions, a C^2-smooth function f of one real variable is convex if and only if $f''(x) \geq 0$ for all x.

Proposition 2.4.6. *Suppose $U \subset \mathbb{C}$ is an open set and $f_\alpha : U \to \mathbb{R} \cup \{-\infty\}$ is a family of subharmonic functions. Let*

$$\varphi(z) = \sup_\alpha f_\alpha(z).$$

If the family is finite, then φ is subharmonic. If the family is infinite, $\varphi(z) \neq \infty$ for all z, and φ is upper-semicontinuous, then φ is subharmonic.

Proof. Suppose $\overline{\Delta_r(a)} \subset U$. For any α,

$$\frac{1}{2\pi}\int_0^{2\pi} \varphi(a + re^{i\theta})\, d\theta \geq \frac{1}{2\pi}\int_0^{2\pi} f_\alpha(a + re^{i\theta})\, d\theta \geq f_\alpha(a).$$

Taking the supremum on the right over α obtains the results. □

Exercise* 2.4.15*: *Prove that if $\varphi\colon \mathbb{R} \to \mathbb{R}$ is a monotonically increasing convex function, $U \subset \mathbb{C}$ is an open set, and $f\colon U \to \mathbb{R}$ is subharmonic, then $\varphi \circ f$ is subharmonic.*

Exercise* 2.4.16*: *Let $U \subset \mathbb{C}$ be open, $\{f_n\}$ a sequence of subharmonic functions uniformly bounded above on compact subsets, and $\{c_n\}$ a sequence of positive real numbers such that $\sum_{n=1}^{\infty} c_n < \infty$. Prove that $f = \sum_{n=1}^{\infty} c_n f_n$ is subharmonic. Make sure to prove the function is upper-semicontinuous.*

Exercise* 2.4.17*: *Suppose $U \subset \mathbb{C}$ is a bounded open set, and $\{p_n\}$ a sequence of points in U. For $z \in U$, define $f(z) = \sum_{n=1}^{\infty} 2^{-n} \log\lvert z - p_n\rvert$, possibly taking on the value $-\infty$.*

a) Show that f is a subharmonic function in U.

b) If $U = \mathbb{D}$ and $p_n = \frac{1}{n}$, show that f is discontinuous at 0 (the natural topology on $\mathbb{R} \cup \{-\infty\}$).

c) If $\{p_n\}$ is dense in U, show that f is nowhere continuous. Hint: Prove that $f^{-1}(-\infty)$ is a small (but dense) set. Another hint: Integrate the partial sums, and use polar coordinates.

There are too many harmonic functions in $\mathbb{C}^n \cong \mathbb{R}^{2n}$. To get the real and imaginary parts of holomorphic functions in $\mathbb{C}^n$, we require a smaller class of functions than all harmonic functions.

Definition 2.4.7. Let $U \subset \mathbb{C}^n$ be open. A C^2-smooth $f\colon U \to \mathbb{R}$ is *pluriharmonic* if for every $a, b \in \mathbb{C}^n$, the function of one variable

$$\xi \mapsto f(a + b\xi)$$

is harmonic (on the set of $\xi \in \mathbb{C}$ where $a + b\xi \in U$). That is, f is harmonic on every complex line.

A function $f\colon U \to \mathbb{R} \cup \{-\infty\}$ is *plurisubharmonic*, sometimes *plush* or *psh* for short, if it is upper-semicontinuous and for every $a, b \in \mathbb{C}^n$, the function of one variable

$$\xi \mapsto f(a + b\xi)$$

is subharmonic (whenever $a + b\xi \in U$).

A harmonic function of one complex variable is in some sense a generalization of an affine linear function of one real variable. Similarly, as far as several complex variables are concerned, a pluriharmonic function is the right generalization to $\mathbb{C}^n$ of an affine linear function on $\mathbb{R}^n$. In the same way plurisubharmonic functions are the correct complex variable generalizations of convex functions. One real variable convex function is like a subharmonic function, and a convex function of several real variables is a function that is convex when restricted to any real line.

Exercise* 2.4.18*: *Let $U \subset \mathbb{C}^n$ be open. A C^2-smooth $f\colon U \to \mathbb{R}$ is pluriharmonic if and only if*

$$\frac{\partial^2 f}{\partial \bar{z}_j \partial z_k} = 0 \qquad \text{on } U \text{ for all } j, k = 1, \ldots, n.$$

Exercise* 2.4.19*: *Show that a pluriharmonic function is harmonic. On the other hand, find an example of a harmonic function that is not pluriharmonic.*

Exercise* 2.4.20*: *Let $U \subset \mathbb{C}^n$ be open. Show that $f : U \to \mathbb{R}$ is pluriharmonic if and only if it is locally the real or imaginary part of a holomorphic function. Hint: Using a previous exercise $\frac{\partial f}{\partial z_k}$ is holomorphic for all k. Assume that U is simply connected, $p \in U$, and $f(p) = 0$. Consider the line integral from p to a nearby $z \in U$:*

$$F(z) = \int_p^z \sum_{k=1}^n \frac{\partial f}{\partial z_k}(z)\, dz_k .$$

Prove that it is path independent, compute derivatives of F, and find out what is $f - F$.

Exercise* 2.4.21*: *Prove the maximum principle. That is, if $U \subset \mathbb{C}^n$ is a domain and $f : U \to \mathbb{R} \cup \{-\infty\}$ is plurisubharmonic and achieves a maximum at $p \in U$, then f is constant.*

Proposition 2.4.8. *Let $U \subset \mathbb{C}^n$ be open. A C^2-smooth $f : U \to \mathbb{R}$ is plurisubharmonic if and only if the complex Hessian matrix*

$$\left[\frac{\partial^2 f}{\partial \bar{z}_j \partial z_k} \right]_{jk}$$

is positive semidefinite at every point.

Proof. First let us suppose that the complex Hessian has a negative eigenvalue at a point $p \in U$. After a translation assume $p = 0$. As f is real-valued, the complex Hessian $\left[\frac{\partial^2 f}{\partial \bar{z}_j \partial z_k}\Big|_0 \right]_{jk}$ is Hermitian. A complex linear change of coordinates acts on the complex Hessian by $*$-congruence, and therefore we can diagonalize, using Sylvester's Law of Inertia again. So assume that $\left[\frac{\partial^2 f}{\partial \bar{z}_j \partial z_k}\Big|_0 \right]_{jk}$ is diagonal. If the complex Hessian has a negative eigenvalue, then one of the diagonal entries is negative. Without loss of generality suppose $\frac{\partial^2 f}{\partial \bar{z}_1 \partial z_1}\Big|_0 < 0$. The function $z_1 \mapsto f(z_1, 0, \ldots, 0)$ has a negative Laplacian and therefore is not subharmonic, and thus f itself is not plurisubharmonic.

For the other direction, suppose the complex Hessian is positive semidefinite at all points. Let $p \in U$. After an affine change of coordinates assume that the line $\xi \mapsto a + b\xi$ is simply setting all but the first variable to zero, that is $a = 0$ and $b = (1, 0, \ldots, 0)$. As the complex Hessian is positive semidefinite, $\frac{\partial^2 f}{\partial \bar{z}_1 \partial z_1} \geq 0$ for all points $(z_1, 0, \ldots, 0)$. We proved above that $\nabla^2 g \geq 0$ implies g is subharmonic, and we are done. ☐

Exercise* 2.4.22*: *Suppose $U \subset \mathbb{C}^n$ is open and $f : U \to \mathbb{C}$ is holomorphic.*

a) Show $\log|f(z)|$ is plurisubharmonic. In fact, it is pluriharmonic away from the zeros of f.
b) Show $|f(z)|^\eta$ is plurisubharmonic for all $\eta > 0$.

Exercise* 2.4.23*: *Show that the set of plurisubharmonic functions on an open set $U \subset \mathbb{C}^n$ is a cone in the sense that if $a, b > 0$ are constants and $f, g : U \to \mathbb{R} \cup \{-\infty\}$ are plurisubharmonic, then $af + bg$ is plurisubharmonic.*

Theorem 2.4.9. *Suppose $U \subset \mathbb{C}^n$ is an open set and $f : U \to \mathbb{R} \cup \{-\infty\}$ is plurisubharmonic. For every $\epsilon > 0$, let $U_\epsilon \subset U$ be the set of points further than ϵ away from ∂U. Then there exists a smooth plurisubharmonic function $f_\epsilon : U_\epsilon \to \mathbb{R}$ such that $f_\epsilon(z) \geq f(z)$, and*

$$f(z) = \lim_{\epsilon \to 0} f_\epsilon(z) \qquad \text{for all } z \in U.$$

That is, f is a limit of smooth plurisubharmonic functions. The idea of the proof is important and useful in many other contexts.

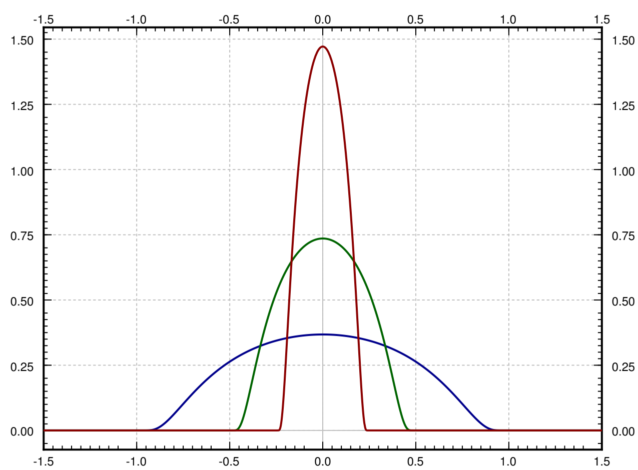
Proof. We smooth f out by convolving with so-called *mollifiers*, or *approximate delta functions*. Many different mollifiers work, but let us use a very specific one for concreteness. For $\epsilon > 0$, define

$$g(z) = \begin{cases} Ce^{-1/(1-\|z\|^2)} & \text{if } \|z\| < 1, \\ 0 & \text{if } \|z\| \geq 1, \end{cases} \qquad \text{and} \qquad g_\epsilon(z) = \frac{1}{\epsilon^{2n}} g(z/\epsilon).$$

It is left as an exercise that g, and so g_ϵ, is smooth. The function g has compact support as it is only nonzero inside the unit ball. The support of g_ϵ is the ϵ-ball. Both are nonnegative. Choose C so that

$$\int_{\mathbb{C}^n} g \, dV = 1, \qquad \text{and therefore} \qquad \int_{\mathbb{C}^n} g_\epsilon \, dV = 1.$$

Here dV is the volume measure. The function g only depends on $\|z\|$. To get an idea of how these functions work, consider the graphs of $e^{-1/(1-x^2)}$, $\frac{1}{0.5} e^{-1/\left(1-(x/0.5)^2\right)}$, and $\frac{1}{0.25} e^{-1/\left(1-(x/0.25)^2\right)}$.



Compare the graphs to the graphs of the Poisson kernel as a function of θ, which is also a type of mollifier. In fact, the idea of integrating against the right approximate delta function with the desired properties is similar to the solution of the Dirichlet problem using the Poisson kernel.

First f is bounded above on compact sets as it is upper semicontinuous. If f is not bounded below, we replace f with $\max\left\{f, \frac{-1}{\epsilon}\right\}$, which is still plurisubharmonic. Therefore, without loss of generality we assume that f is locally bounded.

For $z \in U_\epsilon$, we define f_ϵ as the convolution with g_ϵ:

$$f_\epsilon(z) = (f * g_\epsilon)(z) = \int_{\mathbb{C}^n} f(w) g_\epsilon(z-w) \, dV(w) = \int_{\mathbb{C}^n} f(z-w) g_\epsilon(w) \, dV(w).$$

The two forms of the integral follow easily via change of variables. We are perhaps abusing notation a bit as f is only defined on U, but it is not a problem as long as $z \in U_\epsilon$ as g_ϵ is then zero when f is undefined. By differentiating the first form under the integral, we find that f_ϵ is smooth.

Let us show that f_ϵ is plurisubharmonic. We restrict to a line $\xi \mapsto a + b\xi$. We wish to test subharmonicity by the sub-mean-value property using a disc of radius r around $\xi = 0$:

$$\begin{aligned}
\frac{1}{2\pi}\int_0^{2\pi} f_\epsilon(a + bre^{i\theta})\,d\theta &= \frac{1}{2\pi}\int_0^{2\pi}\int_{\mathbb{C}^n} f\big(a + bre^{i\theta} - w\big)g_\epsilon(w)\,dV(w)\,d\theta \\
&= \int_{\mathbb{C}^n}\left(\frac{1}{2\pi}\int_0^{2\pi} f\big(a - w + bre^{i\theta}\big)\,d\theta\right)g_\epsilon(w)\,dV(w) \\
&\geq \int_{\mathbb{C}^n} f(a-w)g_\epsilon(w)\,dV(w) = f_\epsilon(a).
\end{aligned}$$

For the inequality we used $g_\epsilon \geq 0$. So f_ϵ is plurisubharmonic.

Let us show that $f_\epsilon(z) \geq f(z)$ for all $z \in U_\epsilon$. As $g_\epsilon(w)$ only depends on $|w_1|, \ldots, |w_n|$, we notice that $g_\epsilon(w_1, \ldots, w_n) = g_\epsilon(|w_1|, \ldots, |w_n|)$. Without loss of generality we consider $z = 0$, and we use polar coordinates for the integral.

$$\begin{aligned}
f_\epsilon(0) &= \int_{\mathbb{C}^n} f(-w)g_\epsilon(|w_1|, \ldots, |w_n|)\,dV(w) \\
&= \int_0^\epsilon \cdots \int_0^\epsilon \left(\int_0^{2\pi}\cdots\int_0^{2\pi} f(-r_1e^{i\theta_1}, \ldots, -r_ne^{i\theta_n})\,d\theta_1\cdots d\theta_n\right) \\
&\qquad\qquad g_\epsilon(r_1, \ldots, r_n)\,r_1\cdots r_n\,dr_1\cdots dr_n \\
&\geq \int_0^\epsilon \cdots \int_0^\epsilon \left(\int_0^{2\pi}\cdots\int_0^{2\pi} (2\pi)f(0, -r_2e^{i\theta_2}, \ldots, -r_ne^{i\theta_n})\,d\theta_2\cdots d\theta_n\right) \\
&\qquad\qquad g_\epsilon(r_1, \ldots, r_n)\,r_1\cdots r_n\,dr_1\cdots dr_n \\
&\geq f(0)\int_0^\epsilon \cdots \int_0^\epsilon (2\pi)^n g_\epsilon(r_1, \ldots, r_n)\,r_1\cdots r_n\,dr_1\cdots dr_n \\
&= f(0)\int_{\mathbb{C}^n} g_\epsilon(w)dV(w) = f(0).
\end{aligned}$$

The second equality above follows because integral of g_ϵ only needs to be done over the polydisc of radius ϵ. The penultimate equality follows from the fact that $2\pi = \int_0^{2\pi} d\theta$.

Finally we show $\lim_{\epsilon\to 0} f_\epsilon(z) = f(z)$. For subharmonic, and so for plurisubharmonic, functions, $\limsup_{\zeta\to z} f(\zeta) = f(z)$. So given $\delta > 0$ find an $\epsilon > 0$ such that $f(\zeta) - f(z) \leq \delta$ for all $\zeta \in B_\epsilon(z)$.

$$\begin{aligned}
f_\epsilon(z) - f(z) &= \int_{B_\epsilon(0)} f(z-w)g_\epsilon(w)\,dV(w) - f(z)\int_{B_\epsilon(0)} g_\epsilon(w)\,dV(w) \\
&= \int_{B_\epsilon(0)} \big(f(z-w) - f(z)\big)\,g_\epsilon(w)\,dV(w) \\
&\leq \delta\int_{B_\epsilon(0)} g_\epsilon(w)\,dV(w) = \delta.
\end{aligned}$$

Again we used that g_ϵ is nonnegative. We find $0 \leq f_\epsilon - f(z) \leq \delta$, and so $f_\epsilon(z) \to f(z)$. $\square$

***Exercise* 2.4.24:** *Show that g in the proof above is smooth on all of $\mathbb{C}^n$.*

***Exercise* 2.4.25:**
 a) *Show that for a subharmonic function $\int_0^{2\pi} f(a + re^{i\theta})\, d\theta$ is a monotone function of r (Hint: Try a C^2 function first and use Green's theorem).*
 b) *Use this fact to show that $f_\epsilon(z)$ from Theorem 2.4.9 are monotone decreasing in ϵ.*

***Exercise* 2.4.26:** *Let $U \subset \mathbb{C}^n$ and $V \subset \mathbb{C}^m$ be open. If $g : U \to V$ is holomorphic and $f : V \to \mathbb{R}$ is a C^2 plurisubharmonic function, then $f \circ g$ is plurisubharmonic. Then use this to show that this holds for all plurisubharmonic functions (Hint: monotone convergence).*

***Exercise* 2.4.27:** *Show that plurisubharmonicity is a local property, that is, f is plurisubharmonic if and only if f is plurisubharmonic in some neighborhood of each point.*

***Exercise* 2.4.28:** *Using the computation from Theorem 2.4.9 show that if f is pluriharmonic, then $f_\epsilon = f$ (where it makes sense), obtaining another proof that a pluriharmonic function is C^∞.*

***Exercise* 2.4.29:** *Let the f in Theorem 2.4.9 be continuous and suppose $K \subset\subset U$, in particular for small enough $\epsilon > 0$, $K \subset U_\epsilon$. Show that f_ϵ converges uniformly to f on K.*

***Exercise* 2.4.30:** *Let the f in Theorem 2.4.9 be C^k-smooth for some $k \geq 0$. Show that all derivatives of f_ϵ up to order k converge uniformly on compact sets to the corresponding derivatives of f. See also previous exercise.*

Let us prove the theorem of Radó, which is a complementary result to the Riemann extension theorem. Here on the one hand the function is continuous and vanishes on the set you wish to extend across, but on the other hand you know nothing about this set. It is sometimes covered in a one-variable course, and in several variables it follows directly from the one-variable result.

Theorem 2.4.10 (Radó)**.** *Let $U \subset \mathbb{C}^n$ be open and $f : U \to \mathbb{C}$ a continuous function that is holomorphic on the set*

$$U' = \{ z \in U : f(z) \neq 0 \}.$$

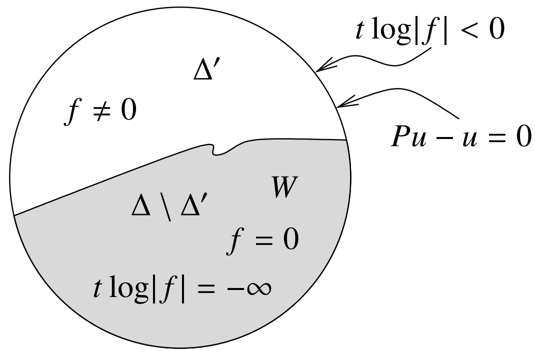
Then $f \in \mathcal{O}(U)$.

Proof. First assume $n = 1$. As the theorem is local, it is enough to prove it for a small disc Δ such that f is continuous on the closure $\overline{\Delta}$, let Δ' be the part of the disc where f is nonzero. If Δ' is empty, then we are done as f is just identically zero and hence holomorphic.

Let u be the real part of f. On Δ', u is a harmonic function. Let Pu be the Poisson integral of u on Δ. Hence Pu equals u on $\partial\Delta$, and Pu is harmonic in all of Δ. Consider the function $Pu(z) - u(z)$ on $\overline{\Delta}$. The function is zero on $\partial\Delta$ and it is harmonic on Δ'. By rescaling f we can without loss of generality assume that $|f(z)| < 1$ for all $z \in \overline{\Delta}$. For any $t > 0$, the function $z \mapsto t \log |f(z)|$ is subharmonic on Δ' and upper-semicontinuous on $\overline{\Delta'}$. Further, it is negative on $\partial\Delta$. The function $z \mapsto -t \log |f(z)|$ is superharmonic (minus a subharmonic function) on Δ', lower-semicontinuous on $\overline{\Delta'}$ and positive on $\partial\Delta$. On the set $\Delta \setminus \Delta'$ where f is zero, the two functions are $-\infty$ and ∞ respectively. Therefore for all $t > 0$ and $z \in \partial\Delta \cup (\Delta \setminus \Delta')$ we have

$$t \log |f(z)| \leq Pu(z) - u(z) \leq -t \log |f(z)| . \tag{2.3}$$

Applying the maximum principle to the subharmonic functions $z \mapsto t \log |f(z)| - \bigl(Pu(z) - u(z)\bigr)$ and $z \mapsto t \log |f(z)| - \bigl(u(z) - Pu(z)\bigr)$ shows that (2.3) holds for all $z \in \Delta'$ and all $t > 0$.



Taking the limit $t \to 0$ shows that $Pu = u$ on Δ'. Let $W = \Delta \setminus \overline{\Delta'}$. On W, $u = 0$ and so $Pu - u$ is harmonic on W and continuous on $\overline{W}$. Furthermore, $Pu - u = 0$ on $\overline{\Delta'} \cup \partial\Delta$, and so $Pu - u = 0$ on ∂W. By the maximum principle, $Pu = u$ on W and therefore on all of $\overline{\Delta}$. Similarly, if v is the imaginary part of f, then $Pv = v$ on $\overline{\Delta}$. In other words, u and v are harmonic on Δ. As Δ is simply connected, let $\tilde{v}$ be the harmonic conjugate of u that equals v at some point of Δ'. As f is holomorphic on Δ', the harmonic functions $\tilde{v}$ and v are equal the nonempty open subset Δ' of Δ and so they are equal everywhere. Consequently, $f = u + iv$ is holomorphic on Δ.

The extension of the proof to several variables is left as an exercise. $\square$

Exercise* 2.4.31*: *Use the one-variable result to extend the theorem to several variables.*

2.5 Hartogs pseudoconvexity

It is worth mentioning explicitly that by the exercises of the previous section, plurisubharmonicity is preserved under holomorphic mappings. That is, if g is holomorphic and f is plurisubharmonic, then $f \circ g$ is plurisubharmonic. In particular, if $\varphi \colon \mathbb{D} \to \mathbb{C}^n$ is an analytic disc and f is plurisubharmonic in a neighborhood of $\varphi(\mathbb{D})$, then $f \circ \varphi$ is subharmonic.

Definition 2.5.1. Let $\mathcal{F}$ be a class of (extended[*])-real-valued functions defined on an open $U \subset \mathbb{R}^n$. If $K \subset U$, define $\widehat{K}$, the *hull* of K with respect to $\mathcal{F}$, as the set

$$\widehat{K} \overset{\text{def}}{=} \left\{ x \in U : f(x) \leq \sup_{y \in K} f(y) \text{ for all } f \in \mathcal{F} \right\}.$$

An open set U is said to be *convex with respect to* $\mathcal{F}$ if for every $K \subset\subset U$, the hull $\widehat{K} \subset\subset U$.[†]

Clearly $K \subset \widehat{K}$. The key is to show that $\widehat{K}$ is not "too large" for U. Keep in mind that the functions in $\mathcal{F}$ are defined on U, so $\widehat{K}$ depends on U not just on K. An easy mistake is to consider

[*]By extended reals we mean $\mathbb{R} \cup \{-\infty, \infty\}$.

[†]Recall that $\subset\subset$ means relatively compact, that is, the closure in the relative (subspace) topology is compact.

functions defined on a larger set, obtaining a smaller $\mathcal{F}$ and hence a larger $\widehat{K}$. Sometimes it is useful to write $\widehat{K}_{\mathcal{F}}$ to denote the dependence on $\mathcal{F}$, especially when talking about several different hulls.

For example, if $U = \mathbb{R}$ and $\mathcal{F}$ is the set of real-valued smooth $f: \mathbb{R} \to \mathbb{R}$ with $f''(x) \geq 0$, then for any $a, b \in \mathbb{R}$ we have $\widehat{\{a, b\}} = [a, b]$. In general, if $\mathcal{F}$ was the set of convex functions, then a domain $U \subset \mathbb{R}^n$ is geometrically convex if and only if it is convex with respect to convex functions, although let us not define what that means except for smooth functions in exercises below.

Exercise* 2.5.1*: *Suppose $U \subset \mathbb{R}^n$ is a domain.*

a) Show that U is geometrically convex if and only if it is convex with respect to the affine linear functions.

b) Suppose U has smooth boundary. Show that U is convex if and only if it is convex with respect to the smooth convex functions on U, that is smooth functions with positive semidefinite Hessian.

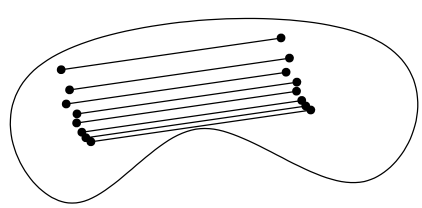
Exercise* 2.5.2*: *Show that any open set $U \subset \mathbb{R}^n$ is convex with respect to real polynomials.*

Theorem 2.5.2 (Kotinuitätssatz—Continuity principle)**.** *Suppose an open set $U \subset \mathbb{C}^n$ is convex with respect to plurisubharmonic functions, then given any collection of closed analytic discs $\Delta_\alpha \subset U$ such that $\bigcup_\alpha \partial\Delta_\alpha \subset\subset U$, we have $\bigcup_\alpha \Delta_\alpha \subset\subset U$.*

Various similar theorems are named the *continuity principle*. Generally what they have in common is the family of analytic discs whose boundaries stay inside a domain, and whose conclusion has to do with extension of holomorphic functions, or with domains of holomorphy.

Proof. Let f be a plurisubharmonic function on U. If $\varphi_\alpha: \overline{\mathbb{D}} \to U$ is the holomorphic (in $\mathbb{D}$) mapping giving the closed analytic disc, then $f \circ \varphi_\alpha$ is subharmonic. By the maximum principle, f on Δ_α must be less than or equal to the supremum of f on $\partial\Delta_\alpha$, so $\overline{\Delta_\alpha}$ is in the hull of $\partial\Delta_\alpha$. In other words $\bigcup_\alpha \Delta_\alpha$ is in the hull of $\bigcup_\alpha \partial\Delta_\alpha$ and therefore $\bigcup_\alpha \Delta_\alpha \subset\subset U$ by convexity. □

Let us illustrate the failure of the continuity principle. If a domain is not convex with respect to plurisubharmonic functions, then you could have discs (denoted by straight line segments) that approach the boundary as in the following picture. In the diagram, the boundaries of the discs are denoted by the dark dots at the end of the segments. In fact, if we replace discs with line segments, this is the standard convexity, see the exercises below.



Exercise* 2.5.3*: *Suppose $U \subset \mathbb{C}^n$ is a domain and $K \subset\subset U$ is a nonempty compact subset. Prove that $U \setminus K$ is not convex with respect to plurisubharmonic functions.*

***Exercise* 2.5.4:** *Suppose $U \subset \mathbb{C}^n$ is a domain with smooth boundary, $p \in \partial U$, and Δ is an affine linear analytic disc with $p \in \Delta$, but $\Delta \setminus \{p\} \subset U$. Prove that U is not convex with respect to the plurisubharmonic functions.*

***Exercise* 2.5.5:** *Prove the corresponding Kotinuitätssatz, and its converse, for geometric convexity: Prove that a domain $U \subset \mathbb{R}^n$ is geometrically convex if and only if whenever $[x_\alpha, y_\alpha] \subset U$ is a collection of straight line segments such that $\bigcup_\alpha \{x_\alpha, y_\alpha\} \subset\subset U$ then $\bigcup_\alpha [x_\alpha, y_\alpha] \subset\subset U$.*

Definition 2.5.3. Let $U \subset \mathbb{C}^n$ be open. An $f : U \to \mathbb{R}$ is an *exhaustion function* for U if

$$\{z \in U : f(z) < r\} \subset\subset U \qquad \text{for every } r \in \mathbb{R}.$$

A domain $U \subset \mathbb{C}^n$ is *Hartogs pseudoconvex* if there exists a continuous plurisubharmonic exhaustion function. The set $\{z \in U : f(z) < r\}$ is called the *sublevel set* of f, or the *r-sublevel set*.

Example 2.5.4: The unit ball $\mathbb{B}_n$ is Hartogs pseudoconvex. The continuous function

$$z \mapsto -\log\bigl(1 - \|z\|\bigr)$$

is an exhaustion function, and it is easy to check directly that it is plurisubharmonic.

Example 2.5.5: The entire $\mathbb{C}^n$ is Hartogs pseudoconvex as $\|z\|^2$ is a continuous plurisubharmonic exhaustion function. Also, because $\|z\|^2$ is plurisubharmonic, then given any $K \subset\subset \mathbb{C}^n$, the hull $\widehat{K}$ with respect to plurisubharmonic functions must be bounded. In other words, $\mathbb{C}^n$ is convex with respect to plurisubharmonic functions.

Theorem 2.5.6. *Suppose $U \subsetneq \mathbb{C}^n$ is a domain. The following are equivalent:*

(i) $-\log \rho(z)$ *is plurisubharmonic, where $\rho(z)$ is the distance from z to ∂U.*

(ii) U has a continuous plurisubharmonic exhaustion function, that is, U is Hartogs pseudoconvex.

(iii) U is convex with respect to plurisubharmonic functions defined on U.

Proof. (i) $\Rightarrow$ (ii): If U is bounded, the function $-\log \rho(z)$ is clearly a continuous exhaustion function. If U is unbounded, take $z \mapsto \max\{-\log \rho(z), \|z\|^2\}$.

(ii) $\Rightarrow$ (iii): Suppose f is a continuous plurisubharmonic exhaustion function. If $K \subset\subset U$, then for some r we have $K \subset \{z \in U : f(z) < r\} \subset\subset U$. But then by definition of the hull $\widehat{K}$ we have $\widehat{K} \subset \{z \in U : f(z) < r\} \subset\subset U$.

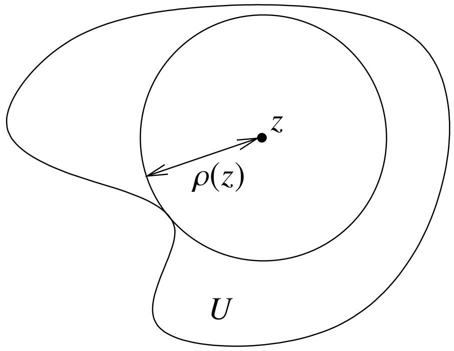
(iii) $\Rightarrow$ (i): For $c \in \mathbb{C}^n$ with $\|c\| = 1$, let $\rho_c(z)$ be the radius of the largest affine disc centered at z in the direction c that still lies in U. That is,

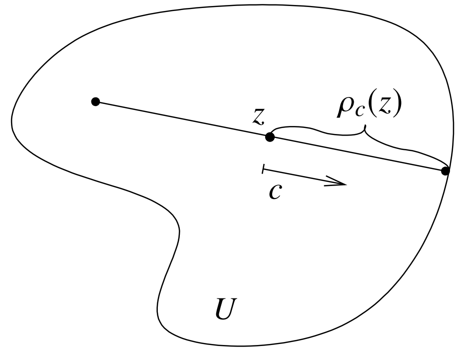
$$\rho_c(z) = \sup\bigl\{\lambda > 0 : z + \zeta c \in U \text{ for all } \zeta \in \lambda\mathbb{D}\bigr\}.$$

As $\rho(z) = \inf_c \rho_c(z)$,

$$-\log \rho(z) = \sup_{\|c\|=1} \bigl(-\log \rho_c(z)\bigr).$$

If we prove that for any $a, b \in \mathbb{C}^n$ the function $\xi \mapsto -\log \rho_c(a + b\xi)$ is subharmonic, then $\xi \mapsto -\log \rho(a + b\xi)$ is subharmonic and we are done. Here is the setup, the disc is drawn as a line:





Suppose $\Delta \subset \mathbb{C}$ is a disc such that $a + b\xi \in U$ for all $\xi \in \overline{\Delta}$. If u is a harmonic function on Δ continuous on $\overline{\Delta}$ such that $-\log \rho_c(a + b\xi) \leq u(\xi)$ on $\partial\Delta$, we must show that the inequality holds on Δ. By Exercise 2.4.13 we may assume u is harmonic on a neighborhood of $\overline{\Delta}$ and so let $u = \operatorname{Re} f$ for a holomorphic function f. Fix a $\xi \in \partial\Delta$. We have $-\log \rho_c(a + b\xi) \leq \operatorname{Re} f(\xi)$, or in other words

$$\rho_c(a + b\xi) \geq e^{-\operatorname{Re} f(\xi)} = \left|e^{-f(\xi)}\right|.$$

Using $\zeta = te^{-f(\xi)}$ in the definition of $\rho_c(a + b\xi)$, the statement above is equivalent to saying that

$$(a + b\xi) + te^{-f(\xi)}c \in U \quad \text{for all } t \in \mathbb{D}.$$

This statement holds whenever $\xi \in \partial\Delta$. We must prove that it also holds for all $\xi \in \Delta$.

The function $\varphi_t(\xi) = (a + b\xi) + te^{-f(\xi)}c$ gives a closed analytic disc with boundary inside U. We have a family of analytic discs, parametrized by t, whose boundaries are in U for all t with $|t| < 1$. For $t = 0$ the entire disc is inside U. As $\varphi_t(\xi)$ is continuous in both t and ξ and $\overline{\Delta}$ is compact, $\varphi_t(\Delta) \subset U$ for t in some neighborhood of 0. Take $0 < t_0 < 1$ such that $\varphi_t(\Delta) \subset U$ for all t with $|t| < t_0$. Then

$$\bigcup_{|t|<t_0} \varphi_t(\partial\Delta) \subset \bigcup_{|t|\leq t_0} \varphi_t(\partial\Delta) \subset\subset U,$$

because continuous functions take compact sets to compact sets. Kontinuitätssatz implies

$$\bigcup_{|t|<t_0} \varphi_t(\Delta) \subset\subset U.$$

By continuity again $\bigcup_{|t|\leq t_0} \varphi_t(\Delta) \subset\subset U$, and so $\bigcup_{|t|<t_0+\epsilon} \varphi_t(\Delta) \subset\subset U$ for some $\epsilon > 0$. Consequently $\varphi_t(\Delta) \subset U$ for all t with $|t| < 1$. Thus $(a + b\xi) + te^{-f(\xi)}c \in U$ for all $\xi \in \Delta$ and all $|t| < 1$. This implies $\rho_c(a+b\xi) \geq e^{-\operatorname{Re} f(\xi)}$ for all $\xi \in \Delta$, which in turn implies $-\log \rho_c(a+b\xi) \leq \operatorname{Re} f(\xi) = u(\xi)$ for all $\xi \in \Delta$. Therefore, $-\log \rho_c(a + b\xi)$ is subharmonic. □

Exercise 2.5.6: *Show that if $U_1 \subset \mathbb{C}^n$ and $U_2 \subset \mathbb{C}^n$ are Hartogs pseudoconvex domains, then so are all the topological components of $U_1 \cap U_2$.*

Exercise 2.5.7: *Show that if $U \subset \mathbb{C}^n$ and $V \subset \mathbb{C}^m$ are Hartogs pseudoconvex domains, then so is $U \times V$.*

Exercise 2.5.8: *Show that every domain $U \subset \mathbb{C}$ is Hartogs pseudoconvex.*

***Exercise* 2.5.9:** *Show that the union $\bigcup_j U_j$ of a nested sequence of Hartogs pseudoconvex domains $U_{j-1} \subset U_j \subset \mathbb{C}^n$ is Hartogs pseudoconvex.*

***Exercise* 2.5.10:** *Let $\mathbb{R}^2 \subset \mathbb{C}^2$ be naturally embedded (that is, it is the set where z_1 and z_2 are real). Show that the set $\mathbb{C}^2 \setminus \mathbb{R}^2$ is not Hartogs pseudoconvex.*

***Exercise* 2.5.11:** *Let $U \subset \mathbb{C}^n$ be a domain and $f \in \mathcal{O}(U)$. Prove that $U' = \{ z \in U : f(z) \neq 0 \}$ is a Hartogs pseudoconvex domain. Hint: See also Exercise 1.6.2.*

***Exercise* 2.5.12:** *Suppose $U, V \subset \mathbb{C}^n$ are biholomorphic domains. Prove that U is Hartogs pseudoconvex if and only if V is Hartogs pseudoconvex.*

***Exercise* 2.5.13:** *Let $U = \{ z \in \mathbb{C}^2 : |z_1| > |z_2| \}$.*
- *a) Prove that U is a Hartogs pseudoconvex domain.*
- *b) Find a closed analytic disc Δ in $\mathbb{C}^2$ such that $0 \in \Delta$ ($0 \notin U$) and $\partial\Delta \subset U$.*
- *c) What do you think would happen if you tried to move Δ a little bit to avoid the intersection with the complement? Think about the continuity principle. Compare with Exercise 2.5.4.*

***Exercise* 2.5.14:** *Let $U \subset \mathbb{C}^n$ be a domain, and $f : \overline{U} \to \mathbb{R}$ be a continuous function, plurisubharmonic and negative on U, and $f = 0 \in \partial U$. Prove that U is Hartogs pseudoconvex.*

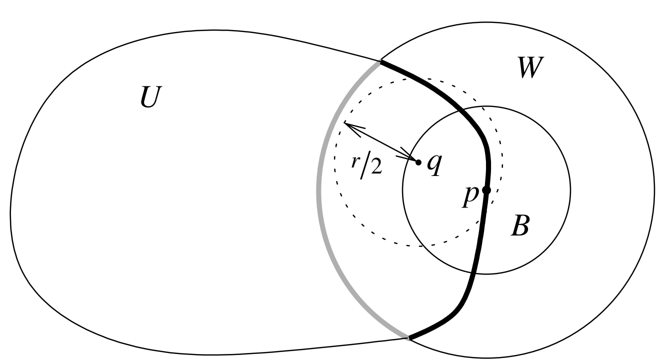
The statement corresponding to Exercise 2.5.9 on nested unions for domains of holomorphy is the *Behnke–Stein theorem*, which follows using this exercise and the solution of the Levi problem. Although historically Behnke–Stein was proved independently and used to solve the Levi problem.

Exercise 2.5.12 says that (Hartogs) pseudoconvexity is a biholomorphic invariant. That is a good indication that we are looking at a correct notion. It also allows us to change variables to more convenient ones when proving a specific domain is (Hartogs) pseudoconvex.

It is not immediately clear from the definition, but Hartogs pseudoconvexity is a local property.

Lemma 2.5.7. *A domain $U \subset \mathbb{C}^n$ is Hartogs pseudoconvex if and only if for every point $p \in \partial U$ there exists a neighborhood W of p such that $W \cap U$ is Hartogs pseudoconvex.*

Proof. One direction is trivial, so consider the other. Suppose $p \in \partial U$, and let W be such that $U \cap W$ is Hartogs pseudoconvex. Intersection of Hartogs pseudoconvex domains is Hartogs pseudoconvex, so assume $W = B_r(p)$. Let $B = B_{r/2}(p)$. If $q \in B \cap U$, the distance from q to the boundary of $W \cap U$ is the same as the distance to ∂U. The setup is illustrated in the following figure.



The part of the boundary ∂U in W is marked by a thick black line, the part of the boundary of $\partial(W \cap U)$ that arises as the boundary of W is marked by a thick gray line. A point $q \in B$ is marked and a ball of radius $r/2$ around q is dotted. No point of distance $r/2$ from q is in ∂W, and the distance of q to ∂U is at most $r/2$ as $p \in \partial U$ and p is the center of B. Let $\operatorname{dist}(x, y)$ denote the euclidean distance function*. Then for $z \in B \cap U$

$$-\log \operatorname{dist}(z, \partial U) = -\log \operatorname{dist}\bigl(z, \partial(U \cap W)\bigr).$$

The right hand side is plurisubharmonic as $U \cap W$ is Hartogs pseudoconvex. Such a ball B exists around every $p \in \partial U$, so near the boundary, $-\log \operatorname{dist}(z, \partial U)$ is plurisubharmonic.

If U is bounded, then ∂U is compact. So there is some $\epsilon > 0$ such that $-\log \operatorname{dist}(z, \partial U)$ is plurisubharmonic if $\operatorname{dist}(z, \partial U) < 2\epsilon$. The function

$$\varphi(z) = \max\bigl\{-\log \operatorname{dist}(z, \partial U), -\log \epsilon\bigr\}$$

is a continuous plurisubharmonic exhaustion function. The proof for unbounded U requires some function of $\|z\|^2$ rather than a constant ϵ, and is left as an exercise. □

***Exercise* 2.5.15:** *Finish the proof of the lemma for unbounded domains.*

It may seem that we defined a totally different concept, but it turns out that Levi and Hartogs pseudoconvexity are one and the same on domains where both concepts make sense. As a consequence of the following theorem we say simply "pseudoconvex" and there is no ambiguity.

Theorem 2.5.8. *Let $U \subset \mathbb{C}^n$ be a domain with smooth boundary. Then U is Hartogs pseudoconvex if and only if U is Levi pseudoconvex.*

Proof. Suppose $U \subset \mathbb{C}^n$ is a domain with smooth boundary that is not Levi pseudoconvex at $p \in \partial U$. As in Theorem 2.3.11, change coordinates so that $p = 0$ and U is defined by

$$\operatorname{Im} z_n > -|z_1|^2 + \sum_{j=2}^{n-1} \epsilon_j |z_j|^2 + O(3).$$

For a small fixed $\lambda > 0$, the closed analytic discs defined by $\xi \in \overline{\mathbb{D}} \mapsto (\lambda\xi, 0, \cdots, 0, is)$ are in U for all small enough $s > 0$. The origin is a limit point of the insides, but not a limit point of their boundaries. Kontinuitätssatz is not satisfied, and U is not convex with respect to the plurisubharmonic functions. Therefore U is not Hartogs pseudoconvex.

Next suppose U is Levi pseudoconvex. Take any $p \in \partial U$. After translation and rotation by a unitary, assume $p = 0$ and write a defining function r as

$$r(z, \bar{z}) = \varphi(z', \bar{z}', \operatorname{Re} z_n) - \operatorname{Im} z_n,$$

where $z' = (z_1, \ldots, z_{n-1})$. Levi pseudoconvexity says

$$\sum_{j=1,\ell=1}^{n} \bar{a}_j a_\ell \frac{\partial^2 r}{\partial \bar{z}_j \partial z_\ell}\Big|_q \geq 0 \quad \text{whenever} \quad \sum_{j=1}^{n} a_j \frac{\partial r}{\partial z_j}\Big|_q = 0, \tag{2.4}$$

*If either x and/or y are sets of points we take the infimum of the euclidean distance over all the points.

for all $q \in \partial U$ near 0. Let s be a small real constant, and let $\widetilde{q} = (q_1, \ldots, q_{n-1}, q_n + is)$. None of the derivatives of r depends on $\operatorname{Im} z_n$, and therefore $\frac{\partial r}{\partial z_\ell}\big|_{\widetilde{q}} = \frac{\partial r}{\partial z_\ell}\big|_q$ and $\frac{\partial^2 r}{\partial \bar{z}_j \partial z_\ell}\big|_{\widetilde{q}} = \frac{\partial^2 r}{\partial \bar{z}_j \partial z_\ell}\big|_q$ for all j and ℓ. So condition (2.4) holds for all $q \in U$ near 0. We will use r to manufacture a plurisubharmonic exhaustion function, that is one with a semidefinite Hessian. Starting with r, we already have what we need in all but one direction.

Let $\nabla_z r|_q = \left(\frac{\partial r}{\partial z_1}\big|_q, \ldots, \frac{\partial r}{\partial z_n}\big|_q\right)$ denote the gradient of r in the holomorphic directions only. Given $q \in U$ near 0, decompose an arbitrary $c \in \mathbb{C}^n$ as $c = a + b$, where $a = (a_1, \ldots, a_n)$ satisfies

$$\sum_{j=1}^n a_j \frac{\partial r}{\partial z_j}\Big|_q = \left\langle a, \overline{\nabla_z r|_q} \right\rangle = 0.$$

Taking the orthogonal decomposition, b is a scalar multiple of $\overline{\nabla_z r|_q}$. By Cauchy–Schwarz,

$$\left| \sum_{j=1}^n c_j \frac{\partial r}{\partial z_j}\Big|_q \right| = \left| \sum_{j=1}^n b_j \frac{\partial r}{\partial z_j}\Big|_q \right| = \left| \left\langle b, \overline{\nabla_z r|_q} \right\rangle \right| = \|b\| \|\nabla_z r|_q\|.$$

As $\nabla_z r|_0 = (0, \ldots, 0, -1/2i)$, then for q sufficiently near 0 we have that $\|\nabla_z r|_q\| \geq 1/3$, and

$$\|b\| = \frac{1}{\|\nabla_z r|_q\|} \left| \sum_{j=1}^n c_j \frac{\partial r}{\partial z_j}\Big|_q \right| \leq 3 \left| \sum_{j=1}^n c_j \frac{\partial r}{\partial z_j}\Big|_q \right|.$$

As $c = a + b$ is the orthogonal decomposition we have that $\|c\| \geq \|b\|$.

The complex Hessian matrix of r is continuous, and so let $M \geq 0$ be an upper bound on its operator norm for q near the origin. Again using Cauchy–Schwarz

$$\begin{aligned}
\sum_{j=1,\ell=1}^n \bar{c}_j c_\ell \frac{\partial^2 r}{\partial \bar{z}_j \partial z_\ell}\Big|_q &= \sum_{j=1,\ell=1}^n (\bar{a}_j + \bar{b}_j)(a_\ell + b_\ell) \frac{\partial^2 r}{\partial \bar{z}_j \partial z_\ell}\Big|_q \\
&= \sum_{j=1,\ell=1}^n \bar{a}_j a_\ell \frac{\partial^2 r}{\partial \bar{z}_j \partial z_\ell}\Big|_q \\
&\qquad + \sum_{j=1,\ell=1}^n \bar{b}_j c_\ell \frac{\partial^2 r}{\partial \bar{z}_j \partial z_\ell}\Big|_q + \sum_{j=1,\ell=1}^n \bar{c}_j b_\ell \frac{\partial^2 r}{\partial \bar{z}_j \partial z_\ell}\Big|_q - \sum_{j=1,\ell=1}^n \bar{b}_j b_\ell \frac{\partial^2 r}{\partial \bar{z}_j \partial z_\ell}\Big|_q \\
&\geq \sum_{j=1,\ell=1}^n \bar{a}_j a_\ell \frac{\partial^2 r}{\partial \bar{z}_j \partial z_\ell}\Big|_q - M\|b\|\|c\| - M\|c\|\|b\| - M\|b\|^2 \\
&\geq -3M\|c\|\|b\|.
\end{aligned}$$

Together with what we know about $\|b\|$, for $q \in U$ near the origin

$$\sum_{j=1,\ell=1}^n \bar{c}_j c_\ell \frac{\partial^2 r}{\partial \bar{z}_j \partial z_\ell}\Big|_q \geq -3M\|c\|\|b\| \geq -3^2 M \|c\| \left| \sum_{j=1}^n c_j \frac{\partial r}{\partial z_j}\Big|_q \right|.$$

For $z \in U$ sufficiently close to 0 define

$$f(z) = -\log\big(-r(z)\big) + A\|z\|^2,$$

where $A > 0$ is some constant we will choose later. The log is there to make f blow up as we approach the boundary. The $A\|z\|^2$ is there to add a constant diagonal matrix to the complex Hessian of f, which we hope is enough to make it positive semidefinite at all z near 0. Compute:

$$\frac{\partial^2 f}{\partial \bar{z}_j \partial z_\ell} = \frac{1}{r^2}\frac{\partial r}{\partial \bar{z}_j}\frac{\partial r}{\partial z_\ell} - \frac{1}{r}\frac{\partial^2 r}{\partial \bar{z}_j \partial z_\ell} + A\delta_j^\ell,$$

where δ_j^ℓ is the Kronecker delta*. Apply the complex Hessian of f to c at $q \in U$ near the origin (recall that r is negative on U and so for $q \in U$, $-r = |r|$):

$$\begin{aligned}\sum_{j=1,\ell=1}^{n} \bar{c}_j c_\ell \frac{\partial^2 f}{\partial \bar{z}_j \partial z_\ell}\Big|_q &= \frac{1}{r^2}\left|\sum_{\ell=1}^{n} c_\ell \frac{\partial r}{\partial z_\ell}\Big|_q\right|^2 + \frac{1}{|r|}\sum_{j=1,\ell=1}^{n} \bar{c}_j c_\ell \frac{\partial^2 r}{\partial \bar{z}_j \partial z_\ell}\Big|_q + A\|c\|^2 \\ &\geq \frac{1}{r^2}\left|\sum_{\ell=1}^{n} c_\ell \frac{\partial r}{\partial z_\ell}\Big|_q\right|^2 - \frac{3^2 M}{|r|}\|c\|\left|\sum_{j=1}^{n} c_j \frac{\partial r}{\partial z_j}\Big|_q\right| + A\|c\|^2.\end{aligned}$$

Now comes a somewhat funky trick. As a quadratic polynomial in $\|c\|$, the right hand side of the inequality is always nonnegative if $A > 0$ and if the discriminant is negative or zero. Let us set the discriminant to zero:

$$0 = \left(\frac{3^2 M}{|r|}\left|\sum_{j=1}^{n} c_j \frac{\partial r}{\partial z_j}\Big|_q\right|\right)^2 - 4A\frac{1}{r^2}\left|\sum_{\ell=1}^{n} c_\ell \frac{\partial r}{\partial z_\ell}\Big|_q\right|^2.$$

All the nonconstant terms go away and $A = \frac{3^4 M^2}{4}$ makes the discriminant zero. Thus for that A,

$$\sum_{j=1,\ell=1}^{n} \bar{c}_j c_\ell \frac{\partial^2 f}{\partial \bar{z}_j \partial z_\ell}\Big|_q \geq 0.$$

In other words, the complex Hessian of f is positive semidefinite at all points $q \in U$ near 0. The function $f(z)$ goes to infinity as z approaches ∂U. So for every $t \in \mathbb{R}$, the t-sublevel set (the set where $f(z) < t$) must be a positive distance away from ∂U near 0.

We have a local continuous plurisubharmonic exhaustion function for U near p. If we intersect with a small ball B centered at p, then we get that $U \cap B$ is Hartogs pseudoconvex. This is true at all $p \in \partial U$, so U is Hartogs pseudoconvex. □

2.6 Holomorphic convexity

Definition 2.6.1. Let $U \subset \mathbb{C}^n$ be a domain. For a set $K \subset U$, define the *holomorphic hull*

$$\widehat{K}_U \overset{\text{def}}{=} \left\{ z \in U : |f(z)| \leq \sup_{w \in K} |f(w)| \text{ for all } f \in \mathcal{O}(U) \right\}.$$

A domain U is *holomorphically convex* if whenever $K \subset\subset U$, then $\widehat{K}_U \subset\subset U$. In other words, U is holomorphically convex if it is convex with respect to moduli of holomorphic functions on U.†

*Recall $\delta_j^\ell = 0$ if $j \neq \ell$ and $\delta_j^\ell = 1$ if $j = \ell$.

†Sometimes simply $\widehat{K}$ is used, but we use $\widehat{K}_U$ to emphasize the dependence on U.

It is a simple exercise (see below) to show that a holomorphically convex domain is Hartogs pseudoconvex. We will prove that holomorphic convexity is equivalent to being a domain of holomorphy. That a Hartogs pseudoconvex domain is holomorphically convex is the Levi problem for Hartogs pseudoconvex domains and is considerably more difficult. The thing is, there are lots of plurisubharmonic functions and they are easy to construct; we can even construct them locally and then piece them together by taking maxima. There are far fewer holomorphic functions, and we cannot just construct them locally and expect the pieces to somehow fit together. As it is so fundamental, let us state it as a theorem.

Theorem 2.6.2 (Solution of the Levi problem)**.** *A domain $U \subset \mathbb{C}^n$ is holomorphically convex if and only it is Hartogs pseudoconvex.*

Proof. The forward direction follows from an exercise below. We skip the proof of the backward direction in order to save some hundred pages or so. See Hörmander's book [H] for the proof. ◻

Exercise* 2.6.1*: *Prove that a holomorphically convex domain is Hartogs pseudoconvex. See Exercise 2.4.22.*

Exercise* 2.6.2*: *Prove that any domain $U \subset \mathbb{C}$ is holomorphically convex by giving a topological description of $\widehat{K}_U$ for any compact $K \subset\subset U$. Hint: Runge may be useful.*

Exercise* 2.6.3*: *Suppose $f : \mathbb{C}^n \to \mathbb{C}$ is holomorphic and U is a topological component of $\{ z \in \mathbb{C}^n : |f(z)| < 1 \}$. Prove that U is a holomorphically convex domain.*

Exercise* 2.6.4*: *Compute the hull $\widehat{K}_{\mathbb{D}^n}$ of the set $K = \{ z \in \mathbb{D}^n : |z_j| = \lambda_j \text{ for } j = 1, \ldots, n \}$, where $0 \leq \lambda_j < 1$. Prove that the unit polydisc is holomorphically convex.*

Exercise* 2.6.5*: *Prove that a geometrically convex domain $U \subset \mathbb{C}^n$ is holomorphically convex.*

Exercise* 2.6.6*: *Prove that the Hartogs figure (see Theorem 2.1.4) is not holomorphically convex.*

Exercise* 2.6.7*: *Let $U \subset \mathbb{C}^n$ be a domain and $f \in \mathcal{O}(U)$. Show that if U is holomorphically convex then $\widetilde{U} = \{ z \in U : f(z) \neq 0 \}$ is holomorphically convex. Hint: First see Exercise 1.6.2.*

Exercise* 2.6.8*: *Suppose $U, V \subset \mathbb{C}^n$ are biholomorphic domains. Prove that U is holomorphically convex if and only if V is holomorphically convex.*

Exercise* 2.6.9*: *In the definition of holomorphic hull of K, replace U with $\mathbb{C}^n$ and $\mathcal{O}(U)$ with holomorphic polynomials on $\mathbb{C}^n$, to get the* polynomial hull *of K. Prove that the polynomial hull of $K \subset\subset \mathbb{C}^n$ is the same as the holomorphic hull $\widehat{K}_{\mathbb{C}^n}$.*

Exercise* 2.6.10*:

a) Prove the Hartogs triangle T (see Exercise 2.1.7) is holomorphically convex.
b) Prove $T \cup B_\epsilon(0)$ (for a small enough $\epsilon > 0$) is not holomorphically convex.

Exercise* 2.6.11*: *Show that if domains $U_1 \subset \mathbb{C}^n$ and $U_2 \subset \mathbb{C}^n$ are holomorphically convex then so are all the topological components of $U_1 \cap U_2$.*

Exercise* 2.6.12*: *Let $n \geq 2$.*

a) Let $U \subset \mathbb{C}^n$ be a domain and $K \subset\subset U$ a nonempty compact subset. Show that $U \setminus K$ is not holomorphically convex.

b) Let $U \subset \mathbb{C}^n$ be a bounded holomorphically convex domain. Prove that $\mathbb{C}^n \setminus U$ is connected.

c) Find an unbounded holomorphically convex domain $U \subset \mathbb{C}^n$ where $\mathbb{C}^n \setminus U$ is disconnected.

Exercise* 2.6.13** (Behnke–Stein again):*** *Show that the union $\bigcup_j U_j$ of a nested sequence of holomorphically convex domains $U_{j-1} \subset U_j \subset \mathbb{C}^n$ is holomorphically convex.*

Using an above exercise we see that $\mathbb{C}^n$ is both holomorphically convex and a domain of holomorphy. In fact, these two notions are equivalent for all other domains in $\mathbb{C}^n$.

Theorem 2.6.3 (Cartan–Thullen)**.** *Let $U \subsetneq \mathbb{C}^n$ be a domain. The following are equivalent:*

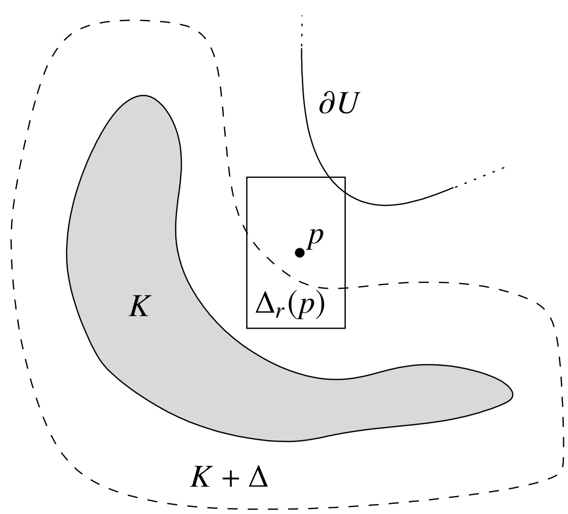
(i) U is a domain of holomorphy.

(ii) For all $K \subset\subset U$, $\operatorname{dist}(K, \partial U) = \operatorname{dist}(\widehat{K}_U, \partial U)$.

(iii) U is holomorphically convex.

Proof. Let us start with (i) $\Rightarrow$ (ii). Suppose there is a $K \subset\subset U$ with $\operatorname{dist}(K, \partial U) > \operatorname{dist}(\widehat{K}_U, \partial U)$. After possibly a rotation by a unitary, there exists a point $p \in \widehat{K}_U$ and a polydisc $\Delta = \Delta_r(0)$ with polyradius $r = (r_1, \ldots, r_n)$ such that $p + \Delta = \Delta_r(p)$ contains a point of ∂U, but

$$K + \Delta = \bigcup_{q \in K} \Delta_r(q) \subset\subset U.$$



If $f \in \mathcal{O}(U)$, then there is an $M > 0$ such that $|f| \leq M$ on $K + \Delta$ as that is a relatively compact set. By the Cauchy estimates for each $q \in K$ we get

$$\left| \frac{\partial^\alpha f}{\partial z^\alpha}(q) \right| \leq \frac{M \alpha!}{r^\alpha}.$$

This inequality therefore holds on $\widehat{K}_U$ and hence at p. The series

$$\sum_{\alpha} \frac{1}{\alpha!} \frac{\partial^\alpha f}{\partial z^\alpha}(p)(z-p)^\alpha$$

converges in $\Delta_r(p)$. Hence f extends to all of $\Delta_r(p)$ and $\Delta_r(p)$ contains points outside of U, in other words, U is not a domain of holomorphy.

The implication (ii) $\Rightarrow$ (iii) is immediate.

Finally we prove (iii) $\Rightarrow$ (i). Suppose U is holomorphically convex. Let $p \in \partial U$. By convexity choose nested compact sets $K_{j-1} \subsetneq K_j \subset\subset U$ such that $\bigcup_j K_j = U$, and $\widehat{(K_j)}_U = K_j$. As the sets exhaust U, we can perhaps pass to a subsequence to ensure that there exists a sequence of points $p_j \in K_j \setminus K_{j-1}$ such that $\lim_{j\to\infty} p_j = p$.

As p_j is not in the hull of K_{j-1}, there is a function $f_j \in \mathcal{O}(U)$ such that $|f_j| < 2^{-j}$ on K_{j-1}, but

$$|f_j(p_j)| > j + \left| \sum_{k=1}^{j-1} f_k(p_j) \right| .$$

Finding such a function is left as an exercise below. For any j, the series $\sum_{k=1}^{\infty} f_k(z)$ converges uniformly on K_j as for all $k > j$, $|f_k| < 2^{-k}$ on K_j. As the K_j exhaust U, the series converges uniformly on compact subsets of U. Consequently,

$$f(z) = \sum_{k=1}^{\infty} f_k(z)$$

is a holomorphic function on U. We bound

$$|f(p_j)| \geq |f_j(p_j)| - \left| \sum_{k=1}^{j-1} f_k(p_j) \right| - \left| \sum_{k=j+1}^{\infty} f_k(p_j) \right| \geq j - \sum_{k=j+1}^{\infty} 2^{-k} \geq j-1.$$

So $\lim_{j\to\infty} f(p_j) = \infty$. Clearly there cannot be any open $W \subset \mathbb{C}^n$ containing p to which f extends (see definition of domain of holomorphy). As any connected open W such that $W \setminus U \neq \emptyset$ must contain a point of ∂U, we are done. □

By Exercise 2.6.8, holomorphic convexity is a biholomorphic invariant. Thus, being a domain of holomorphy is also a biholomorphic invariant. This fact is not easy to prove from the definition of a domain of holomorphy, as the biholomorphism is defined only on the interior of our domains.

Holomorphic convexity is an intrinsic notion; that is, it does not require knowing anything about points outside of U. Therefore, it is a much better way to think about domains of holomorphy. In fact holomorphic convexity generalizes easily to more complicated complex manifolds*, while the notion of a domain of holomorphy only makes sense for domains in $\mathbb{C}^n$.

*Manifolds with complex structure, that is, "manifolds with multiplication by i on the tangent space."

Exercise* 2.6.14*: *Prove the existence of the function $f_j \in \mathcal{O}(U)$ as indicated in the proof above.*

Exercise* 2.6.15*: *Extend the proof to show that if $U \subset \mathbb{C}^n$ is holomorphically convex then there exists a single function $f \in \mathcal{O}(U)$, that does not extend through any point $p \in \partial U$.*

Exercise* 2.6.16*: *We know $U = \mathbb{C}^2 \setminus \{z \in \mathbb{C}^2 : z_1 = 0\}$ is a domain of holomorphy. Use part (ii) of the theorem to show that if $W \subset \mathbb{C}^2$ is a domain of holomorphy and $U \subset W$, then either $W = U$ or $W = \mathbb{C}^2$. Hint: Suppose $L \subset W$ is a complex line and K is a circle in L. What is $\widehat{K}_W$?*

In the following series of exercises, which you should most definitely do in sequence, you will solve the Levi problem (and more) for complete Reinhardt domains. Recall that a domain U is a complete Reinhardt domain if whenever $(z_1, \ldots, z_n)$ is in U and $r_j = |z_j|$, then the entire closed polydisc $\overline{\Delta_r(0)} \subset U$. We say a complete Reinhardt domain U is *logarithmically convex* if there exists a (geometrically) convex $C \subset \mathbb{R}^n$ such that $z \in U$ if and only if $(\log|z_1|, \ldots, \log|z_n|) \in C$.

Exercise* 2.6.17*: *Prove that a logarithmically convex complete Reinhardt domain is the intersection of sets of the form*

$$\left\{z \in \mathbb{C}^n : \alpha_1 \log|z_1| + \cdots + \alpha_n \log|z_n| < \beta\right\} = \left\{z \in \mathbb{C}^n : |z_1|^{\alpha_1} \cdots |z_n|^{\alpha_n} < e^{\beta}\right\}$$

for some nonnegative $\alpha_1, \ldots, \alpha_n$, and $\beta \in \mathbb{R}$.

Exercise* 2.6.18*: *Prove that a complete Reinhardt domain that is Hartogs pseudoconvex is logarithmically convex.*

Exercise* 2.6.19*: *For each $k \in \mathbb{N}_0$, let $\ell_m^k \in \mathbb{N}_0$ be the smallest nonnegative integer such that $\ell_m^k \geq k\alpha_m$. Prove that the domain of convergence of the power series*

$$\sum_{k=0}^{\infty} e^{-k\beta} z_1^{\ell_1^k} \cdots z_n^{\ell_n^k}$$

is precisely the set $\left\{z \in \mathbb{C}^n : |z_1|^{\alpha_1} \cdots |z_n|^{\alpha_n} < e^{\beta}\right\}$. Hint: That it diverges outside is easy, what's hard is that it converges inside. Perhaps useful is to notice $\frac{\ell_m^k}{k} - \alpha_m \leq \frac{1}{k}$, and furthermore notice that if z is in the set, there is some $\epsilon > 0$ such that $(1+\epsilon)|z_1|^{\alpha_1} \cdots |z_n|^{\alpha_n} = e^{\beta}$.

Exercise* 2.6.20*: *Prove that a logarithmically convex Reinhardt domain is holomorphically convex and therefore it is a domain of holomorphy.*

Exercise* 2.6.21*: *Prove that a complete Reinhardt domain is a domain of holomorphy if and only if it is the domain of convergence of some power series at the origin. Hint: There is a function that does not extend past any boundary point of a holomorphically convex domain.*

We (you) have proved the following proposition.

Proposition 2.6.4. *Let $U \subset \mathbb{C}^n$ be a complete Reinhardt domain. Then the following are equivalent:*

(i) U is logarithmically convex.

(ii) U is a domain of holomorphy.

(iii) U is a domain of convergence of some power series at the origin.

(iv) U is Hartogs pseudoconvex.

Chapter 3

CR functions

3.1 Real-analytic functions and complexification

Definition 3.1.1. Let $U \subset \mathbb{R}^n$ be open. A function $f : U \to \mathbb{C}$ is *real-analytic* (or simply *analytic* if clear from context) if at each point $p \in U$, the function f has a convergent power series that converges (absolutely) to f in some neighborhood of p. A common notation for real-analytic is C^ω.

Before we discuss the connection to holomorphic functions let us prove a simple lemma.

Lemma 3.1.2. *Let $\mathbb{R}^n \subset \mathbb{C}^n$ be the natural inclusion and $V \subset \mathbb{C}^n$ a domain such that $V \cap \mathbb{R}^n \neq \emptyset$. Suppose $f, g : V \to \mathbb{C}$ are holomorphic functions such that $f = g$ on $V \cap \mathbb{R}^n$. Then $f = g$ on V.*

Proof. Considering $f - g$ we may assume that $g = 0$. Let $z = x + iy$ as usual so that $\mathbb{R}^n$ is given by $y = 0$. Our assumption is that $f = 0$ when $y = 0$, so the derivative of f with respect to x_j is zero. So when $y = 0$, the Cauchy–Riemann equations say

$$0 = \frac{\partial f}{\partial x_j} = -i \frac{\partial f}{\partial y_j}.$$

Therefore, on $y = 0$,

$$\frac{\partial f}{\partial z_j} = 0.$$

The derivative $\frac{\partial f}{\partial z_j}$ is holomorphic and $\frac{\partial f}{\partial z_j} = 0$ on $y = 0$. Then by induction all derivatives of f at $p \in \mathbb{R}^n \cap V$ vanish, it has a zero power series. Hence f is identically zero in a neighborhood of p in $\mathbb{C}^n$ and by the identity theorem it is zero on all of V. $\square$

Let us return to $\mathbb{R}^n$ for a moment. We write a power series in $\mathbb{R}^n$ in multinomial notation as usual. Suppose that for some $a \in \mathbb{R}^n$ and some polyradius $r = (r_1, \ldots, r_n)$, the series

$$\sum_\alpha c_\alpha (x - a)^\alpha$$

converges whenever $|x_j - a_j| \leq r_j$ for all j. Here convergence is absolute convergence. That is,

$$\sum_\alpha |c_\alpha| \, |x - a|^\alpha$$

converges. If we replace $x_j \in \mathbb{R}$ with $z_j \in \mathbb{C}$ such that $|z_j - a_j| \leq |x_j - a_j|$, then the series still converges. Hence the series

$$\sum_{\alpha} c_\alpha (z-a)^\alpha$$

converges absolutely in $\Delta_r(a) \subset \mathbb{C}^n$.

Proposition 3.1.3 (Complexification part I)**.** *Suppose $U \subset \mathbb{R}^n$ is a domain and $f : U \to \mathbb{C}$ is real-analytic. Let $\mathbb{R}^n \subset \mathbb{C}^n$ be the natural inclusion. Then there exists a domain $V \subset \mathbb{C}^n$ such that $U \subset V$ and a unique holomorphic function $F : V \to \mathbb{C}$ such that $F|_U = f$.*

In particular, among many other things that follow from this proposition, we can now conclude that a real-analytic function is C^∞. Be careful and notice that U is a domain in $\mathbb{R}^n$, but it is not an open set when considered as a subset of $\mathbb{C}^n$. Furthermore, V may be a very "thin" neighborhood around U. There is no way of finding V just from knowing U. You need to also know f. As an example, consider $f(x) = \frac{1}{\epsilon^2+x^2}$ for $\epsilon > 0$, which is real-analytic on $\mathbb{R}$, but the complexification is not holomorphic at $\pm \epsilon i$.

Proof. We already proved the local version. But we must prove that if we extend our f near every point, we always get the same function. That follows from the lemma above; any two such functions are equal on $\mathbb{R}^n$, and hence equal. There is a subtle topological technical point in this, so let us elaborate. A key topological fact is that we define V as a union of the polydiscs where the series converges. If a point p is in two different such polydiscs, we need to show that the two definitions of F are the same at p. But the intersection of two polydiscs is always connected, and in this case contains a piece of $\mathbb{R}^n$ as well, and we may apply the lemma above. □

Exercise* 3.1.1*: *Prove the identity theorem for real-analytic functions. That is, if $U \subset \mathbb{R}^n$ is a domain, $f : U \to \mathbb{R}$ a real-analytic function and f is zero on a nonempty open subset of U, then f is identically zero.*

Exercise* 3.1.2*: *Suppose $U \subset \mathbb{R}^n$ is a domain and $f : U \to \mathbb{R}$ a real-analytic function. Suppose that $W \subset U$ is a nonempty open subset and $f|_W$ is harmonic. Prove that f is harmonic.*

Exercise* 3.1.3*: *Let $(0,1) \subset \mathbb{R}$. Construct an fixed real-analytic function on $(0,1)$ that does not complexify to the rectangle $(0,1) + i(-\epsilon,\epsilon) \subset \mathbb{C}$ for any $\epsilon > 0$. Why does this not contradict the proposition?*

Recall that a polynomial $P(x)$ in n real variables $(x_1,\ldots,x_n)$ is homogeneous of degree d if $P(sx) = s^d P(x)$ for all $s \in \mathbb{R}$ and $x \in \mathbb{R}^n$. That is, a homogeneous polynomial of degree d is a polynomial whose every monomial is of total degree d. If f is real-analytic near $a \in \mathbb{R}^n$, then write the power series of f at a as

$$\sum_{j=0}^{\infty} f_j(x-a),$$

where f_j is a homogeneous polynomial of degree j. The f_d is called the *degree d homogeneous part* of f at a.

There is usually a better way to complexify real-analytic functions in $\mathbb{C}^n$. Suppose $U \subset \mathbb{C}^n \cong \mathbb{R}^{2n}$, and $f : U \to \mathbb{C}$ is real-analytic. Assume $a = 0 \in U$ for simplicity. Writing $z = x + iy$, near 0,

$$f(x,y) = \sum_{j=0}^{\infty} f_j(x,y) = \sum_{j=0}^{\infty} f_j\left(\frac{z+\bar{z}}{2}, \frac{z-\bar{z}}{2i}\right).$$

The polynomial f_j becomes a homogeneous polynomial of degree j in the variables z and $\bar{z}$. The series becomes a power series in z and $\bar{z}$. As mentioned before, we simply write the function as $f(z,\bar{z})$, and we consider the power series representation in z and $\bar{z}$ rather than in x and y. In multinomial notation we write a power series at $a \in \mathbb{C}^n$ as

$$\sum_{\alpha,\beta} c_{\alpha,\beta}(z-a)^{\alpha}(\bar{z}-\bar{a})^{\beta}.$$

Notice that a holomorphic function is real-analytic, but not vice-versa. A holomorphic function is a real-analytic function that does not depend on $\bar{z}$.

Before we discuss complexification in terms of z and $\bar{z}$ we need the following lemma.

Lemma 3.1.4. *Let $V \subset \mathbb{C}^n \times \mathbb{C}^n$ be a domain, let the coordinates be $(z,\zeta) \in \mathbb{C}^n \times \mathbb{C}^n$, let*

$$D = \left\{(z,\zeta) \in \mathbb{C}^n \times \mathbb{C}^n : \zeta = \bar{z}\right\},$$

and suppose $D \cap V \neq \emptyset$. Suppose $f, g : V \to \mathbb{C}$ are holomorphic functions such that $f = g$ on $D \cap V$. Then $f = g$ on all of V.

The set D is sometimes called the *diagonal*.

Proof. Again assume without loss of generality that $g = 0$. Whenever $(z,\bar{z}) \in V$, we have $f(z,\bar{z}) = 0$, which is really f composed with the map that takes z to $(z,\bar{z})$. Using the chain rule

$$0 = \frac{\partial}{\partial \bar{z}_j}\Big[f(z,\bar{z})\Big] = \frac{\partial f}{\partial \zeta_j}(z,\bar{z}).$$

Let us do this again with the z_j

$$0 = \frac{\partial}{\partial z_j}\Big[f(z,\bar{z})\Big] = \frac{\partial f}{\partial z_j}(z,\bar{z}).$$

Either way, we get another holomorphic function in z and ζ that is zero on D. By induction, for all α and β we get

$$0 = \frac{\partial^{|\alpha|+|\beta|}}{\partial z^{\alpha}\partial \bar{z}^{\beta}}\Big[f(z,\bar{z})\Big] = \frac{\partial^{|\alpha|+|\beta|} f}{\partial z^{\alpha}\partial \zeta^{\beta}}(z,\bar{z}).$$

All holomorphic derivatives in z and ζ of f are zero on every point $(z,\bar{z})$, so the power series is zero at every point $(z,\bar{z})$, and so f is identically zero in a neighborhood of any point $(z,\bar{z})$. The lemma follows by the identity theorem. □

Let f be a real-analytic function. Suppose the series (in multinomial notation)

$$f(z,\bar{z}) = \sum_{\alpha,\beta} c_{\alpha,\beta}(z-a)^\alpha(\bar{z}-\bar{a})^\beta$$

converges in a polydisc $\Delta_r(a) \subset \mathbb{C}^n$. By convergence we mean absolute convergence as we discussed before: that is,

$$\sum_{\alpha,\beta} |c_{\alpha,\beta}|\,|z-a|^\alpha|\bar{z}-\bar{a}|^\beta$$

converges. The series still converges if we replace $\bar{z}_j$ with ζ_j where $|\zeta_j - \bar{a}| \leq |\bar{z}_j - \bar{a}|$. So the series

$$F(z,\zeta) = \sum_{\alpha,\beta} c_{\alpha,\beta}(z-a)^\alpha(\zeta-\bar{a})^\beta$$

converges for all $(z,\zeta) \in \Delta_r(a) \times \Delta_r(\bar{a})$.

Putting together the above discussion with the lemma we obtain:

Proposition 3.1.5 (Complexification part II)**.** *Suppose $U \subset \mathbb{C}^n$ is a domain and $f\colon U \to \mathbb{C}$ is real-analytic. Then there exists a domain $V \subset \mathbb{C}^n \times \mathbb{C}^n$ such that*

$$\bigl\{(z,\zeta) : \zeta = \bar{z} \text{ and } z \in U\bigr\} \subset V,$$

and a unique holomorphic function $F\colon V \to \mathbb{C}$ such that $F(z,\bar{z}) = f(z,\bar{z})$ for all $z \in U$.

The function f can be thought of as the restriction of F to the set where $\zeta = \bar{z}$. We will abuse notation and write simply $f(z,\zeta)$ both for f and its extension. The reason for this abuse is evident from the computations above. What we are calling f is a function of $(z,\bar{z})$ if thinking of it as a function on the diagonal where $\zeta = \bar{z}$, or it is a function of z if thinking of it as just the function $z \mapsto f(z,\bar{z})$, or it is the function $(z,\zeta) \mapsto f(z,\zeta)$. We have the following commutative diagram:

$$\begin{tikzcd}
U \subset \mathbb{C}^n \arrow[rr, "z \mapsto (z,\bar{z})"] \arrow[dr, "f"'] & & V \subset \mathbb{C}^n \times \mathbb{C}^n \arrow[dl, "f\ \ (=F)"] \\
& \mathbb{C} &
\end{tikzcd}$$

All three ways of going from one place to another in the diagram we are calling f. The arrow from V was called F in the proposition. The notation plays well with differentiation and the Wirtinger operators. Differentiating f (really the F in the proposition) in ζ_j and evaluating at $(z,\bar{z})$ is the same thing as evaluating at $(z,\bar{z})$ and then differentiating in $\bar{z}_j$ using the Wirtinger operator:

$$\frac{\partial F}{\partial \zeta_j}(z,\bar{z}) = \frac{\partial f}{\partial \zeta_j}(z,\bar{z}) = \frac{\partial}{\partial \bar{z}_j}\Bigl[f(z,\bar{z})\Bigr] = \frac{\partial f}{\partial \bar{z}_j}(z,\bar{z}).$$

If we squint our mind's eye, we can't quite see the difference between $\bar{z}$ and ζ. We already used this idea for smooth functions, but for real-analytic functions we can treat z and $\bar{z}$ as truly independent variables. So the abuse of notation is entirely justified, at least once it is understood well.

Remark 3.1.6. The domain V in the proposition is not simply U times the conjugate of U. In general, it is much smaller. For example, a real-analytic $f: \mathbb{C}^n \to \mathbb{C}$ does not necessarily complexify to all of $\mathbb{C}^n \times \mathbb{C}^n$. That is because the domain of convergence for a real-analytic function on $\mathbb{C}^n$ is not necessarily all of $\mathbb{C}^n$. In one dimension, the function

$$f(z, \bar{z}) = \frac{1}{1 + |z|^2}$$

is real-analytic on $\mathbb{C}$, but it is not a restriction to the diagonal of a holomorphic function on all of $\mathbb{C}^2$. The problem is that the complexified function

$$f(z, \zeta) = \frac{1}{1 + z\zeta}$$

is undefined on the set where $z\zeta = -1$, which by a fluke never happens when $\zeta = \bar{z}$.

Remark 3.1.7. This form of complexification is sometimes called *polarization* due to its relation to the polarization identities*. That is, suppose A is a Hermitian matrix, we recover A and therefore the sesquilinear form $\langle Az, w \rangle$ for $z, w \in \mathbb{C}^n$, by simply knowing the values of

$$\langle Az, z \rangle = z^* Az = \sum_{j,k=1}^{n} a_{jk} \, \bar{z}_j z_k$$

for all $z \in \mathbb{C}^n$. In fact, under the hood Proposition 3.1.5 is polarization in an infinite-dimensional Hilbert space, but we digress.

The idea of treating $\bar{z}$ as a separate variable is very powerful, and as we have just seen it is completely natural when speaking about real-analytic functions. This is one of the reasons why real-analytic functions play a special role in several complex variables.

***Exercise* 3.1.4:** *Let $U \subset \mathbb{C}^n$ be an open set and $\varphi: U \to \mathbb{R}$ a pluriharmonic function. Prove that φ is real-analytic.*

***Exercise* 3.1.5:** *Let $U \subset \mathbb{C}^n$ be an open set, $z_0 \in U$. Suppose $\varphi: U \to \mathbb{R}$ is a pluriharmonic function. You know that φ is real-analytic. Using complexification, write down a formula for a holomorphic function near z_0 whose real part is φ.*

***Exercise* 3.1.6:** *Let $U \subset \mathbb{C}^n$ be a domain, and suppose $f, g \in \mathcal{O}(U)$. Suppose that $f = \bar{g}$ on U. Use complexification (complexify $f - \bar{g}$) to show that both f and g are constant.*

Example 3.1.8: Not every C^∞ smooth function is real-analytic. For example, on the real line

$$f(x) = \begin{cases} e^{-1/x^2} & \text{if } x > 0, \\ 0 & \text{if } x \leq 0. \end{cases}$$

The function $f: \mathbb{R} \to \mathbb{R}$ is C^∞ and $f^{(k)}(0) = 0$ for all k. So the Taylor series of f at the origin does not converge to f in any neighborhood of the origin; it converges to the zero function but not to f. Because of this, there is no neighborhood V of the origin in $\mathbb{C}$ such that f is the restriction to $V \cap \mathbb{R}$ of a holomorphic function in V.

*Such as $4\langle z, w \rangle = \|z + w\|^2 - \|z - w\|^2 + i\big(\|z + iw\|^2 - \|z - iw\|^2\big)$.

Exercise* 3.1.7*: *Prove the statements of the above example.*

Definition 3.1.9. A real hypersurface $M \subset \mathbb{R}^n$ is said to be real-analytic if locally at every point it is the graph of a real-analytic function. That is near every point (locally), after perhaps relabeling coordinates M can be written as a graph

$$y = \varphi(x),$$

where φ is real-analytic, $(x, y) \in \mathbb{R}^{n-1} \times \mathbb{R} = \mathbb{R}^n$.

Compare this definition to Definition 2.2.1. In fact, we could define a real-analytic hypersurface as in Definition 2.2.1 and then prove an analogue of Lemma 2.2.8 to show that this would be identical to the definition above. The definition we gave is sufficient and so we avoid the complication and leave it to the interested reader.

Exercise* 3.1.8*: *Show that the definition above is equivalent to an analogue of Definition 2.2.1. That is, state the alternative definition of real-analytic hypersurface and then prove the analogue of Lemma 2.2.8.*

A mapping to $\mathbb{R}^m$ is real-analytic if all the components are real-analytic functions. Via complexification we give a simple proof of the following result.

Proposition 3.1.10. *Let $U \subset \mathbb{R}^n$, $V \subset \mathbb{R}^k$ be open and let $f : U \to V$ and $g : V \to \mathbb{R}^m$ be real-analytic. Then $g \circ f$ is real-analytic.*

Proof. Let $x \in \mathbb{R}^n$ be our coordinates in U and $y \in \mathbb{R}^k$ be our coordinates in V. We complexify $f(x)$ and $g(y)$ by allowing x to be a complex vector in a small neighborhood of U in $\mathbb{C}^n$ and y to be a complex vector in a small neighborhood of V in $\mathbb{C}^k$. So we treat f and g as holomorphic functions. On a certain neighborhood of U in $\mathbb{C}^n$, the composition $f \circ g$ makes sense and it is holomorphic as composition of holomorphic mappings is holomorphic. Restricting the complexified $f \circ g$ back to $\mathbb{R}^n$ we obtain a real-analytic function. □

The proof demonstrates a very simple application of complexification. Many properties of holomorphic functions are easy to prove because holomorphic functions are solutions to certain PDE (the Cauchy–Riemann equations). There is no PDE that defines real-analytic functions, so complexification provides a useful tool to transfer certain properties of holomorphic functions to real-analytic functions. We must be careful, however. Hypotheses on real-analytic functions only give us hypotheses on certain points of the complexified holomorphic functions.

Exercise* 3.1.9*: *Demonstrate the point about complexification we made just above. Find a nonconstant bounded real-analytic $f : \mathbb{R}^n \to \mathbb{R}$, that happens to complexify to $\mathbb{C}^n$.*

Exercise* 3.1.10*: *Let $U \subset \mathbb{R}^n$ be open. Let $\varphi : (0, 1) \to U$ be a real-analytic function (curve), and let $f : U \to \mathbb{R}$ be real-analytic. Suppose that $(f \circ \varphi)(t) = 0$ for all $t \in (0, \epsilon)$ for some $\epsilon > 0$. Prove that f is zero on the image $\varphi\big((0, 1)\big)$.*

3.2 CR functions

We first need to know what it means for a function $f\colon X \to \mathbb{C}$ to be smooth if X is not an open set, for example, if X is a hypersurface.

Definition 3.2.1. Let $X \subset \mathbb{R}^n$ be a set. The function $f\colon X \to \mathbb{C}$ is smooth (resp. real-analytic) if for each point $p \in X$ there is a neighborhood $U \subset \mathbb{R}^n$ of p and a smooth (resp. real-analytic) $F\colon U \to \mathbb{C}$ such that $F(q) = f(q)$ for $q \in X \cap U$.

For an arbitrary set X, issues surrounding this definition can be very subtle. It is very natural, however, if X is nice, such as a hypersurface, or if X is a closure of a domain with smooth boundary.

Proposition 3.2.2. *If $M \subset \mathbb{R}^n$ is a smooth (resp. real-analytic) real hypersurface, then $f\colon M \to \mathbb{C}$ is smooth (resp. real-analytic) if and only if whenever near any point if we write M in coordinates $(x, y) \in \mathbb{R}^{n-1} \times \mathbb{R}$ as*

$$y = \varphi(x)$$

for a smooth (resp. real-analytic) function φ, then the function $f\big(x, \varphi(x)\big)$ is a smooth (resp. real-analytic) function of x.

***Exercise* 3.2.1:** *Prove the proposition.*

***Exercise* 3.2.2:** *Prove that if M is a smooth or real-analytic hypersurface, and $f\colon M \to \mathbb{C}$ is smooth or real-analytic, then the function F from the definition is never unique, even for a fixed neighborhood U.*

***Exercise* 3.2.3:** *Suppose $M \subset \mathbb{R}^n$ is a smooth hypersurface, $f\colon M \to \mathbb{C}$ is a smooth function, $p \in M$, and $X_p \in T_pM$. Prove that X_p is well-defined. That is, suppose U is a neighborhood of p, $F\colon U \to \mathbb{C}$ and $G\colon U \to \mathbb{C}$ are smooth functions that both equal f on $U \cap M$. Prove that $X_pF = X_pG$.*

Due to the last exercise, we can apply vectors of T_pM to a smooth function on a hypersurface by simply applying them to any smooth extension. Similarly we can apply vectors of $\mathbb{C} \otimes T_pM$ to smooth function on M, as $\mathbb{C} \otimes T_pM$ is simply the complex span of vectors in T_pM.

Definition 3.2.3. Let $M \subset \mathbb{C}^n$ be a smooth real hypersurface. Then a smooth function $f\colon M \to \mathbb{C}$ is a *smooth CR function* if

$$X_p f = 0$$

for all $p \in M$ and all vectors $X_p \in T_p^{(0,1)}M$.

Remark 3.2.4. One only needs one derivative (rather than C^∞) in the above definition. One can even define a continuous CR function if the derivative is taken in the distribution sense, but we digress.

Remark 3.2.5. When $n = 1$, a real hypersurface $M \subset \mathbb{C}$ is a curve and $T_p^{(0,1)}M$ is trivial. Therefore, all functions $f\colon M \to \mathbb{C}$ are CR functions.

Proposition 3.2.6. *Let $M \subset U$ be a smooth (resp. real-analytic) real hypersurface in an open $U \subset \mathbb{C}^n$. Suppose $F: U \to \mathbb{C}$ is a holomorphic function, then the restriction $f = F|_M$ is a smooth (resp. real-analytic) CR function.*

Proof. First let us prove that f is smooth. The function F is smooth and defined on a neighborhood of any point, and so it can be used in the definition. Similarly for real-analytic.

Let us show f is CR at some $p \in M$. Differentiating f with vectors in $\mathbb{C} \otimes T_p M$ is the same as differentiating F. As $T_p^{(0,1)} M \subset T_p^{(0,1)} \mathbb{C}^n$ we have

$$X_p f = X_p F = 0 \qquad \text{for all} \quad X_p \in T_p^{(0,1)} M. \qquad \square$$

On the other hand, not every smooth CR function is a restriction of a holomorphic function.

Example 3.2.7: Take the smooth function $f: \mathbb{R} \to \mathbb{R}$ we defined before that is not real-analytic at the origin. Take $M \subset \mathbb{C}^2$ be the set defined by $\operatorname{Im} z_2 = 0$. M is a real-analytic real hypersurface. Clearly $T_p^{(0,1)} M$ is one-complex-dimensional, and at each $p \in M$, $\frac{\partial}{\partial \bar{z}_1}\big|_p$ is tangent and spans $T_p^{(0,1)} M$. Define $g: M \to \mathbb{C}$ by

$$g(z_1, z_2, \bar{z}_1, \bar{z}_2) = f(\operatorname{Re} z_2).$$

Then g is CR as it is independent of $\bar{z}_1$. If $G: U \subset \mathbb{C}^2 \to \mathbb{C}$ is a holomorphic function where U is some open set containing the origin, then G restricted to M must be real-analytic (a power series in $\operatorname{Re} z_1$, $\operatorname{Im} z_1$, and $\operatorname{Re} z_2$) and therefore G cannot equal to g on M.

Exercise* 3.2.4*: *Suppose $M \subset \mathbb{C}^n$ is a smooth hypersurface and $f: M \to \mathbb{C}$ is a CR function that is a restriction of a holomorphic function $F: U \to \mathbb{C}$ defined in some neighborhood $U \subset \mathbb{C}^n$ of M. Show that F is unique, that is if $G: U \to \mathbb{C}$ is another holomorphic function such that $G|_M = f = F|_M$, then $G = F$.*

Exercise* 3.2.5*: *Show that there is no maximum principle of CR functions. In fact, find a smooth hypersurface $M \subset \mathbb{C}^n$, $n \geq 2$, and a smooth CR function f on M such that $|f|$ attains a strict maximum at a point.*

Exercise* 3.2.6*: *Suppose $M \subset \mathbb{C}^n$, $n \geq 2$, is the hypersurface given by* $\operatorname{Im} z_n = 0$. *Show that any smooth CR function on M is holomorphic in the variables $z_1, \ldots, z_{n-1}$. Use this to show that for no smooth CR function f on M can $|f|$ attain a* strict *maximum on M. But show that there do exist nonconstant functions such that $|f|$ attains a (nonstrict) maximum M.*

Real-analytic CR functions on a real-analytic hypersurface M always extend to holomorphic functions of a neighborhood of M. To prove this we wish to complexify everything, that is treat the zs and $\bar{z}$s as separate variables. The standard way of writing a hypersurface as a graph is not as convenient for this setting, so let us prove that for a real-analytic hypersurface, we can write it as a graph of a holomorphic function in the complexified variables. That is, using variables (z, w), we will write M as a graph of $\bar{w}$ over z, $\bar{z}$, and w. This allows us to easily eliminate $\bar{w}$ in any real-analytic expression.

Proposition 3.2.8. *Suppose $M \subset \mathbb{C}^n$ is a real-analytic hypersurface and $p \in M$. Then after a translation and rotation by a unitary matrix, $p = 0$, and near the origin in coordinates $(z, w) \in \mathbb{C}^{n-1} \times \mathbb{C}$, the hypersurface M is given by*

$$\bar{w} = \Phi(z, \bar{z}, w),$$

where $\Phi(z, \zeta, w)$ is a holomorphic function defined on a neighborhood of the origin in $\mathbb{C}^{n-1} \times \mathbb{C}^{n-1} \times \mathbb{C}$, such that Φ, $\frac{\partial \Phi}{\partial z_j}$, $\frac{\partial \Phi}{\partial \zeta_j}$ vanish at the origin for all j, and $w = \bar{\Phi}\bigl(\zeta, z, \Phi(z, \zeta, w)\bigr)$ for all z, ζ, and w.

A local basis for $T^{(0,1)}M$ vector fields is given by

$$\frac{\partial}{\partial \bar{z}_j} + \frac{\partial \Phi}{\partial \bar{z}_j} \frac{\partial}{\partial \bar{w}} \qquad \left(= \frac{\partial}{\partial \bar{z}_j} + \frac{\partial \Phi}{\partial \zeta_j} \frac{\partial}{\partial \bar{w}} \right), \qquad j = 1, \ldots, n-1.$$

Finally, let $\mathcal{M}$ be the set in $(z, \zeta, w, \omega) \in \mathbb{C}^{n-1} \times \mathbb{C}^{n-1} \times \mathbb{C} \times \mathbb{C}$ coordinates given near the origin by $\omega = \Phi(z, \zeta, w)$. Then $\mathcal{M}$ is the unique complexification *of M near the origin in the sense that if $f(z, \bar{z}, w, \bar{w})$ is a real-analytic function vanishing on M near the origin, then $f(z, \zeta, w, \omega)$ vanishes on $\mathcal{M}$ near the origin.*

Again as a slight abuse of notation Φ refers to both the function $\Phi(z, \zeta, w)$ and $\Phi(z, \bar{z}, w)$.

Proof. Translate and rotate such that M is given by

$$\operatorname{Im} w = \varphi(z, \bar{z}, \operatorname{Re} w),$$

where φ is $O(2)$. Write the defining function as $r(z, w, \bar{z}, \bar{w}) = -\frac{w - \bar{w}}{2i} + \varphi\bigl(z, \bar{z}, \frac{w + \bar{w}}{2}\bigr)$. Complexifying, consider $r(z, w, \zeta, \omega)$ as a holomorphic function of $2n$ variables, and let $\mathcal{M}$ be the set defined by $r(z, w, \zeta, \omega) = 0$. The derivative of r in ω (that is $\bar{w}$) does not vanish near the origin. Use the implicit function theorem for holomorphic functions to write $\mathcal{M}$ near the origin as

$$\omega = \Phi(z, \zeta, w).$$

Restrict to the diagonal, $\bar{w} = \omega$ and $\bar{z} = \zeta$, to get $\bar{w} = \Phi(z, \bar{z}, w)$. This is order 2 in the z and the $\bar{z}$ since φ is $O(2)$.

Because r is real-valued, then $r(z, \bar{z}, w, \bar{w}) = \overline{r(z, \bar{z}, w, \bar{w})} = \bar{r}(\bar{z}, z, \bar{w}, w)$. Complexify to obtain $r(z, \zeta, w, \omega) = \bar{r}(\zeta, z, \omega, w)$ for all (z, ζ, w, ω) near the origin. If $r(z, \zeta, w, \omega) = 0$, then

$$0 = \overline{r(z, \zeta, w, \omega)} = \overline{\bar{r}(\zeta, z, \omega, w)} = r(\bar{\zeta}, \bar{z}, \bar{\omega}, \bar{w}) = 0.$$

So, $(z, \zeta, w, \omega) \in \mathcal{M}$ if and only if $(\bar{\zeta}, \bar{z}, \bar{\omega}, \bar{w}) \in \mathcal{M}$. Near the origin, $(z, \zeta, w, \omega) \in \mathcal{M}$ if and only if $\omega = \Phi(z, \zeta, w)$, and hence if and only if $\bar{w} = \Phi(\bar{\zeta}, \bar{z}, \bar{\omega})$. Conjugating, we get that $\mathcal{M}$ is also given by

$$w = \bar{\Phi}(\zeta, z, \omega).$$

As $\bigl(z, \zeta, w, \Phi(z, \zeta, w)\bigr) \in \mathcal{M}$, then for all z, ζ, and w,

$$w = \bar{\Phi}\bigl(\zeta, z, \Phi(z, \zeta, w)\bigr).$$

The vector field $X_j = \frac{\partial}{\partial \bar{z}_j} + \frac{\partial \Phi}{\partial \bar{z}_j} \frac{\partial}{\partial \omega}$ annihilates the function $\Phi(z,\bar{z},w) - \bar{w}$, but that is not enough. The vector field must annihilate a real defining function such as the real part of $\Phi(z,\bar{z},w) - \bar{w}$. So X_j must also annihilate the conjugate $\bar{\Phi}(\bar{z},z,\bar{w}) - w$, at least on M. Compute, for $(z,w) \in M$,

$$\begin{aligned} X_j \left[\bar{\Phi}(\bar{z},z,\bar{w}) - w \right] &= \frac{\partial \bar{\Phi}}{\partial \bar{z}_j}(\bar{z},z,\bar{w}) + \frac{\partial \Phi}{\partial \bar{z}_j}(z,\bar{z},w) \frac{\partial \bar{\Phi}}{\partial \bar{w}}(\bar{z},z,\bar{w}) \\ &= \frac{\partial \bar{\Phi}}{\partial \bar{z}_j}\big(\bar{z},z,\Phi(z,\bar{z},\bar{w})\big) + \frac{\partial \Phi}{\partial \bar{z}_j}(z,\bar{z},w) \frac{\partial \bar{\Phi}}{\partial \bar{w}}\big(\bar{z},z,\Phi(z,\bar{z},\bar{w})\big) \\ &= \frac{\partial}{\partial \bar{z}_j} \Big[\bar{\Phi}\big(\bar{z},z,\Phi(z,\bar{z},w)\big) \Big] = \frac{\partial}{\partial \bar{z}_j} \Big[w \Big] = 0. \end{aligned}$$

The last claim of the proposition is left as an exercise. □

Why do we say the last claim in the proposition proves the “uniqueness” of the complexification? Suppose we defined a complexification $\mathcal{M}'$ by another holomorphic equation $f = 0$. By the claim, $\mathcal{M} \subset \mathcal{M}'$, at least near the origin. If the derivative df is nonzero at the origin, then $f\big(z,\zeta,w,\Phi(z,\zeta,w)\big) = 0$ implies that $\frac{\partial f}{\partial \omega}$ is nonzero at the origin. Using the holomorphic implicit function theorem we can uniquely solve $f = 0$ for ω near the origin, that unique solution is Φ, and hence $\mathcal{M}' = \mathcal{M}$ near the origin.

As an example, recall that the sphere (minus a point) in $\mathbb{C}^2$ is biholomorphic to the hypersurface given by $\operatorname{Im} w = |z|^2$. That is, $\frac{w-\bar{w}}{2i} = z\bar{z}$. Solving for $\bar{w}$ and using ζ and ω obtains the equation for the complexification $\omega = -2iz\zeta + w$. Then $\Phi(z,\zeta,w) = -2iz\zeta + w$, and $\bar{\Phi}(\zeta,z,\omega) = 2i\zeta z + \omega$. Let us check that Φ is the right sort of function: $\bar{\Phi}\big(z,\zeta,\Phi(z,\zeta,w)\big) = 2i\zeta z + (-2iz\zeta + w) = w$. The CR vector field is given by $\frac{\partial}{\partial \bar{z}} + 2iz\frac{\partial}{\partial \bar{w}}$.

***Exercise* 3.2.7:** *Finish the proof of the proposition: Let $M \subset \mathbb{C}^n$ be a real-analytic hypersurface given by $\bar{w} = \Phi(z,\bar{z},w)$ near the origin, as in the proposition. Let $f(z,\bar{z},w,\bar{w})$ be a real-analytic function such that $f = 0$ on M. Prove that the complexified $f(z,\zeta,w,\omega)$ vanishes on $\mathcal{M}$.*

***Exercise* 3.2.8:** *In the proposition we only rotated and translated. Sometimes the following change of coordinates is also done. Prove that one can change coordinates (no longer linear) so that the Φ in the proposition is such that $\Phi(z,0,w) = \Phi(0,\zeta,w) = w$ for all z, ζ, and w. These coordinates are called* normal coordinates.

***Exercise* 3.2.9:** *Suppose Φ is a holomorphic function defined on a neighborhood of the origin in $\mathbb{C}^{n-1} \times \mathbb{C}^{n-1} \times \mathbb{C}$.*

a) Show that $\bar{w} = \Phi(z,\bar{z},w)$ defines a real-analytic hypersurface near the origin if and only $w = \bar{\Phi}\big(\zeta,z,\Phi(z,\zeta,w)\big)$ for all z, ζ, and w. Hint: one direction was proved already.

b) As an example, show that $\bar{w} = z\bar{z}$ does not satisfy the above condition, nor does it define a real hypersurface.

Let us prove that real-analytic CR functions on real-analytic hypersurfaces are restrictions of holomorphic functions. To motivate the proof, consider a real-analytic function f on the circle $|z|^2 = z\bar{z} = 1$ (f is vacuously CR). This f is a restriction of a real-analytic function on a neighborhood of the circle, that we write $f(z,\bar{z})$. On the circle $\bar{z} = 1/z$. Thus, $F(z) = f\bigl(z, 1/z\bigr)$ is a holomorphic function defined on a neighborhood of the circle and equal to f on the circle. Our strategy then is to solve for one of the barred variables via Proposition 3.2.8, and hope the CR conditions take care of the rest of the barred variables in more than one dimension.

Theorem 3.2.9 (Severi)**.** *Suppose $M \subset \mathbb{C}^n$ is a real-analytic hypersurface and $p \in M$. For any real-analytic CR function $f \colon M \to \mathbb{C}$, there exists a holomorphic function $F \in \mathcal{O}(U)$ for a neighborhood U of p such that $F(q) = f(q)$ for all $q \in M \cap U$.*

Proof. Write M near p as $\bar{w} = \Phi(z,\bar{z},w)$. Let $\mathcal{M}$ be the set in the $2n$ variables (z,w,ζ,ω) given by $\omega = \Phi(z,\zeta,w)$. Take f and consider any real-analytic extension of f to a neighborhood of p and write it $f(z,w,\bar{z},\bar{w})$. Complexify[*] as before to $f(z,w,\zeta,\omega)$. On $\mathcal{M}$ we have $f(z,w,\zeta,\omega) = f\bigl(z,w,\zeta,\Phi(z,\zeta,w)\bigr)$. Let

$$F(z,w,\zeta) = f\bigl(z,w,\zeta,\Phi(z,\zeta,w)\bigr).$$

Clearly $F(z,w,\bar{z})$ equals f on M. As f is a CR function, it is annihilated by $\frac{\partial}{\partial \bar{z}_j} + \frac{\partial \Phi}{\partial \bar{z}_j}\frac{\partial}{\partial \bar{w}}$ on M. So

$$\frac{\partial F}{\partial \zeta_j} + \frac{\partial \Phi}{\partial \zeta_j}\frac{\partial F}{\partial \omega} = \frac{\partial F}{\partial \zeta_j} = 0$$

on $M \subset \mathcal{M}$. We have a real analytic function $\frac{\partial F}{\partial \zeta_j}(z,w,\bar{z})$ that is zero on M, so $\frac{\partial F}{\partial \zeta_j}(z,w,\zeta) = 0$ on $\mathcal{M}$ (Proposition 3.2.8 again). As $\frac{\partial F}{\partial \zeta_j}$ is a function only of z, w, and ζ (and not of ω), $\frac{\partial F}{\partial \zeta_j} = 0$ for all (z,w,ζ) in a neighborhood of the origin. Consequently, F does not depend on ζ, and F is actually a holomorphic function of z and w only and $F = f$ on M. □

The most important place where we find CR functions that aren't necessarily real-analytic is as boundary values of holomorphic functions.

Proposition 3.2.10. *Suppose $U \subset \mathbb{C}^n$ is an open set with smooth boundary. Suppose $f \colon \overline{U} \to \mathbb{C}$ is a smooth function, holomorphic on U. Then $f|_{\partial U}$ is a smooth CR function.*

Proof. The function $f|_{\partial U}$ is clearly smooth.

Suppose $p \in \partial U$. If $X_p \in T_p^{(0,1)} \partial U$ is such that

$$X_p = \sum_{j=1}^n a_j \frac{\partial}{\partial \bar{z}_j}\Big|_p ,$$

take $\{q_k\}$ in U that approaches p, then take

$$X_{q_k} = \sum_{j=1}^n a_j \frac{\partial}{\partial \bar{z}_j}\Big|_{q_k} .$$

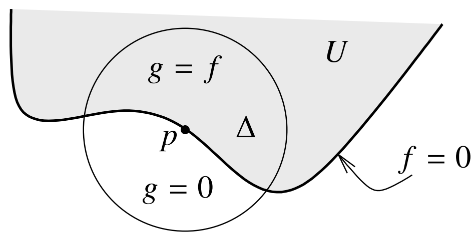
Then $X_{q_k} f = 0$ for all k and by continuity $X_p f = 0$. □

[*] At this point f stands for three distinct objects: the function on M, its real-analytic extension to a neighborhood in $\mathbb{C}^n$, and its complexification to a neighborhood of $(p,\bar{p})$ in $\mathbb{C}^n \times \mathbb{C}^n$.

The boundary values of a holomorphic function define the function uniquely. That is, if two holomorphic functions continuous up to the (smooth) boundary are equal on an open set of the boundary, then they are equal in the domain:

Proposition 3.2.11. *Suppose $U \subset \mathbb{C}^n$ is a domain with smooth boundary and $f \colon \overline{U} \to \mathbb{C}$ is a continuous function, holomorphic on U. If $f = 0$ on a nonempty open subset of ∂U, then $f = 0$ on all of U.*

Proof. Take $p \in \partial U$ such that $f = 0$ on a neighborhood of p in ∂U. Consider a small neighborhood Δ of p such that f is zero on $\partial U \cap \Delta$. Define $g \colon \Delta \to \mathbb{C}$ by setting $g(z) = f(z)$ if $z \in U$ and $g(z) = 0$ otherwise. It is not hard to see that g is continuous, and it is clearly holomorphic where it is not zero. Radó's theorem (Theorem 2.4.10) says that g is holomorphic, and as it is zero on a nonempty open subset of Δ, it is identically zero on Δ, meaning f is zero on a nonempty open subset of U, and we are done by identity.



□

Exercise* 3.2.10*: *Find a domain $U \subset \mathbb{C}^n$, $n \geq 2$, with smooth boundary and a smooth CR function $f \colon \partial U \to \mathbb{C}$ such that there is no holomorphic function on U or $\mathbb{C}^n \setminus U$ continuous up to the boundary and whose boundary values are f.*

Exercise* 3.2.11*:
a) Suppose $U \subset \mathbb{C}^n$ is a bounded open set with smooth boundary, $f \colon \overline{U} \to \mathbb{C}$ is a continuous function, holomorphic in U, and $f|_{\partial U}$ is real-valued. Show that f is constant.
b) Find a counterexample to the statement if you allow U to be unbounded.

Exercise* 3.2.12*: *Find a smooth CR function on the sphere $S^{2n-1} \subset \mathbb{C}^n$ that is not a restriction of a holomorphic function of a neighborhood of S^{2n-1}.*

Exercise* 3.2.13*: *Show a global version of Severi. Given a real-analytic hypersurface $M \subset \mathbb{C}^n$ and a real-analytic CR function $f \colon M \to \mathbb{C}$, show that there exists a neighborhood U of M, and an $F \in \mathcal{O}(U)$ such that $F|_U = f$.*

A problem we tackle next is to try to extend a smooth CR function from the boundary of a domain into a holomorphic function inside. This is a PDE problem where the PDE are the Cauchy–Riemann equations, and the function on the boundary is the boundary condition. Cauchy–Riemann equations are *overdetermined*, that is, there are too many equations. Not every data on the boundary gives a solution. Proposition 3.2.10 says that the data being CR is a necessary condition for a solution (it is not sufficient in general). Proposition 3.2.11 says that a solution is unique if it exists.

3.3 Approximation of CR functions

The following theorem (proved circa 1980) holds in much more generality, but we state its simplest version. One of the simplifications we make is that we consider only smooth CR functions here, although the theorem holds even for continuous CR functions where the CR conditions are interpreted in the sense of distributions.

Theorem 3.3.1 (Baouendi–Trèves)**.** *Suppose $M \subset \mathbb{C}^n$ is a smooth real hypersurface. Let $p \in M$ be fixed and let $z = (z_1, \dots, z_n)$ be holomorphic coordinates near p. Then there exists a compact neighborhood $K \subset M$ of p, such that for any smooth CR function $f : M \to \mathbb{C}$, there exists a sequence $\{p_j\}$ of polynomials in z such that*

$$p_j(z) \to f(z) \qquad \textit{uniformly in } K.$$

A key point is that K cannot be chosen arbitrarily, it depends on p and M. On the other hand it does not depend on f. Given M and $p \in M$ there is a K such that *every* CR function on M is approximated uniformly on K by holomorphic polynomials. The theorem applies in one dimension, although in that case the theorem of Mergelyan (see Theorem B.31) is much more general.

Example 3.3.2: Let us show that K cannot possibly be arbitrary. For simplicity $n = 1$. Let $S^1 \subset \mathbb{C}$ be the unit circle (boundary of the disc), then any smooth function on S^1 is a smooth CR function. Let f be a nonconstant real function such as $\operatorname{Re} z$. Suppose for contradiction that we could take $K = S^1$ in the theorem. Then $f(z) = \operatorname{Re} z$ could be uniformly approximated on S^1 by holomorphic polynomials. By the maximum principle, the polynomials would converge on $\mathbb{D}$ to a holomorphic function on $\mathbb{D}$ continuous on $\overline{\mathbb{D}}$. This function would have nonconstant real boundary values, which is impossible. Clearly K cannot be the entire circle.

The example is easily extended to $\mathbb{C}^n$ by considering $M = S^1 \times \mathbb{C}^{n-1}$, then $\operatorname{Re} z_1$ is a smooth CR function on M that cannot be approximated uniformly on $S^1 \times \{0\}$ by holomorphic polynomials.

The technique of the above example will be used later in a more general situation, to extend CR functions using Baouendi–Trèves.

Remark 3.3.3. It is important to note the difference between Baouendi–Trèves (and similar theorems in complex analysis) and the Weierstrass approximation theorem. In Baouendi–Trèves we obtain an approximation by holomorphic polynomials, while Weierstrass gives us polynomials in the real variables, or in z and $\bar{z}$. For example, via Weierstrass, any continuous function is uniformly approximable on S^1 via polynomials in $\operatorname{Re} z$ and $\operatorname{Im} z$, and therefore by polynomials in z and $\bar{z}$. These polynomials do not in general converge anywhere but on S^1.

Exercise* 3.3.1*: *Let $z = x + iy$ as usual in $\mathbb{C}$. Find a sequence of polynomials in x and y that converge uniformly to e^{x-y} on S^1, but diverge everywhere else.*

The proof is an ingenious use of the standard technique used to prove the Weierstrass approximation theorem. Also, as we have seen mollifiers before, the technique will not be completely foreign even to the reader who does not know the Weierstrass approximation theorem. Basically what we do is use the standard convolution argument, this time against a holomorphic function.

Letting $z = x + iy$ we only do the convolution in the x variables keeping $y = 0$. Then we use the fact that the function is CR to show that we get an approximation even for other y.

In the formulas below, given any vector $v = (v_1, \ldots, v_n)$, it will be useful to write

$$[v]^2 \overset{\text{def}}{=} v_1^2 + \cdots + v_n^2.$$

The following lemma is a neat application of ideas from several complex variables to solve a problem that does not at first seems to involve holomorphic functions.

Lemma 3.3.4. *Let W be the set of $n \times n$ complex matrices A such that*

$$\|(\operatorname{Im} A)x\| < \|(\operatorname{Re} A)x\|$$

for all nonzero $x \in \mathbb{R}^n$ and $\operatorname{Re} A$ *is positive definite. Then for all $A \in W$,*

$$\int_{\mathbb{R}^n} e^{-[Ax]^2} \det A \, dx = \pi^{n/2}.$$

Proof. Suppose A has real entries and A is positive definite (so A is also invertible). By a change of coordinates

$$\int_{\mathbb{R}^n} e^{-[Ax]^2} \det A \, dx = \int_{\mathbb{R}^n} e^{-[x]^2} \, dx = \left(\int_{\mathbb{R}} e^{-x_1^2} \, dx_1 \right) \cdots \left(\int_{\mathbb{R}} e^{-x_n^2} \, dx_n \right) = (\sqrt{\pi})^n.$$

Next suppose A is any matrix in W. There is some $\epsilon > 0$ such that $\|(\operatorname{Im} A)x\|^2 \leq (1-\epsilon^2)\|(\operatorname{Re} A)x\|^2$ for all $x \in \mathbb{R}^n$. That is because we only need to check this for x in the unit sphere, which is compact (exercise). By reality of $\operatorname{Re} A$, $\operatorname{Im} A$, and x we get $[(\operatorname{Re} A)x]^2 = \|(\operatorname{Re} A)x\|^2$ and $[(\operatorname{Im} A)x]^2 = \|(\operatorname{Im} A)x\|^2$. So

$$\left| e^{-[Ax]^2} \right| = e^{-\operatorname{Re}[Ax]^2} \leq e^{-[(\operatorname{Re} A)x]^2 + [(\operatorname{Im} A)x]^2} \leq e^{-\epsilon^2 [(\operatorname{Re} A)x]^2}.$$

Therefore the integral exists for all A in W by a similar computation as above.

The expression

$$\int_{\mathbb{R}^n} e^{-[Ax]^2} \det A \, dx$$

is a well-defined holomorphic function in the entries of A, thinking of W as a domain (see exercises below) in $\mathbb{C}^{n^2}$. We have a holomorphic function that is constantly equal to $\pi^{n/2}$ on $W \cap \mathbb{R}^{n^2}$ and hence it is equal to $\pi^{n/2}$ everywhere on W. ☐

***Exercise* 3.3.2:** *Prove the existence of $\epsilon > 0$ in the proof above.*

***Exercise* 3.3.3:** *Show that $W \subset \mathbb{C}^{n^2}$ in the proof above is a domain (open and connected).*

***Exercise* 3.3.4:** *Prove that we can really differentiate under the integral to show that the integral is holomorphic in the entries of A.*

***Exercise* 3.3.5:** *Show that some hypotheses are needed for the lemma. In particular take $n = 1$ and find the exact set of A (now just a complex number) for which the theorem is true.*

Given an $n \times n$ matrix A, let $\|A\|$ denote the operator norm,

$$\|A\| = \sup_{\|v\|=1} \|Av\| = \sup_{v \in \mathbb{C}^n, v \neq 0} \frac{\|Av\|}{\|v\|}.$$

Exercise* 3.3.6*: *Let W be as in Lemma 3.3.4. Let B be an $n \times n$ real matrix such that $\|B\| < 1$. Show that $I + iB \in W$.*

We will be using differential forms, and the following lemma says that as far as the exterior derivative is concerned, all CR functions behave as restrictions of holomorphic functions.

Lemma 3.3.5. *Let $M \subset \mathbb{C}^n$ be a smooth real hypersurface, $f \colon M \to \mathbb{C}$ be a smooth CR function, and $(z_1, \ldots, z_n)$ be the holomorphic coordinates of $\mathbb{C}^n$. Then at each point $p \in M$, the exterior derivative df is a linear combination of $dz_1, \ldots, dz_n$, thinking of $z_1, \ldots, z_n$ as functions on M. In particular,*

$$d(f \wedge dz) = df \wedge dz = 0.$$

Recall the notation $dz = dz_1 \wedge dz_2 \wedge \cdots \wedge dz_n$.

Proof. After a complex affine change of coordinates we simply need to show this at the origin. Let $\xi_1, \ldots, \xi_n$ be the new holomorphic coordinates and suppose the $T_0^{(1,0)} M$ tangent space is spanned by $\frac{\partial}{\partial \xi_1}\big|_0, \ldots, \frac{\partial}{\partial \xi_{n-1}}\big|_0$, and such that $\frac{\partial}{\partial \operatorname{Re} \xi_n}\big|_0$ is tangent and $\frac{\partial}{\partial \operatorname{Im} \xi_n}\big|_0$ is normal. At the origin, the CR conditions are $\frac{\partial f}{\partial \bar{\xi}_j}(0) = 0$ for all j, so

$$df(0) = \frac{\partial f}{\partial \xi_1}(0)\, d\xi_1(0) + \cdots + \frac{\partial f}{\partial \xi_{n-1}}(0)\, d\xi_{n-1}(0) + \frac{\partial f}{\partial \operatorname{Re} \xi_n}(0)\, d(\operatorname{Re} \xi_n)(0).$$

Also, at the origin $d\xi_n(0) = d(\operatorname{Re} \xi_n)(0) + i d(\operatorname{Im} \xi_n)(0) = d(\operatorname{Re} \xi_n)(0)$. So $df(0)$ is a linear combination of $d\xi_1(0), \ldots, d\xi_n(0)$. As ξ is a complex affine function of z, then each $d\xi_j$ is a linear combination of dz_1 through dz_n, and the claim follows. So if f is a CR function, then $d(f\, dz) = df \wedge dz = 0$ since $dz_j \wedge dz_j = 0$. □

Proof of the theorem of Baouendi–Trèves. Suppose $M \subset \mathbb{C}^n$ is a smooth real hypersurface, and without loss of generality suppose $p = 0 \in M$. Let $z = (z_1, \ldots, z_n)$ be the holomorphic coordinates, write $z = x + iy$, $y = (y', y_n)$, and suppose M is given by

$$y_n = \psi(x, y'),$$

where ψ is $O(2)$. The variables (x, y') parametrize M near 0:

$$z_j = x_j + i y_j, \qquad \text{for } j = 1, \ldots, n-1, \quad \text{and} \quad z_n = x_n + i\psi(x, y').$$

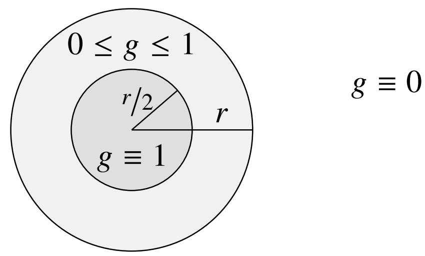
Define

$$\varphi(x, y') = \big(y_1, \ldots, y_{n-1}, \psi(x, y')\big).$$

Write $(x, y') \mapsto z = x + i\varphi(x, y')$ as the parametrization. That is, think of z as a function of (x, y').

Let $r > 0$ and $d > 0$ be small numbers to be determined later. Assume they are small enough so that f and φ are defined and smooth on some neighborhood of the set where $\|x\| \leq r$ and $\|y'\| \leq d$.

There exists a smooth $g \colon \mathbb{R}^n \to [0,1]$ such that $g \equiv 1$ on $B_{r/2}(0)$ and $g \equiv 0$ outside of $B_r(0)$. Explicit formula can be given. Alternatively we obtain such a g by use of mollifiers on a function that is identically one on $B_{3r/4}(0)$ and zero elsewhere. Such a g is commonly called a *cutoff function*.



***Exercise* 3.3.7:** *Find an explicit formula for g without using mollifiers.*

Let
$$K' = \left\{ (x, y') : \|x\| \leq r/4, \|y'\| \leq d \right\}.$$
Let $K = z(K')$, that is the image of K' under the mapping $z(x, y')$.

Consider the CR function f to be a function of (x, y') and write $f(x, y')$. For $\ell \in \mathbb{N}$, let α_ℓ be a differential n-form defined (thinking of $w \in \mathbb{C}^n$ as a constant parameter) by

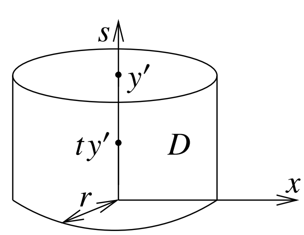
$$\begin{aligned} \alpha_\ell(x, y') &= \left(\frac{\ell}{\pi}\right)^{n/2} e^{-\ell[w-z]^2} g(x) f(x, y')\, dz \\ &= \left(\frac{\ell}{\pi}\right)^{n/2} e^{-\ell[w-x-i\varphi(x,y')]^2} g(x) f(x, y') \\ &\qquad (dx_1 + i dy_1) \wedge \cdots \wedge (dx_{n-1} + i dy_{n-1}) \wedge \left(dx_n + i d\psi(x, y')\right). \end{aligned}$$

The key is the exponential, which looks like the bump function mollifier, except that now we have w and z possibly complex. The exponential is also holomorphic in w, and that will give us entire holomorphic approximating functions.

Fix y' with $0 < \|y'\| < d$ and let D be defined by

$$D = \left\{ (x, s) \in \mathbb{R}^n \times \mathbb{R}^{n-1} : \|x\| < r \text{ and } s = t y' \text{ for } t \in (0, 1) \right\}.$$

D is an $(n+1)$-dimensional "cylinder." That is, we take a ball in the x directions and then take a single fixed point y' in the s variables and make a cylinder. Here is a rough diagram:



We orient D in the standard way as if it sat in the (x,t) variables in $\mathbb{R}^n \times \mathbb{R}$. Stokes' theorem says

$$\int_D d\alpha_\ell(x,s) = \int_{\partial D} \alpha_\ell(x,s).$$

Since $g(x) = 0$ if $\|x\| \geq r$, α_ℓ is zero on the sides of the cylinder D, so the integral over ∂D only needs to consider the top and bottom of the cylinder. And because of g, the integral over the top and bottom can be taken over $\mathbb{R}^n$. As is usual in these sorts of arguments, we do the slight abuse of notation where we ignore that f and φ are undefined where g is identically zero:

$$\begin{aligned}\int_{\partial D} \alpha_\ell(x,s) =& \left(\frac{\ell}{\pi}\right)^{n/2} \int_{x\in\mathbb{R}^n} e^{-\ell[w-x-i\varphi(x,y')]^2} g(x) f(x,y')\, dx_1 \wedge \cdots \wedge dx_{n-1} \wedge \big(dx_n + i d_x\psi(x,y')\big) \\ & - \left(\frac{\ell}{\pi}\right)^{n/2} \int_{x\in\mathbb{R}^n} e^{-\ell[w-x-i\varphi(x,0)]^2} g(x) f(x,0)\, dx_1 \wedge \cdots \wedge dx_{n-1} \wedge \big(dx_n + i d_x\psi(x,0)\big),\end{aligned} \tag{3.1}$$

where d_x means the derivative in the x directions only. That is, $d_x\psi = \frac{\partial\psi}{\partial x_1}dx_1 + \cdots + \frac{\partial\psi}{\partial x_n}dx_n$.

We will show that as $\ell \to \infty$, the left hand side of (3.1) goes to zero uniformly for $w \in K$ and the first term on the right hand side goes to $f(\tilde{x}, y')$ if $w = z(\tilde{x}, y')$ is in M. Hence, we define entire functions that we will show approximate f:

$$f_\ell(w) = \left(\frac{\ell}{\pi}\right)^{n/2} \int_{x\in\mathbb{R}^n} e^{-\ell[w-x-i\varphi(x,0)]^2} g(x) f(x,0)\, dx_1 \wedge \cdots \wedge dx_{n-1} \wedge \big(dx_n + i d_x\psi(x,0)\big).$$

Clearly each f_ℓ is holomorphic and defined for all $w \in \mathbb{C}^n$.

In the next claim it is important that f is a CR function.

Claim 3.3.6. *We have*

$$d\alpha_\ell(x,s) = \left(\frac{\ell}{\pi}\right)^{n/2} e^{-\ell[w-z(x,s)]^2} f(x,s)\, dg(x) \wedge dz(x,s),$$

and for sufficiently small $r > 0$ and $d > 0$,

$$\lim_{\ell\to\infty} \left(\frac{\ell}{\pi}\right)^{n/2} \int_{(x,s)\in D} e^{-\ell[w-z(x,s)]^2} f(x,s)\, dg(x) \wedge dz(x,s) = 0$$

uniformly as a function of $w \in K$ and $y' \in B_d(0)$ (recall that D depends on y').

Proof. The function $(x,s) \mapsto e^{-\ell[w-z(x,s)]^2}$ is CR (as a function on M), and so is $f(x,s)$. Therefore, using Lemma 3.3.5,

$$d\alpha_\ell(x,s) = \left(\frac{\ell}{\pi}\right)^{n/2} e^{-\ell[w-z(x,s)]^2} f(x,s)\, dg(x) \wedge dz(x,s).$$

Since dg is zero for $\|x\| \leq r/2$, the integral

$$\int_D d\alpha_\ell(x,s) = \left(\frac{\ell}{\pi}\right)^{n/2} \int_D e^{-\ell[w-z(x,s)]^2} f(x,s)\, dg(x) \wedge dz(x,s)$$

is only evaluated for the subset of D where $\|x\| > r/2$.

Suppose $w \in K$ and $(x,s) \in D$ with $\|x\| > r/2$. Let $w = z(\tilde{x}, \tilde{s})$. We need to estimate

$$\left| e^{-\ell[w-z(x,s)]^2} \right| = e^{-\ell \operatorname{Re}[w-z(x,s)]^2}.$$

Then

$$-\operatorname{Re}[w-z]^2 = -\|\tilde{x}-x\|^2 + \|\varphi(\tilde{x},\tilde{s}) - \varphi(x,s)\|^2.$$

By the mean value theorem

$$\|\varphi(\tilde{x},\tilde{s}) - \varphi(x,s)\| \le \|\varphi(\tilde{x},\tilde{s}) - \varphi(x,\tilde{s})\| + \|\varphi(x,\tilde{s}) - \varphi(x,s)\| \le a\|\tilde{x}-x\| + A\|\tilde{s}-s\|,$$

where a and A are

$$a = \sup_{\|\hat{x}\| \le r, \|\hat{y}'\| \le d} \left\| \left[\frac{\partial \varphi}{\partial x}(\hat{x}, \hat{y}') \right] \right\|, \qquad A = \sup_{\|\hat{x}\| \le r, \|\hat{y}'\| \le d} \left\| \left[\frac{\partial \varphi}{\partial y'}(\hat{x}, \hat{y}')) \right] \right\|.$$

Here $\left[\frac{\partial \varphi}{\partial x}\right]$ and $\left[\frac{\partial \varphi}{\partial y'}\right]$ are the derivatives (matrices) of φ with respect to x and y' respectively, and the norm we are taking is the operator norm. Because $\left[\frac{\partial \varphi}{\partial x}\right]$ is zero at the origin, we pick r and d small enough (and hence K small enough) so that $a \le 1/4$. We furthermore pick d possibly even smaller to ensure that $d \le \frac{r}{32A}$. We have that $r/2 \le \|x\| \le r$, but $\|\tilde{x}\| \le r/4$ (recall $w \in K$), so

$$\frac{r}{4} \le \|\tilde{x} - x\| \le \frac{5r}{4}.$$

Also, $\|\tilde{s} - s\| \le 2d$ by triangle inequality.

Therefore,

$$\begin{aligned}
-\operatorname{Re}[w-z(x,s)]^2 &\le -\|\tilde{x}-x\|^2 + a^2\|\tilde{x}-x\|^2 + A^2\|\tilde{s}-s\|^2 + 2aA\|\tilde{x}-x\|\|\tilde{s}-s\| \\
&\le \frac{-15}{16}\|\tilde{x}-x\|^2 + A^2\|\tilde{s}-s\|^2 + \frac{A}{2}\|\tilde{x}-x\|\|\tilde{s}-s\| \\
&\le \frac{-r^2}{64}.
\end{aligned}$$

In other words,

$$\left| e^{-\ell[w-z(x,s)]^2} \right| \le e^{-\ell r^2/64},$$

or

$$\left| \left(\frac{\ell}{\pi}\right)^{n/2} \int_{(x,s)\in D} e^{-\ell[w-z(x,s)]^2} f(x,s)\, dg(x) \wedge dz(x,s) \right| \le C\ell^{n/2} e^{-\ell r^2/64},$$

for some constant C. Note that D depends on y'. The set of all y' with $\|y'\| \le d$, is a compact set, so we can make C large enough to not depend on the y' that was chosen. The claim follows. □

Claim 3.3.7. *For the given $r > 0$ and $d > 0$,*

$$\lim_{\ell\to\infty} \left(\frac{\ell}{\pi}\right)^{n/2} \int_{x\in\mathbb{R}^n} e^{-\ell[\tilde{x}+i\varphi(\tilde{x},y')-x-i\varphi(x,y')]^2} g(x) f(x,y') dx_1 \wedge \cdots \wedge dx_{n-1} \wedge \left(dx_n + i d_x\psi(x,y')\right) = f(\tilde{x}, y')$$

uniformly in $(\tilde{x}, y') \in K'$.

That is, we look at (3.1) and we plug in $w = z(\tilde{x}, y') \in K$. The g (as usual) makes sure we never evaluate f, ψ, or φ at points where they are not defined.

Proof. The change of variables formula implies

$$dx_1 \wedge \cdots \wedge dx_{n-1} \wedge \left(dx_n + i d_x\psi(x, y')\right) = d_x z(x, y') = \det\left[\frac{\partial z}{\partial x}(x, y')\right] dx,$$

where $\left[\frac{\partial z}{\partial x}(x, y')\right]$ is the matrix corresponding to the derivative of the mapping z with respect to the x variables evaluated at (x, y').

Let us change variables of integration via $\xi = \sqrt{\ell}(x - \tilde{x})$:

$$\left(\frac{\ell}{\pi}\right)^{n/2} \int_{x\in\mathbb{R}^n} e^{-\ell[\tilde{x}+i\varphi(\tilde{x},y')-x-i\varphi(x,y')]^2} g(x) f(x, y') \det\left[\frac{\partial z}{\partial x}(x, y')\right] dx =$$
$$\left(\frac{1}{\pi}\right)^{n/2} \int_{\xi\in\mathbb{R}^n} e^{-\left[\xi+i\sqrt{\ell}\left(\varphi\left(\tilde{x}+\frac{\xi}{\sqrt{\ell}},y'\right)-\varphi(\tilde{x},y')\right)\right]^2} g\left(\tilde{x} + \frac{\xi}{\sqrt{\ell}}\right) f\left(\tilde{x} + \frac{\xi}{\sqrt{\ell}}, y'\right) \det\left[\frac{\partial z}{\partial x}\left(\tilde{x} + \frac{\xi}{\sqrt{\ell}}, y'\right)\right] d\xi.$$

We now wish to take a limit as $\ell \to \infty$ and for this we apply the dominated convergence theorem. So we need to dominate the integrand. The second half of the integrand is uniformly bounded independent of ℓ as

$$x \mapsto g(x) f(x, y') \det\left[\frac{\partial z}{\partial x}(x, y')\right]$$

is a continuous function with compact support (because of g). Hence it is enough to worry about the exponential term. We also only consider those ξ where the integrand is not zero. Recall that r and d are small enough that

$$\sup_{\|\hat{x}\|\leq r, \|\hat{y}'\|\leq d} \left\| \left[\frac{\partial \varphi}{\partial x}(\hat{x}, \hat{y}')\right] \right\| \leq \frac{1}{4},$$

and as $\|\tilde{x}\| \leq r/4$ (as $(\tilde{x}, y') \in K$) and $\left\|\tilde{x} + \frac{\xi}{\sqrt{\ell}}\right\| \leq r$ (because g is zero otherwise) then

$$\left\|\varphi\left(\tilde{x} + \frac{\xi}{\sqrt{\ell}}, y'\right) - \varphi(\tilde{x}, y')\right\| \leq \frac{1}{4}\left\|\tilde{x} + \frac{\xi}{\sqrt{\ell}} - \tilde{x}\right\| = \frac{\|\xi\|}{4\sqrt{\ell}}.$$

So under the same conditions we have

$$\begin{aligned}\left|e^{-\left[\xi+i\sqrt{\ell}\left(\varphi\left(\tilde{x}+\frac{\xi}{\sqrt{\ell}},y'\right)-\varphi(\tilde{x},y')\right)\right]^2}\right| &= e^{-\operatorname{Re}\left[\xi+i\sqrt{\ell}\left(\varphi\left(\tilde{x}+\frac{\xi}{\sqrt{\ell}},y'\right)-\varphi(\tilde{x},y')\right)\right]^2} \\ &= e^{-\|\xi\|^2+\ell\left\|\varphi\left(\tilde{x}+\frac{\xi}{\sqrt{\ell}},y'\right)-\varphi(\tilde{x},y')\right\|^2} \\ &\leq e^{-(15/16)\|\xi\|^2}.\end{aligned}$$

And that is integrable. Therefore we take the pointwise limit under the integral to obtain

$$\left(\frac{1}{\pi}\right)^{n/2} \int_{\xi\in\mathbb{R}^n} e^{-\left[\xi+i\left[\frac{\partial\varphi}{\partial x}(\tilde{x},y')\right]\xi\right]^2} g(\tilde{x}) f(\tilde{x}, y') \det\left[\frac{\partial z}{\partial x}(\tilde{x}, y')\right] d\xi.$$

Notice how in the exponent we actually have an expression for the derivative in the ξ direction with y' fixed. If $(\tilde{x}, y') \in K'$, then $g(\tilde{x}) = 1$ and so we can ignore g.

Let $A = I + i\left[\frac{\partial\varphi}{\partial x}(\tilde{x}, y')\right]$. Lemma 3.3.4 says

$$\left(\frac{1}{\pi}\right)^{n/2} \int_{\xi\in\mathbb{R}^n} e^{-\left[\xi + i\left[\frac{\partial\varphi}{\partial x}(\tilde{x},y')\right]\xi\right]^2} f(\tilde{x}, y') \det\left[\frac{\partial z}{\partial x}(\tilde{x}, y')\right] d\xi = f(\tilde{x}, y').$$

That the convergence is uniform in $(\tilde{x}, y') \in K'$ is left as an exercise. □

Exercise* 3.3.8*: *In the above claim, finish the proof that the convergence is uniform in* $(\tilde{x}, y') \in K'$. *Hint: It may be easier to use the form of the integral before the change of variables and prove that the sequence is uniformly Cauchy.*

We are essentially done with the proof of the theorem. The two claims together with (3.1) show that f_ℓ are entire holomorphic functions that approximate f uniformly on K. Entire holomorphic functions can be approximated by polynomials uniformly on compact subsets; simply take the partial sums of Taylor series at the origin. □

Exercise* 3.3.9*: *Explain why being approximable on K by (holomorphic) polynomials does not necessarily mean that f is real-analytic.*

Exercise* 3.3.10*: *Suppose $M \subset \mathbb{C}^n$ is given by* $\operatorname{Im} z_n = 0$. *Use the standard Weierstrass approximation theorem to show that for any $K \subset\subset M$ an arbitrary smooth CR function $f : M \to \mathbb{C}$ can be uniformly approximated by holomorphic polynomials on K.*

3.4 Extension of CR functions

We will now apply the so-called "technique of analytic discs" together with Baouendi–Trèves to prove the Lewy extension theorem. Lewy's original proof was different and predates Baouendi–Trèves. A local extension theorem of this type was first proved by Helmut Knesser in 1936.

Theorem 3.4.1 (Lewy)**.** *Suppose $M \subset \mathbb{C}^n$ is a smooth real hypersurface and $p \in M$. There exists a neighborhood U of p with the following property. Suppose $r : U \to \mathbb{R}$ is a smooth defining function for $M \cap U$, denote by $U_- \subset U$ the set where r is negative and $U_+ \subset U$ the set where r is positive. Let $f : M \to \mathbb{R}$ be a smooth CR function. Then:*

(i) If the Levi form with respect to r has a positive eigenvalue at p, then f extends to a holomorphic function on U_- continuous up to M (that is, continuous on $\{z \in U : r(z) \leq 0\}$).

(ii) If the Levi form with respect to r has a negative eigenvalue at p, then f extends to a holomorphic function on U_+ continuous up to M (that is, continuous on $\{z \in U : r(z) \geq 0\}$).

(iii) If the Levi form with respect to r has eigenvalues of both signs at p, then f extends to a function holomorphic on U.

So if the Levi form has eigenvalues of both signs, then near p all CR functions are restrictions of holomorphic functions. The function r can be any defining function for M. Either we can extend it to all of U or we could take a smaller U such that r is defined on U. As we noticed before, once we pick sides (where r is positive and where it is negative), then the number of positive eigenvalues and the number of negative eigenvalues of the Levi form is fixed. A different r may flip U_- and U_+, but the conclusion of the theorem is exactly the same.

Proof. We prove the first item, and the second item follows by considering $-r$. Suppose $p = 0$ and M is given in some neighborhood Ω of the origin as

$$\operatorname{Im} w = |z_1|^2 + \sum_{j=2}^{n-1} \epsilon_j |z_j|^2 + E(z_1, z', \bar{z}_1, \bar{z}', \operatorname{Re} w),$$

where $z' = (z_2, \ldots, z_{n-1})$, $\epsilon_j = -1, 0, 1$, and E is $O(3)$. Let Ω_- be given by

$$0 > r = |z_1|^2 + \sum_{j=2}^{n-1} \epsilon_j |z_j|^2 + E(z_1, z', \bar{z}_1, \bar{z}', \operatorname{Re} w) - \operatorname{Im} w.$$

The (real) Hessian of the function

$$z_1 \mapsto |z_1|^2 + E(z_1, 0, \bar{z}_1, 0, 0)$$

is positive definite in a neighborhood of the origin and the function has a strict minimum at 0. There is some small disc $D \subset \mathbb{C}$ such that this function is strictly positive on ∂D.

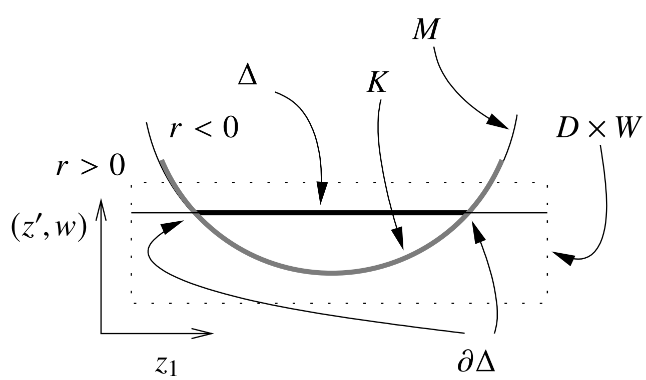
Therefore, for $(z', w) \in W$ in some small neighborhood $W \subset \mathbb{C}^{n-1}$ of the origin, the function

$$z_1 \mapsto |z_1|^2 + \sum_{j=2}^{n} \epsilon_j |z_j|^2 + E(z_1, z', \bar{z}_1, \bar{z}', \operatorname{Re} w) - \operatorname{Im} w$$

is still strictly positive on ∂D.

We wish to apply Baouendi–Trèves and so let K be the compact neighborhood of the origin from the theorem. Take D and W small enough such that $(D \times W) \cap M \subset K$. Find the polynomials p_j that approximate f uniformly on K. Consider $z_1 \in D$ and $(z', w) \in W$ such that $(z_1, z', w) \in \Omega_-$. Let $\Delta = \bigl(D \times \{(z', w)\}\bigr) \cap \Omega_-$. Denote by $\partial\Delta$ the boundary of Δ in the subspace topology of $\mathbb{C} \times \{(z', w)\}$.

The set Ω_+ where $r > 0$ is open and it contains $(\partial D) \times \{(z', w)\}$. Therefore, $\partial\Delta$ contains no points of $(\partial D) \times \{(z', w)\}$. Consequently, $\partial\Delta$ contains only points where $r = 0$, that is $\partial\Delta \subset M$, and also $\partial\Delta \subset D \times W$. As $(D \times W) \cap M \subset K$ and we have $\partial\Delta \subset K$.



As $p_j \to f$ uniformly on K then $p_j \to f$ uniformly on $\partial\Delta$. As p_j are holomorphic, then by the maximum principle, p_j converge uniformly on all of Δ. In fact, as (z_1, z', w) was an arbitrary point in $(D \times W) \cap \Omega_-$, the polynomials p_j converge uniformly on $(D \times W) \cap \overline{\Omega_-}$. Let $U = D \times W$, then $U_- = (D \times W) \cap \Omega_-$. Notice U depends on K, but not on f. So p_j converge to a continuous function F on $\overline{U_-} \cap U$ and F is holomorphic on U_-. Clearly F equals f on $M \cap U$.

To prove the last item, pick a side, and then use one of the first two items to extend the function to that side. Via the tomato can principle (Theorem 2.3.11) the function also extends across M and therefore to a whole neighborhood of p. □

If you were wondering what happened to the analytic discs we promised, the Δ in the above is an analytic disc (simply connected) for a small enough U, but it was not necessary to prove that fact.

We state the next corollary for a strongly convex domain, even though it holds with far more generality. It is a simpler version of the *Hartogs–Bochner**. Later, in Exercise 4.3.4, you will prove it for strongly pseudoconvex domains. However, the theorem is true for any bounded domain with connected smooth boundary with no assumptions on the Levi form, but a different approach would have to be taken.

Corollary 3.4.2. *Suppose $U \subset \mathbb{C}^n$, $n \geq 2$, is a bounded domain with smooth boundary that is strongly convex and $f\colon \partial U \to \mathbb{C}$ is a smooth CR function, then there exists a continuous function $F\colon \overline{U} \to \mathbb{C}$ holomorphic in U such that $F|_{\partial U} = f$.*

Proof. A strongly convex domain is strongly pseudoconvex, so f must extend to the inside locally near any point. The extension is locally unique as any two extensions have the same boundary values. Therefore, there exists a set $K \subset\subset U$ such that f extends to $U \setminus K$. Via an exercise below we can assume that K is strongly convex and therefore we can apply the special case of Hartogs phenomenon that you proved in Exercise 2.1.6 to find an extension holomorphic in U. □

Exercise* 3.4.1*: *Prove the existence of the strongly convex K above.*

Exercise* 3.4.2*: *Show by example that the corollary is not true when $n = 1$. Explain where in the proof have we used that $n \geq 2$.*

Exercise* 3.4.3*: *Suppose $f\colon \partial\mathbb{B}_2 \to \mathbb{C}$ is a smooth CR function. Write down an explicit formula for the extension F.*

Exercise* 3.4.4*: *Suppose a smooth hypersurface $M \subset \mathbb{C}^3$ is defined by $\operatorname{Im} w = |z_1|^2 - |z_2|^2 + O(3)$ and f is a real-valued smooth CR function on M. Show that $|f|$ does not attain a maximum at the origin.*

Exercise* 3.4.5*: *Suppose $M \subset \mathbb{C}^n$, $n \geq 3$, is a real-analytic hypersurface such that the Levi form at $p \in M$ has eigenvalues of both signs. Show that every smooth CR function f on M is in fact real-analytic in a neighborhood of p.*

*What is called Hartogs–Bochner is the C^1 version of this theorem where the domain is only assumed to be bounded and the boundary connected, and it was proved by neither Hartogs nor Bochner, but by Martinelli in 1961.

Exercise* 3.4.6*: *Let $M \subset \mathbb{C}^3$ be defined by $\operatorname{Im} w = |z_1|^2 - |z_2|^2$.*

a) Show that an arbitrary compact subset $K \subset\subset M$ will work for the conclusion Baouendi–Trèves.

b) Use this to show that every smooth CR function $f: M \to \mathbb{C}$ is a restriction of an entire holomorphic function $F: \mathbb{C}^3 \to \mathbb{C}$.

Exercise* 3.4.7*: *Find an $M \subset \mathbb{C}^n$, $n \geq 2$, such that near some $p \in M$, for every neighborhood W of p in M, there is a CR function $f: W \to \mathbb{C}$ that does not extend holomorphically to either side of M at p.*

Exercise* 3.4.8*: *Suppose $f: \partial\mathbb{B}_n \to \mathbb{C}$ is a smooth function, $n \geq 2$. Prove that f is a CR function if and only if*

$$\int_0^{2\pi} f(e^{i\theta}v)\, e^{ik\theta} d\theta = 0 \qquad \text{for all } v \in \partial\mathbb{B}_n \text{ and all } k \in \mathbb{N}.$$

Exercise* 3.4.9*: *Prove the third item in the Lewy extension theorem without the use of the tomato can principle. That is, prove in a more elementary way that if $M \subset U \subset \mathbb{C}^n$ is a smooth hypersurface in an open set U and $f: U \to \mathbb{C}$ is continuous and holomorphic in $U \setminus M$, then f is holomorphic.*

Remark 3.4.3. Studying solutions to nonhomogeneous CR equations of the form $Xf = \psi$ for a CR vector field X, and the fact that such conditions can guarantee that a function must be real-analytic, led Lewy to a famous, very surprising, and rather simple example of a linear partial differential equation with smooth coefficients that has no solution on any open set*. The example is surprising because when a linear PDE has real-analytic coefficients, a solution always exists by the theorem of Cauchy–Kowalevski.

*Lewy, Hans, *An example of a smooth linear partial differential equation without solution*, Annals of Mathematics, **66** (1957), 155–158.

Chapter 4

The $\bar{\partial}$-problem

4.1 The generalized Cauchy integral formula

Before we get into the $\bar{\partial}$-problem, let us prove a more general version of Cauchy's formula using Stokes' theorem (really Green's theorem). This version is called the *Cauchy–Pompeiu integral formula*. We will only need the theorem for smooth functions, but as it is often applied in less regular contexts and it is just an application of Stokes' theorem, let us state it that way. In applications, the boundary is often only piecewise smooth, and again that is all we need for Stokes.

Theorem 4.1.1 (Cauchy–Pompeiu)**.** *Let $U \subset \mathbb{C}$ be a bounded open set with piecewise C^1-smooth boundary ∂U oriented positively, and let $f : \overline{U} \to \mathbb{C}$ be a continuous function with bounded continuous partial derivatives in U. Then for $z \in U$:*

$$f(z) = \frac{1}{2\pi i} \int_{\partial U} \frac{f(\zeta)}{\zeta - z} \, d\zeta + \frac{1}{2\pi i} \int_{U} \frac{\frac{\partial f}{\partial \bar{z}}(\zeta)}{\zeta - z} \, d\zeta \wedge d\bar{\zeta}.$$

If f is holomorphic, then the second term is zero, and we obtain the standard Cauchy formula. If $\zeta = x+iy$, then the standard orientation on $\mathbb{C}$ is the one corresponding to the area form $dA = dx \wedge dy$. The form $d\zeta \wedge d\bar{\zeta}$ is the area form up to a scalar. That is,

$$d\zeta \wedge d\bar{\zeta} = (dx + i\, dy) \wedge (dx - i\, dy) = (-2i) dx \wedge dy = (-2i) dA.$$

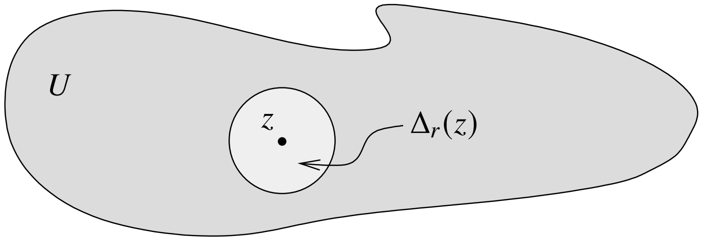
As we want to use Stokes, let us note that if $z = x + iy$, then we can write the standard exterior derivative in terms of z and $\bar{z}$:

$$d\psi = \frac{\partial \psi}{\partial x} dx + \frac{\partial \psi}{\partial y} dy = \frac{\partial \psi}{\partial z} dz + \frac{\partial \psi}{\partial \bar{z}} d\bar{z}.$$

Exercise* 4.1.1*: *Observe the singularity in the second term, and prove that the integral still makes sense (the function is integrable). Hint: polar coordinates.*

Exercise* 4.1.2*: *Why can we not differentiate in $\bar{z}$ under the integral in the second term? Notice that it would lead to an impossible result.*

Proof. Fix $z \in U$. We wish to apply Stokes' theorem[*], but the integrand is not smooth at z. Let $\Delta_r(z)$ be a small disc such that $\Delta_r(z) \subset\subset U$. Stokes now applies on $U \setminus \Delta_r(z)$.



Via Stokes we get

$$\int_{\partial U} \frac{f(\zeta)}{\zeta - z}\, d\zeta - \int_{\partial \Delta_r(z)} \frac{f(\zeta)}{\zeta - z}\, d\zeta = \int_{U \setminus \Delta_r(z)} d\left(\frac{f(\zeta)}{\zeta - z}\, d\zeta\right) = \int_{U \setminus \Delta_r(z)} \frac{\frac{\partial f}{\partial \bar{\zeta}}(\zeta)}{\zeta - z}\, d\bar{\zeta} \wedge d\zeta.$$

The second equality follows because holomorphic derivatives in ζ have a $d\zeta$ and when we wedge them with $d\zeta$ we just get zero. We now wish to let the radius r go to zero. Via the exercise above, $\frac{\frac{\partial f}{\partial \bar{\zeta}}(\zeta)}{\zeta - z}\, d\bar{\zeta} \wedge d\zeta$ is integrable over all of U. Therefore,

$$\lim_{r \to 0} \int_{U \setminus \Delta_r(z)} \frac{\frac{\partial f}{\partial \bar{\zeta}}(\zeta)}{\zeta - z}\, d\bar{\zeta} \wedge d\zeta = \int_U \frac{\frac{\partial f}{\partial \bar{\zeta}}(\zeta)}{\zeta - z}\, d\bar{\zeta} \wedge d\zeta = -\int_U \frac{\frac{\partial f}{\partial \bar{\zeta}}(\zeta)}{\zeta - z}\, d\zeta \wedge d\bar{\zeta}.$$

The second equality is simply swapping the order of the $d\zeta$ and $d\bar{\zeta}$. By continuity of f,

$$\lim_{r \to 0} \frac{1}{2\pi i} \int_{\partial \Delta_r(z)} \frac{f(\zeta)}{\zeta - z}\, d\zeta = \lim_{r \to 0} \frac{1}{2\pi} \int_0^{2\pi} f(z + re^{i\theta})\, d\theta = f(z).$$

The theorem follows. □

Exercise* 4.1.3*:

a) Let $U \subset \mathbb{C}$ be a bounded open set with piecewise C^1-smooth boundary and suppose $f : \overline{U} \to \mathbb{C}$ is a C^1-smooth function such that $\int_U \frac{\frac{\partial f}{\partial \bar{z}}(\zeta)}{\zeta - z}\, dA(\zeta) = 0$ for every $z \in \partial U$. Prove that $f|_{\partial U}$ are the boundary values of a holomorphic function in U.

b) Given arbitrary $\epsilon > 0$, find a C^1 function f on the closed unit disc $\overline{\mathbb{D}}$, such that $\frac{\partial f}{\partial \bar{z}}$ is identically zero outside an ϵ-neighborhood of the origin, yet $f|_{\partial \mathbb{D}}$ are not the boundary values of a holomorphic function.

Exercise* 4.1.4*: *Let $U \subset \mathbb{C}$ and f be as in the theorem, but let $z \notin \overline{U}$. Show that*

$$\frac{1}{2\pi i} \int_{\partial U} \frac{f(\zeta)}{\zeta - z}\, d\zeta + \frac{1}{2\pi i} \int_U \frac{\frac{\partial f}{\partial \bar{z}}(\zeta)}{\zeta - z}\, d\zeta \wedge d\bar{\zeta} = 0.$$

[*]We are really using Green's theorem, which is the generalized Stokes' theorem in 2 dimensions, see Theorem B.2.

4.2 Simple case of the $\bar{\partial}$-problem

For a smooth function ψ, consider the exterior derivative in terms of z and $\bar{z}$,

$$d\psi = \frac{\partial \psi}{\partial z_1} dz_1 + \cdots + \frac{\partial \psi}{\partial z_n} dz_n + \frac{\partial \psi}{\partial \bar{z}_1} d\bar{z}_1 + \cdots + \frac{\partial \psi}{\partial \bar{z}_n} d\bar{z}_n.$$

Let us give a name to the two parts of the derivative:

$$\partial \psi \overset{\text{def}}{=} \frac{\partial \psi}{\partial z_1} dz_1 + \cdots + \frac{\partial \psi}{\partial z_n} dz_n, \qquad \bar{\partial} \psi \overset{\text{def}}{=} \frac{\partial \psi}{\partial \bar{z}_1} d\bar{z}_1 + \cdots + \frac{\partial \psi}{\partial \bar{z}_n} d\bar{z}_n.$$

Then $d\psi = \partial\psi + \bar{\partial}\psi$. Notice ψ is holomorphic if and only if $\bar{\partial}\psi = 0$.

The so-called *inhomogeneous $\bar{\partial}$-problem* ($\bar{\partial}$ is pronounced “dee bar”) is to solve the equation

$$\bar{\partial}\psi = g,$$

for ψ, given a one-form

$$g = g_1 d\bar{z}_1 + \cdots + g_n d\bar{z}_n.$$

Such a g is called a $(0,1)$*-form*. The fact that the partial derivatives of ψ commute, forces certain compatibility conditions on g for us to have any hope of getting a solution (see below).

Exercise* 4.2.1: Find an explicit example of a g in $\mathbb{C}^2$ such that no corresponding ψ can exist.*

On any open set where $g = 0$, ψ is holomorphic. So for a general g, what we are doing is finding a function that is not holomorphic in a very specific way.

Theorem 4.2.1. *Suppose g is a $(0,1)$-form on $\mathbb{C}^n$, $n \geq 2$, given by*

$$g = g_1 d\bar{z}_1 + \cdots + g_n d\bar{z}_n,$$

where $g_j \colon \mathbb{C}^n \to \mathbb{C}$ are compactly supported smooth functions satisfying the compatibility conditions

$$\frac{\partial g_k}{\partial \bar{z}_\ell} = \frac{\partial g_\ell}{\partial \bar{z}_k} \qquad \text{for all } k, \ell = 1, 2, \ldots, n. \tag{4.1}$$

Then there exists a unique compactly supported smooth function $\psi \colon \mathbb{C}^n \to \mathbb{C}$ such that

$$\bar{\partial}\psi = g.$$

The compatibility conditions on g are necessary, but the compactness is not. However in that case the boundary of the set where the equation lives would come into play. Let us not worry about this, and prove this simple compactly supported version always has a solution. Without the compact support condition the solution is clearly not unique. Given any holomorphic f, $\bar{\partial}(\psi + f) = g$. But since the difference of any two solutions ψ_1 and ψ_2 is holomorphic, and the only holomorphic compactly supported function is 0, then the compactly supported solution ψ is unique.

Proof. We really have n smooth functions, $g_1, \ldots, g_n$, so the equation $\bar{\partial}\psi = g$ is the n equations

$$\frac{\partial \psi}{\partial \bar{z}_k} = g_k,$$

where the functions g_k satisfy the compatibility conditions (4.1).

We claim that the following is an explicit solution:

$$\begin{aligned}\psi(z) &= \frac{1}{2\pi i}\int_{\mathbb{C}} \frac{g_1(\zeta, z_2, \ldots, z_n)}{\zeta - z_1} d\zeta \wedge d\bar{\zeta} \\ &= \frac{1}{2\pi i}\int_{\mathbb{C}} \frac{g_1(\zeta + z_1, z_2, \ldots, z_n)}{\zeta} d\zeta \wedge d\bar{\zeta}.\end{aligned}$$

To show that the singularity does not matter for integrability is the same idea as for the generalized Cauchy formula.

Let us check we have the solution. We use the generalized Cauchy formula on the z_1 variable. Take R large enough so that $g_j(\zeta, z_2, \ldots, z_n)$ is zero when $|\zeta| \geq R$ for all j. For any j we get

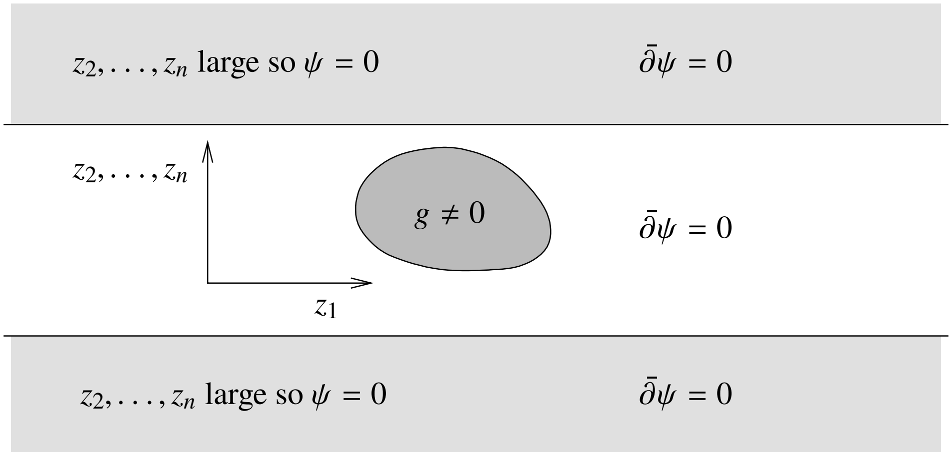
$$\begin{aligned}g_j(z_1, \ldots, z_n) &= \frac{1}{2\pi i}\int_{|\zeta|=R} \frac{g_j(\zeta, z_2, \ldots, z_n)}{\zeta - z_1} d\zeta + \frac{1}{2\pi i}\int_{|\zeta|\leq R} \frac{\frac{\partial g_j}{\partial \bar{z}_1}(\zeta, z_2, \ldots, z_n)}{\zeta - z_1} d\zeta \wedge d\bar{\zeta} \\ &= \frac{1}{2\pi i}\int_{\mathbb{C}} \frac{\frac{\partial g_j}{\partial \bar{z}_1}(\zeta, z_2, \ldots, z_n)}{\zeta - z_1} d\zeta \wedge d\bar{\zeta}.\end{aligned}$$

Using the second form of the definition of ψ, the compatibility conditions (4.1), and the above computation we get

$$\begin{aligned}\frac{\partial \psi}{\partial \bar{z}_j}(z) &= \frac{1}{2\pi i}\int_{\mathbb{C}} \frac{\frac{\partial g_1}{\partial \bar{z}_j}(\zeta + z_1, z_2, \ldots, z_n)}{\zeta} d\zeta \wedge d\bar{\zeta} \\ &= \frac{1}{2\pi i}\int_{\mathbb{C}} \frac{\frac{\partial g_j}{\partial \bar{z}_1}(\zeta + z_1, z_2, \ldots, z_n)}{\zeta} d\zeta \wedge d\bar{\zeta} \\ &= \frac{1}{2\pi i}\int_{\mathbb{C}} \frac{\frac{\partial g_j}{\partial \bar{z}_1}(z_1, z_2, \ldots, z_n)}{\zeta - z_1} d\zeta \wedge d\bar{\zeta} = g_j(z).\end{aligned}$$

***Exercise* 4.2.2:** *Show that we were allowed to differentiate under the integral in the computation above.*

That ψ has compact support follows because g_1 has compact support together with analytic continuation. In particular, ψ is holomorphic for very large z since $\bar{\partial}\psi = g = 0$ when z is large. When $z_2, \ldots, z_n$ are large, then ψ is identically zero simply from its definition. See the following diagram:



As $\bar{\partial}\psi = 0$ on the light gray and white areas in the diagram, ψ is holomorphic there. As ψ is zero on the light gray region it is zero also on the white region by analytic continuation. That is, ψ is zero on the unbounded component of the set where $g = 0$, that is, ψ has compact support. □

The first part of the proof still works when $n = 1$, so we do get a solution ψ. However the last bit of the proof does not work in one dimension, so ψ does not have compact support.

Exercise* 4.2.3*:

a) *Show that if g is supported in $K \subset\subset \mathbb{C}^n$, $n \geq 2$, then ψ is supported in the complement of the unbounded component of $\mathbb{C}^n \setminus K$. In particular, show that if K is the support of g and $\mathbb{C}^n \setminus K$ is connected, then the support of ψ is K.*

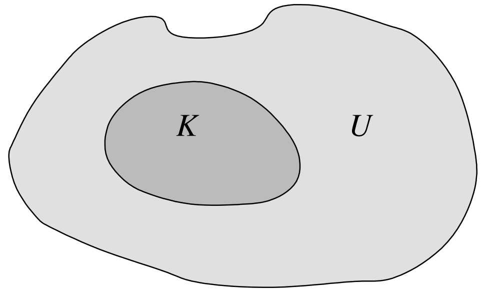
b) *Find an explicit example where the support of ψ is strictly larger than the support of g.*

Exercise* 4.2.4*: *Find an example of a smooth function $g\colon \mathbb{C} \to \mathbb{C}$ with compact support, such that no solution $\psi\colon \mathbb{C} \to \mathbb{C}$ to $\frac{\partial \psi}{\partial \bar{z}} = g$ (at least one of which always exists) is of compact support.*

4.3 The general Hartogs phenomenon

We can now prove the general Hartogs phenomenon as an application of the solution of the compactly supported inhomogeneous $\bar{\partial}$-problem. We proved special versions of this phenomenon using Hartogs figures before. The proof of the theorem has a complicated history as Hartogs' original proof from 1906 contained gaps. A fully working proof was finally supplied by Fueter in 1939 for $n = 2$ and independently by Bochner and Martinelli for higher n in the early 40s. The proof we give is the standard one given nowadays due to Leon Ehrenpreis from 1961.

Theorem 4.3.1 (Hartogs phenomenon)**.** *Let $U \subset \mathbb{C}^n$ be a domain, $n \geq 2$, and let $K \subset\subset U$ be a compact set such that $U \setminus K$ is connected. Every holomorphic $f\colon U \setminus K \to \mathbb{C}$ extends uniquely to a holomorphic function on U.*



The idea of the proof is extending in some way and then using the solution to the $\bar{\partial}$-problem to correct the result to make it holomorphic.

Proof. First find a smooth function φ that is 1 in a neighborhood of K and is compactly supported in U (exercise below). Let $f_0 = (1-\varphi)f$ on $U \setminus K$ and $f_0 = 0$ on K. The function f_0 is smooth on U and it is holomorphic and equal to f near the boundary of U, where φ is 0. We let $g = \bar{\partial} f_0$ on U, that is $g_k = \frac{\partial f_0}{\partial \bar{z}_k}$, and we let $g = 0$ outside U. As g_k are identically zero near ∂U, we find that each g_k is smooth on $\mathbb{C}^n$. The compatibility conditions (4.1) are satisfied because partial derivatives commute. Let us see why g_k is compactly supported. The only place to check is on $U \setminus K$ as elsewhere we have $g_k = 0$ automatically. Note that f is holomorphic on $U \setminus K$ and compute

$$\frac{\partial f_0}{\partial \bar{z}_k} = \frac{\partial}{\partial \bar{z}_k}\big((1-\varphi)f\big) = \frac{\partial f}{\partial \bar{z}_k} - \varphi \frac{\partial f}{\partial \bar{z}_k} - \frac{\partial \varphi}{\partial \bar{z}_k} f = -\frac{\partial \varphi}{\partial \bar{z}_k} f.$$

And $\frac{\partial \varphi}{\partial \bar{z}_k}$ is compactly supported in $U \setminus K$ by construction. Now apply the solution of the compactly supported $\bar{\partial}$-problem to find a compactly supported function ψ such that $\bar{\partial}\psi = g$. Set $F = f_0 - \psi$. Let us check that F is the desired extension. It is holomorphic:

$$\frac{\partial F}{\partial \bar{z}_k} = \frac{\partial f_0}{\partial \bar{z}_k} - \frac{\partial \psi}{\partial \bar{z}_k} = g_k - g_k = 0.$$

Next, Exercise 4.2.3 and the fact that $U \setminus K$ is connected reveals that ψ must be compactly supported in U. This means that F agrees with f near the boundary (in particular on an open set) and thus everywhere in $U \setminus K$ since $U \setminus K$ is connected. □

The hypotheses in the theorem are ideal. Clearly, no such theorem is true in one dimension, and similarly if $U \setminus K$ is disconnected, a simple counterexample can be constructed. See the exercise below.

Exercise* 4.3.1*: *Show that φ exists. Hint: Use mollifiers.*

Exercise* 4.3.2*: *Suppose $U \subset \mathbb{C}^n$ is a domain and $K \subset U$ is any compact set (perhaps $U \setminus K$ is disconnected). Prove that given $f \in \mathcal{O}(U \setminus K)$ there exists an $F \in \mathcal{O}(U)$ that equals to f on the intersection of U and the unbounded component of $\mathbb{C}^n \setminus K$.*

Exercise* 4.3.3*: *Suppose $U \subset \mathbb{C}^n$ is a domain and $K \subset U$ is a compact set such that $U \setminus K$ is disconnected. Find a counterexample to the conclusion to Hartogs.*

One of many consequences of the Hartogs phenomenon is that the zero set of a holomorphic function f is never compact in dimension 2 or higher. If it were compact, $\frac{1}{f}$ would provide a contradiction, see also Exercise 1.6.2.

Corollary 4.3.2. *Suppose $U \subset \mathbb{C}^n$, $n \geq 2$, is a domain and $f: U \to \mathbb{C}$ is holomorphic. If the zero set $f^{-1}(0)$ is not empty, then it is not compact.*

What is usually called the Hartogs–Bochner theorem (which is the C^1 version) was first stated in the real-analytic case by Severi in 1931.

Corollary 4.3.3 (Severi)**.** *Suppose $U \subset \mathbb{C}^n$, $n \geq 2$, is a bounded domain with connected real-analytic boundary and $f: \partial U \to \mathbb{C}$ is a real-analytic CR function. Then there exists some neighborhood $U' \subset \mathbb{C}^n$ of $\overline{U}$ and a holomorphic function $F: U' \to \mathbb{C}$ for which $F|_{\partial U} = f$.*

Proof. By Severi's result (Theorem 3.2.9) f extends to a small neighborhood of ∂U near each point. Because the local extension is unique for a hypersurface, we obtain an extension in a single neighborhood of ∂U. We write this neighborhood as $U' \setminus K$ for some compact K and a connected U' such that $\overline{U} \subset U'$. Consider the topological components of $\mathbb{C}^n \setminus K$. As ∂U is connected and U is bounded, the unbounded component of $\mathbb{C}^n \setminus K$ must contain all of ∂U. By boundedness of U, all the other components are relatively compact in U. If we add them to K, then K is still compact and $U' \setminus K$ is connected. We apply the Hartogs phenomenon. ◻

Exercise* 4.3.4** (Hartogs–Bochner again):*** *Let $U \subset \mathbb{C}^n$, $n \geq 2$, be a bounded domain with connected strongly pseudoconvex smooth boundary and let $f: \partial U \to \mathbb{C}$ be a smooth CR function. Prove that there exists a continuous function $F: \overline{U} \to \mathbb{C}$ holomorphic in U such that $F|_{\partial U} = f$. Note: Strong pseudoconvexity is not needed (bounded with smooth boundary will do), but that is more difficult to prove.*

Exercise* 4.3.5*: *Suppose $U \subset \mathbb{C}^n$, $n \geq 2$, is a bounded domain of holomorphy. Show that $\mathbb{C}^n \setminus U$ is connected using the Hartogs phenomenon.*

Exercise* 4.3.6*: *Suppose $W \subset U \subset \mathbb{C}^n$, $n \geq 3$, are domains such that for each fixed $z_3^0, z_4^0, \ldots, z_n^0$,*

$$\left\{(z_1, z_2) \in \mathbb{C}^2 : (z_1, z_2, z_3^0, \ldots, z_n^0) \in U \setminus W\right\} \quad \subset\subset \quad \left\{(z_1, z_2) \in \mathbb{C}^2 : (z_1, z_2, z_3^0, \ldots, z_n^0) \in U\right\}.$$

Prove that every $f \in \mathcal{O}(W)$ extends to a holomorphic function on U. Note: The fact that W is connected is important.

Exercise* 4.3.7*:

a) *Prove that if $n \geq 2$, no domain of the form $U = \mathbb{C}^n \setminus K$ for a compact K is biholomorphic to a bounded domain.*
b) *Prove that every domain of the form $U = \mathbb{C} \setminus K$ for a compact K with nonempty interior is biholomorphic to a bounded domain.*

Exercise* 4.3.8*: *Suppose $U \subset \mathbb{C}^n$, $n \geq 2$, is a domain such that for some affine $A: \mathbb{C}^2 \to \mathbb{C}^n$ the set $A^{-1}\left(\mathbb{C}^n \setminus U\right)$ has a bounded topological component. Prove that U is not a domain of holomorphy.*

Chapter 5

Integral kernels

5.1 The Bochner–Martinelli kernel

A generalization of the Cauchy's formula to several variables is called the Bochner–Martinelli integral formula, which in fact reduces to Cauchy's formula when $n = 1$. And just like for Cauchy's formula, we will prove the formula for all smooth functions using Stokes theorem.

First, let us define the *Bochner–Martinelli kernel*:

$$\omega(\zeta, z) \overset{\text{def}}{=} \frac{(n-1)!}{(2\pi i)^n} \sum_{j=1}^{n} \frac{\bar{\zeta}_j - \bar{z}_j}{\lVert \zeta - z \rVert^{2n}} \, d\bar{\zeta}_1 \wedge d\zeta_1 \wedge \cdots \wedge \widehat{d\bar{\zeta}_j} \wedge d\zeta_j \wedge \cdots \wedge d\bar{\zeta}_n \wedge d\zeta_n.$$

The notation $\widehat{d\bar{\zeta}_j}$ means that this term is simply left out.

Theorem 5.1.1 (Bochner–Martinelli)**.** *Let $U \subset \mathbb{C}^n$ be a bounded open set with smooth boundary and let $f \colon \overline{U} \to \mathbb{C}$ be a smooth function, then for $z \in U$:*

$$f(z) = \int_{\partial U} f(\zeta)\omega(\zeta, z) - \int_U \bar{\partial} f(\zeta) \wedge \omega(\zeta, z).$$

In particular, if $f \in \mathcal{O}(U)$, then

$$f(z) = \int_{\partial U} f(\zeta)\omega(\zeta, z).$$

Recall that if $\zeta = x + iy$ are the coordinates in $\mathbb{C}^n$, the orientation that we assigned to $\mathbb{C}^n$ in this book* is the one corresponding to the volume form

$$dV = dx_1 \wedge dy_1 \wedge dx_2 \wedge dy_2 \wedge \cdots \wedge dx_n \wedge dy_n.$$

With this orientation,

$$d\zeta_1 \wedge d\bar{\zeta}_1 \wedge d\zeta_2 \wedge d\bar{\zeta}_2 \wedge \cdots \wedge d\zeta_n \wedge d\bar{\zeta}_n = (-2i)^n dV,$$

and hence

$$d\bar{\zeta}_1 \wedge d\zeta_1 \wedge d\bar{\zeta}_2 \wedge d\zeta_2 \wedge \cdots \wedge d\bar{\zeta}_n \wedge d\zeta_n = (2i)^n dV.$$

*Again, there is no canonical orientation of $\mathbb{C}^n$, and not all authors follow this (perhaps more prevalent) convention.

Exercise* 5.1.1*: *Similarly to the Cauchy–Pompeiu formula, note the singularity in the second term of the Bochner–Martinelli formula, and prove that the integral still makes sense (the function is integrable).*

Exercise* 5.1.2*: *Check that for $n = 1$, the Bochner–Martinelli formula becomes the standard Cauchy–Pompeiu formula.*

We will, again, apply Stokes formula, but we need to apply it to forms of higher degree. As before, we split the derivatives into the holomorphic and antiholomorphic parts. We work with multiindices. For α and β with $|\alpha| = p$ and $|\beta| = q$, a differential form

$$\eta = \sum_{\substack{|\alpha|=p \\ |\beta|=q}} \eta_{\alpha\beta}\, dz^\alpha \wedge d\bar{z}^\beta$$

is called a (p,q)-*form* or a differential form of *bidegree* (p,q). Define

$$\partial\eta \overset{\text{def}}{=} \sum_{\substack{|\alpha|=p \\ |\beta|=q}} \sum_{j=1}^{n} \frac{\partial\eta_{\alpha\beta}}{\partial z_j} dz_j \wedge dz^\alpha \wedge d\bar{z}^\beta, \qquad \text{and} \qquad \bar{\partial}\eta \overset{\text{def}}{=} \sum_{\substack{|\alpha|=p \\ |\beta|=q}} \sum_{j=1}^{n} \frac{\partial\eta_{\alpha\beta}}{\partial \bar{z}_j} d\bar{z}_j \wedge dz^\alpha \wedge d\bar{z}^\beta.$$

It is not difficult to see that $d\eta = \partial\eta + \bar{\partial}\eta$ as before.

Proof of Bochner–Martinelli. The structure of the proof is essentially the same as that of the Cauchy–Pompeiu theorem for $n = 1$, although some of the formulas are somewhat more involved.

Let $z \in U$ be fixed. Suppose $r > 0$ is small enough such that $\overline{B_r(z)} \subset U$. We orient ∂U and $\partial B_r(z)$ both positively. Notice that $f(\zeta)\omega(\zeta,z)$ contains all the holomorphic $d\zeta_j$. Therefore,

$$\begin{aligned} d\big(f(\zeta)\omega(\zeta,z)\big) &= \bar{\partial}\big(f(\zeta)\omega(\zeta,z)\big) \\ &= \bar{\partial} f(\zeta) \wedge \omega(\zeta,z) \\ &\quad + f(\zeta)\frac{(n-1)!}{(2\pi i)^n} \sum_{j=1}^{n} \frac{\partial}{\partial\bar{\zeta}_j}\left[\frac{\bar{\zeta}_j - \bar{z}_j}{\|\zeta - z\|^{2n}}\right] d\bar{\zeta}_1 \wedge d\zeta_1 \wedge \cdots \wedge d\bar{\zeta}_n \wedge d\zeta_n. \end{aligned}$$

We compute

$$\sum_{j=1}^{n} \frac{\partial}{\partial\bar{\zeta}_j}\left[\frac{\bar{\zeta}_j - \bar{z}_j}{\|\zeta - z\|^{2n}}\right] = \sum_{j=1}^{n} \left(\frac{1}{\|\zeta - z\|^{2n}} - n\frac{\left|\zeta_j - z_j\right|^2}{\|\zeta - z\|^{2n+2}}\right) = 0.$$

Therefore $d\big(f(\zeta)\omega(\zeta,z)\big) = \bar{\partial} f(\zeta) \wedge \omega(\zeta,z)$. We apply Stokes as before:

$$\int_{\partial U} f(\zeta)\omega(\zeta,z) - \int_{\partial B_r(z)} f(\zeta)\omega(\zeta,z) = \int_{U\setminus\overline{B_r(z)}} d\big(f(\zeta)\omega(\zeta,z)\big) = \int_{U\setminus\overline{B_r(z)}} \bar{\partial} f(\zeta) \wedge \omega(\zeta,z).$$

Again, due to the integrability, which you showed in an above exercise, the right hand side converges to the integral over U as $r \to 0$. Just as for the Cauchy–Pompeiu formula, we now need to show that the integral over $\partial B_r(z)$ goes to $f(z)$ as $r \to 0$.

So

$$\int_{\partial B_r(z)} f(\zeta)\omega(\zeta,z) = f(z)\int_{\partial B_r(z)} \omega(\zeta,z) + \int_{\partial B_r(z)} \big(f(\zeta)-f(z)\big)\omega(\zeta,z).$$

To finish the proof, we will show that $\int_{\partial B_r(z)} \omega(\zeta,z) = 1$, and that the second term goes to zero. We apply Stokes again and note that the volume of $B_r(z)$ is $\frac{\pi^n}{n!} r^{2n}$.

$$\begin{aligned}
\int_{\partial B_r(z)} \omega(\zeta,z) &= \int_{\partial B_r(z)} \frac{(n-1)!}{(2\pi i)^n} \sum_{j=1}^n \frac{\bar{\zeta}_j - \bar{z}_j}{\lVert \zeta - z\rVert^{2n}}\, d\bar{\zeta}_1 \wedge d\zeta_1 \wedge \cdots \wedge \widehat{d\bar{\zeta}_j} \wedge d\zeta_j \wedge \cdots \wedge d\bar{\zeta}_n \wedge d\zeta_n \\
&= \frac{(n-1)!}{(2\pi i)^n} \frac{1}{r^{2n}} \int_{\partial B_r(z)} \sum_{j=1}^n (\bar{\zeta}_j - \bar{z}_j) d\bar{\zeta}_1 \wedge d\zeta_1 \wedge \cdots \wedge \widehat{d\bar{\zeta}_j} \wedge d\zeta_j \wedge \cdots \wedge d\bar{\zeta}_n \wedge d\zeta_n \\
&= \frac{(n-1)!}{(2\pi i)^n} \frac{1}{r^{2n}} \int_{B_r(z)} d\left(\sum_{j=1}^n (\bar{\zeta}_j - \bar{z}_j) d\bar{\zeta}_1 \wedge d\zeta_1 \wedge \cdots \wedge \widehat{d\bar{\zeta}_j} \wedge d\zeta_j \wedge \cdots \wedge d\bar{\zeta}_n \wedge d\zeta_n \right) \\
&= \frac{(n-1)!}{(2\pi i)^n} \frac{1}{r^{2n}} \int_{B_r(z)} n\, d\bar{\zeta}_1 \wedge d\zeta_1 \wedge \cdots \wedge d\bar{\zeta}_n \wedge d\zeta_n \\
&= \frac{(n-1)!}{(2\pi i)^n} \frac{1}{r^{2n}} \int_{B_r(z)} n (2i)^n dV = 1.
\end{aligned}$$

Next, we tackle the second term. Via the same computation as above we find

$$\begin{aligned}
\int_{\partial B_r(z)} \big(f(\zeta)-f(z)\big)\omega(\zeta,z) = \frac{(n-1)!}{(2\pi i)^n} \frac{1}{r^{2n}} \Bigg(&\int_{B_r(z)} \big(f(\zeta)-f(z)\big) n\, d\bar{\zeta}_1 \wedge d\zeta_1 \wedge \cdots \wedge d\bar{\zeta}_n \wedge d\zeta_n \\
&+ \int_{B_r(z)} \sum_{j=1}^n \frac{\partial f}{\partial \bar{\zeta}_j}(\zeta)(\bar{\zeta}_j - \bar{z}_j)\, d\bar{\zeta}_1 \wedge d\zeta_1 \wedge \cdots \wedge d\bar{\zeta}_n \wedge d\zeta_n \Bigg).
\end{aligned}$$

As U is bounded, $\lvert f(\zeta) - f(z)\rvert \leq M\lVert \zeta - z\rVert$ and $\left\lvert \frac{\partial f}{\partial \bar{\zeta}_j}(\zeta)(\bar{\zeta}_j - \bar{z}_j)\right\rvert \leq M\lVert \zeta - z\rVert$ for some M. So for all $\zeta \in \partial B_r(z)$, we have $\lvert f(\zeta) - f(z)\rvert \leq Mr$ and $\left\lvert \frac{\partial f}{\partial \bar{\zeta}_j}(\zeta)(\bar{\zeta}_j - \bar{z}_j)\right\rvert \leq Mr$. Hence

$$\left\lvert \int_{\partial B_r(z)} \big(f(\zeta)-f(z)\big)\omega(\zeta,z) \right\rvert \leq \frac{(n-1)!}{(2\pi)^n} \frac{1}{r^{2n}} \left(\int_{B_r(z)} n 2^n Mr\, dV + \int_{B_r(z)} n 2^n Mr\, dV \right) = 2Mr.$$

Therefore, this term goes to zero as $r \to 0$. □

One drawback of the Bochner–Martinelli formula is that the kernel is not holomorphic in z unless $n = 1$. That is, it does not simply produce holomorphic functions. If we differentiate in $\bar{z}$ underneath the ∂U integral, we do not necessarily obtain zero. On the other hand, we have an explicit formula and this formula does not depend on U. This is not be the case of the Bergman and Szegö kernels, which we will see next, although those are holomorphic in the right way.

***Exercise* 5.1.3:** *Prove that if $z \notin \overline{U}$, then rather than $f(z)$ in the formula you obtain*

$$\int_{\partial U} f(\zeta)\omega(\zeta,z) - \int_U \bar{\partial} f(\zeta) \wedge \omega(\zeta,z) = 0.$$

***Exercise* 5.1.4:** *Suppose f is holomorphic on a neighborhood of $\overline{B_r(z)}$.*

a) Using the Bochner–Martinelli formula, prove that

$$f(z) = \frac{1}{V\big(B_r(z)\big)} \int_{B_r(z)} f(\zeta)\, dV(\zeta),$$

where $V\big(B_r(z)\big)$ is the volume of $B_r(z)$.

b) Use part a) to prove the maximum principle for holomorphic functions.

***Exercise* 5.1.5:** *Use Bochner–Martinelli for the solution of $\bar{\partial}$ with compact support. That is, suppose g is a smooth compactly supported $(0,1)$-form on $\mathbb{C}^n$, $n \geq 2$, $g = g_1 d\bar{z}_1 + \cdots + g_n d\bar{z}_n$, and $\frac{\partial g_k}{\partial \bar{z}_\ell} = \frac{\partial g_\ell}{\partial \bar{z}_k}$ for all k, ℓ. Prove that*

$$\psi(z) = -\int_{\mathbb{C}^n} g(\zeta) \wedge \omega(\zeta,z)$$

is a compactly supported smooth solution to $\bar{\partial}\psi = g$. Hint: Look at the previous proof.

5.2 The Bergman kernel

Let $U \subset \mathbb{C}^n$ be a domain. Define

$$A^2(U) \overset{\text{def}}{=} \mathcal{O}(U) \cap L^2(U).$$

That is, let $A^2(U)$ denote the space of holomorphic functions $f \in \mathcal{O}(U)$ such that

$$\|f\|^2_{A^2(U)} \overset{\text{def}}{=} \|f\|^2_{L^2(U)} = \int_U |f(z)|^2 dV < \infty.$$

The space $A^2(U)$ is called the *Bergman space* of U. The inner product is the $L^2(U)$ inner product

$$\langle f, g\rangle \overset{\text{def}}{=} \int_U f(z)\overline{g(z)}\, dV.$$

We need to prove that $A^2(U)$ is complete, in other words that it is a Hilbert space. We first have to prove that we can bound the uniform norm on compact sets via the $A^2(U)$ norm.

Lemma 5.2.1. *Let $U \subset \mathbb{C}^n$ be a domain and $K \subset\subset U$ compact. Then there exists a constant C_K, such that for any $f \in A^2(U)$ we have*

$$\|f\|_K = \sup_{z\in K} |f(z)| \leq C_K \|f\|_{A^2(U)}.$$

Proof. As K is compact there exists an $r > 0$ such that for all $z \in K$ we have $\overline{\Delta_r(z)} \subset U$. Fix $z \in K$, apply Exercise 1.2.8 and Cauchy–Schwarz again:

$$\begin{aligned} |f(z)| &= \left| \frac{1}{V\big(\Delta_r(z)\big)} \int_{\Delta_r(z)} f(\xi)\, dV(\xi) \right| \\ &\leq \frac{1}{\pi^n r^{2n}} \sqrt{\int_{\Delta_r(z)} 1^2 \, dV(\xi)} \sqrt{\int_{\Delta_r(z)} |f(\xi)|^2 \, dV(\xi)} \\ &= \frac{1}{\pi^{n/2} r^n} \|f\|_{A^2(\Delta_r(z))} \leq \frac{1}{\pi^{n/2} r^n} \|f\|_{A^2(U)}. \end{aligned}$$

□

A sequence $\{f_j\}$ of functions in $A^2(U)$ converging in $L^2(U)$ to some $f \in L^2(U)$ converges uniformly on compact sets. Therefore $f \in \mathcal{O}(U)$. Consequently, $A^2(U)$ is a closed subspace of $L^2(U)$, and hence complete. For a bounded domain $A^2(U)$ is always infinite-dimensional, see exercise below. However, there do exist unbounded domains for which either $A^2(U)$ is trivial (just the zero function) or even finite-dimensional. When $n = 1$, however, $A^2(U)$ is either trivial, or infinite-dimensional.

Exercise* 5.2.1*: *Show that if $U \subset \mathbb{C}^n$ is bounded, then $A^2(U)$ is infinite-dimensional.*

Exercise* 5.2.2*:

a) Show that $A^2(\mathbb{C}^n)$ is trivial (it is just the zero function).

b) Show that $A^2(\mathbb{D} \times \mathbb{C})$ is trivial.

c) Find an example of an unbounded domain U for which $A^2(U)$ is infinite-dimensional. Hint: Think in one dimension for simplicity.

Exercise* 5.2.3*: *Show that $A^2(\mathbb{D})$ can be identified with $A^2(\mathbb{D} \setminus \{0\})$, that is, every function in the latter can be extended to a function in the former.*

Again using the lemma, we notice that point evaluation is a bounded linear functional, that is take $K = \{z\}$, then the linear operator

$$f \mapsto f(z)$$

is a bounded linear functional. By Riesz–Fisher theorem, there exists a $k_z \in A^2(U)$, such that

$$f(z) = \langle f, k_z \rangle.$$

Define the *Bergman kernel* for U as

$$K_U(z, \bar{\zeta}) \overset{\text{def}}{=} \overline{k_z(\zeta)}.$$

The function K_U is defined as $(z, \bar{\zeta})$ vary over $U \times U^*$, where $U^* = \{\zeta \in \mathbb{C}^n : \bar{\zeta} \in U\}$.

Then for all $f \in A^2(U)$ we have

$$f(z) = \int_U f(\zeta) K_U(z, \bar{\zeta})\, dV(\zeta). \tag{5.1}$$

This last equation is sometimes called the *reproducing property* of the kernel.

It should be noted that the Bergman kernel depends on U, which is why we write it as $K_U(z, \bar{\zeta})$.

Proposition 5.2.2. *The Bergman kernel $K_U(z,\bar{\zeta})$ is holomorphic in z and antiholomorphic in ζ, and furthermore*

$$\overline{K_U(z,\bar{\zeta})} = K_U(\zeta,\bar{z}).$$

Proof. As each k_z is in $A^2(U)$ it is holomorphic in ζ. Hence, K_U is antiholomorphic in ζ. Thus if we prove $\overline{K_U(z,\bar{\zeta})} = K_U(\zeta,\bar{z})$, then we find that K_U is holomorphic in z.

As $\overline{K_U(z,\bar{\zeta})} = k_z(\zeta)$ is in $A^2(U)$, then

$$\begin{aligned}\overline{K_U(z,\bar{\zeta})} &= \int_U \overline{K_U(z,\bar{w})} K_U(\zeta,\bar{w})\, dV(w) \\ &= \overline{\left(\int_U \overline{K_U(\zeta,\bar{w})} K_U(z,\bar{w})\, dV(w)\right)} = \overline{\overline{K_U(\zeta,\bar{z})}} = K_U(\zeta,\bar{z}).\end{aligned}$$

□

In particular, by the proposition K_U is analytic; thinking of $\bar{\zeta}$ as the variable, then K_U is a holomorphic function of $2n$ variables.

Example 5.2.3: Let us compute the Bergman kernel (and the Szegö kernel of the next section while we're at it) explicitly for the unit disc $\mathbb{D} \subset \mathbb{C}$. Let $f \in \mathcal{O}(\mathbb{D}) \cap C(\overline{D})$, that is holomorphic in $\mathbb{D}$ and continuous up to the boundary. Let $z \in \mathbb{D}$. Then

$$f(z) = \frac{1}{2\pi i} \int_{\partial \mathbb{D}} \frac{f(\zeta)}{\zeta - z}\, d\zeta.$$

On the unit circle $\zeta\bar{\zeta} = 1$. Let ds be the arc-length measure on the circle. Parametrize the circle as $\zeta = e^{is}$. Then $d\zeta = ie^{is} ds$, and

$$\begin{aligned} f(z) &= \frac{1}{2\pi i} \int_{\partial \mathbb{D}} \frac{f(\zeta)}{\zeta - z}\, d\zeta \\ &= \frac{1}{2\pi i} \int_{\partial \mathbb{D}} \frac{f(\zeta)}{1 - z\bar{\zeta}} \bar{\zeta}\, d\zeta \\ &= \frac{1}{2\pi} \int_{\partial \mathbb{D}} \frac{f(\zeta)}{1 - z\bar{\zeta}}\, ds. \end{aligned}$$

The integral is now a regular line integral of a function whose singularity, which used to be inside the unit disc, disappeared (we "reflected it" to the outside). The kernel $\frac{1}{2\pi}\frac{1}{1-z\bar{\zeta}}$ is called the *Szegö kernel*, which we will briefly mention next. We apply Stokes to the second integral above.

$$\begin{aligned} \frac{1}{2\pi i} \int_{\partial \mathbb{D}} \frac{f(\zeta)}{1 - z\bar{\zeta}} \bar{\zeta}\, d\zeta &= \frac{1}{2\pi i} \int_{\mathbb{D}} f(\zeta) \frac{\partial}{\partial \bar{\zeta}} \left[\frac{\bar{\zeta}}{1 - z\bar{\zeta}} \right] d\bar{\zeta} \wedge d\zeta \\ &= \frac{1}{\pi} \int_{\mathbb{D}} \frac{f(\zeta)}{(1 - z\bar{\zeta})^2}\, dA(\zeta). \end{aligned}$$

The Bergman kernel in the unit disc is, therefore,

$$K_U(z,\bar{\zeta}) = \frac{1}{\pi} \frac{1}{(1 - z\bar{\zeta})^2}.$$

The fact that this really is the Bergman kernel follows from the exercises below. That is, that K_U is the unique conjugate symmetric reproducing function that is in $A^2(U)$ for a fixed ζ. We have only shown the formula for functions continuous up to the boundary, but those are dense in $A^2(U)$.

Example 5.2.4: On the other hand you found in an exercise that $A^2(\mathbb{C}^n) = \{0\}$. Therefore we have trivially $K_{\mathbb{C}^n}(z,\bar{\zeta}) \equiv 0$.

In general it is difficult to compute the kernel explicitly for just any U, although we have the following formula for it.

Proposition 5.2.5. *Suppose $\{\varphi_j(z)\}_{j\in I}$ is a complete orthonormal system for $A^2(U)$. Then*

$$K_U(z,\bar{\zeta}) = \sum_{j\in I} \varphi_j(z)\overline{\varphi_j(\zeta)},$$

with uniform convergence on compact subsets of $U \times U^*$.

Proof. For any fixed $\zeta \in U$, the function $z \mapsto K_U(z,\bar{\zeta})$ is in $A^2(U)$ and so expand this function in terms of the basis and use the reproducing property of K_U

$$K_U(z,\bar{\zeta}) = \sum_{j\in I}\left(\int_U K_U(w,\bar{\zeta})\overline{\varphi_j(w)}\,dV(w)\right)\varphi_j(z) = \sum_{j\in I}\overline{\varphi_j(\zeta)}\varphi_j(z).$$

The convergence is in L^2 as a function of z, for a fixed ζ. Let $K \subset\subset U$ be a compact set. Via Lemma 5.2.1, L^2 convergence in $A^2(U)$ is uniform convergence on compact sets. Therefore for a fixed ζ the convergence is uniform in $z \in K$. In particular we get pointwise convergence so

$$\sum_{j\in I}\left|\varphi_j(z)\right|^2 = \sum_{j\in I}\varphi_j(z)\overline{\varphi_j(z)} = K_U(z,\bar{z}) \le C_K < \infty,$$

where C_K is the supremum of $K_U(z,\bar{\zeta})$ on $K \times K^*$. Hence for $(z,\bar{\zeta}) \in K \times K^*$,

$$\sum_{j\in I}\left|\varphi_j(z)\overline{\varphi_j(\zeta)}\right| \le \sqrt{\sum_{j\in I}\left|\varphi_j(z)\right|^2}\sqrt{\sum_{j\in I}\left|\varphi_j(\zeta)\right|^2} \le C_K < \infty.$$

And so the convergence is uniform on $K \times K^*$. □

***Exercise* 5.2.4:**
a) Show that if $U \subset \mathbb{C}^n$ is bounded then for all $z \in U$ we find $K_U(z,\bar{z}) > 0$.
b) Why can this fail if U is unbounded? Find a (trivial) counterexample.

***Exercise* 5.2.5:** *Show that given a domain $U \subset \mathbb{C}^n$, the Bergman kernel is the unique function $K_U(z,\bar{\zeta})$ such that*
1) for a fixed ζ, $K_U(z,\bar{\zeta})$ is in $A^2(U)$,
2) $\overline{K_U(z,\bar{\zeta})} = K_U(\zeta,\bar{z})$,
3) the reproducing property (5.1) holds.

***Exercise* 5.2.6:** *Let $U \subset \mathbb{C}^n$ be a bounded domain with smooth boundary. Show that the functions in $A^2(U) \cap C(\overline{U})$ are dense in $A^2(U)$. In particular this exercise says we only need to check the reproducing property on functions continuous up to the boundary to show we have the Bergman kernel.*

Exercise* 5.2.7*: *Let $U, V \subset \mathbb{C}^n$ be two domains with a biholomorphism $f \colon U \to V$. Show that*

$$K_U(z, \bar{\zeta}) = \det Df(z)\, \overline{\det Df(\zeta)}\, K_V\big(f(z), \overline{f(\zeta)}\big).$$

Exercise* 5.2.8*: *Show that the Bergman kernel for the polydisc is*

$$K_{\mathbb{D}^n}(z, \bar{\zeta}) = \frac{1}{\pi^n} \prod_{j=1}^{n} \frac{1}{(1 - z_j \bar{\zeta}_j)^2}.$$

Exercise* 5.2.9** (Hard):*** *Show that for the unit ball $\mathbb{B}_n$, then for some constants c_α, the set of all monomials $\frac{z^\alpha}{c_\alpha}$ gives a complete orthonormal system. Hint: To show orthonormality compute the integral using polar coordinates in each variable separately, that is let $z_j = r_j e^{i\theta_j}$ where $\theta \in [0, 2\pi]^n$ and $\sum_j r_j^2 < 1$. Then show completeness by showing that if $f \in A^2(\mathbb{B}_n)$ is orthogonal to all z^α then $f = 0$. Finding $c_\alpha = \sqrt{\frac{\pi^n \alpha!}{(n+|\alpha|)!}}$ requires the classical β function of special function theory.*

Exercise* 5.2.10*: *Using the previous exercise, show that the Bergman kernel for the unit ball is*

$$K_{\mathbb{B}_n}(z, \bar{\zeta}) = \frac{n!}{\pi^n} \frac{1}{(1 - \langle z, \zeta \rangle)^{n+1}},$$

where $\langle z, \zeta \rangle$ is the standard inner product on $\mathbb{C}^n$.

5.3 The Szegö kernel

We can use the same techniques to create a reproducing kernel on the boundary by starting with $L^2(\partial U, d\sigma)$ instead of $L^2(U)$ to obtain a kernel where we integrate over the boundary rather than the domain itself. We will give a quick overview here though we will leave out the details.

Let $U \subset \mathbb{C}^n$ be a bounded domain with smooth boundary. Let $C(\overline{U}) \cap \mathcal{O}(U)$ be the holomorphic functions in U continuous up to the boundary. The restriction of $f \in C(\overline{U}) \cap \mathcal{O}(U)$ to ∂U is a continuous function and hence $f|_{\partial U}$ is in $L^2(\partial U, d\sigma)$, where $d\sigma$ is the surface measure on ∂U. Taking a closure of these restrictions in $L^2(\partial U)$ obtains the Hilbert space $H^2(\partial U)$, which is called the *Hardy space*. The inner product in this case is the $L^2(\partial U, d\sigma)$ inner product:

$$\langle f, g \rangle \overset{\text{def}}{=} \int_{\partial U} f(z) \overline{g(z)}\, d\sigma(z).$$

Exercise* 5.3.1*: *Show that monomials z^α are a complete orthonormal system in $H^2(\partial \mathbb{B}_n)$.*

Exercise* 5.3.2*: *Show that for any bounded $U \subset \mathbb{C}^n$ with smooth boundary, $H^2(\partial U)$ is infinite-dimensional.*

Let

$$Pf(z) = \int_{\partial U} f(\zeta)\, P(z,\zeta)\, d\sigma(\zeta)$$

be the Poisson integral, that is, $P(z,\zeta)$ is the Poisson kernel, which reproduces harmonic functions. As holomorphic functions are harmonic we find that if $f \in C(\overline{U}) \cap \mathcal{O}(U)$, then $Pf = f$.

For $z \in U$ we can also find $Pf(z)$ for $f \in H^2(\partial U)$. For each $z \in U$,

$$f \mapsto Pf(z)$$

defines a continuous linear functional. So again we find a $s_z \in H^2(\partial U)$ such that

$$Pf(z) = \langle f, s_z \rangle.$$

Hence for $z \in U$ and $\zeta \in \partial U$ we define

$$S_U(z,\bar{\zeta}) \overset{\text{def}}{=} \overline{s_z(\zeta)},$$

although for a fixed z this is a function only defined almost everywhere as it is an element of $L^2(\partial U, d\sigma)$. The function S_U is the *Szegö kernel*. If $f \in H^2(\partial U)$, then

$$Pf(z) = \int_{\partial U} f(\zeta)\, S_U(z,\bar{\zeta})\, d\sigma(\zeta).$$

As functions in $H^2(\partial U)$ extend to $\overline{U}$, then often $f \in H^2(\partial U)$ is considered a function on $\overline{U}$ where values in U are given by Pf. Similarly we extend $S(z,\bar{\zeta})$ to a function on $U \times \overline{U}^*$ (where the values on the boundary are defined only almost everywhere). We state without proof that if $\{\varphi_j\}$ is a complete orthonormal system for $H^2(\partial U)$, then

$$S_U(z,\bar{\zeta}) = \sum_j \varphi_j(z)\overline{\varphi_j(\zeta)} \tag{5.2}$$

for $(z,\bar{\zeta}) \in U \times U^*$ uniformly on compact subsets. As before, from this formula it can be seen that S is conjugate symmetric, and so it extends to $(U \times \overline{U}^*) \cup (\overline{U} \times U^*)$.

Example 5.3.1: For the unit disc, we computed before that for $f \in C(\overline{\mathbb{D}}) \cap \mathcal{O}(\mathbb{D})$:

$$f(z) = \frac{1}{2\pi} \int_{\partial\mathbb{D}} \frac{f(\zeta)}{1 - z\bar{\zeta}}\, ds.$$

That is, $S_{\mathbb{D}}(z,\zeta) = \frac{1}{\pi}\frac{1}{1-z\bar{\zeta}}$.

***Exercise* 5.3.3:** *Using (5.2) compute $S_{\mathbb{B}_n}$.*

Chapter 6

Complex analytic varieties

6.1 The ring of germs

Definition 6.1.1. Let p be a point in a topological space X. Let Y be a set and $U, V \subset X$ be open neighborhoods of p. We say that two functions $f: U \to Y$ and $g: V \to Y$ are equivalent if there exists a neighborhood W of p such that $f|_W = g|_W$.

An equivalence class of functions defined in a neighborhood of p is called a *germ of a function*. Usually it is denoted by (f, p), but we simply say f when the context is clear.

The set of germs of complex-valued functions forms a commutative ring, see exercise below to check the details. For example, to multiply (f, p) and (g, p), take two representatives f and g defined on a common neighborhood multiply them and then consider the germ (fg, p). Similarly $(f, p) + (g, p)$ is defined as $(f + g, p)$. It is easy to check that these operations are well-defined.

***Exercise* 6.1.1:** *Let X be a topological space and $p \in X$. Let $\mathcal{F}$ be a class of complex-valued functions defined on open subsets of such that whenever $f: U \to \mathbb{C}$ is in $\mathcal{F}$ and $W \subset U$ is open, then $f|_W \in \mathcal{F}$, and such that whenever f and g are two functions in $\mathcal{F}$, and W is an open set where both are defined, then $fg|_W$ and $(f + g)|_W$ are also in $\mathcal{F}$. Assume that all constant functions are in $\mathcal{F}$. Show that the ring operations defined above on a set of germs at p of functions from $\mathcal{F}$ are well-defined, and that the set of germs at p of functions from $\mathcal{F}$ is a commutative ring.*

***Exercise* 6.1.2:** *Let $X = Y = \mathbb{R}$ and $p = 0$. Consider the ring of germs of continuous functions (or smooth functions if you wish). Show that for every continuous $f: \mathbb{R} \to \mathbb{R}$ and every neighborhood W of 0, there exists a $g: \mathbb{R} \to \mathbb{R}$ such that $(f, 0) = (g, 0)$, but $g|_W \neq f|_W$.*

Germs are particularly useful for holomorphic functions because of the identity theorem. In particular, the behavior of the above exercise does not happen for holomorphic functions. Furthermore, for holomorphic functions, the ring of germs is the same as the ring of convergent power series, see exercise below. No similar result is true for only smooth functions.

Definition 6.1.2. Let $p \in \mathbb{C}^n$. Write ${}_n\mathcal{O}_p = \mathcal{O}_p$ as the ring of germs at p of holomorphic functions.

The ring of germs $\mathcal{O}_p$ has many nice properties, and it is generally a "nicer" ring than the ring $\mathcal{O}(U)$ for some open U, and so it is easier to work with if we are interested in local properties.

Exercise* 6.1.3*:

a) Show that $\mathcal{O}_p$ is an integral domain (has no zero divisors).

b) Prove the ring of germs at $0 \in \mathbb{R}$ of smooth real-valued functions is not an integral domain.

Exercise* 6.1.4*: *Show that the units (elements with multiplicative inverse) of $\mathcal{O}_p$ are the germs of functions which do not vanish at p.*

Exercise* 6.1.5*:

a) (easy) Show that given a germ $(f,p) \in \mathcal{O}_p$, there exists a fixed open neighborhood U of p and a representative $f : U \to \mathbb{C}$ such that any other representative g can be analytically continued from p to a holomorphic function U.

b) (easy) Given two representatives $f : U \to \mathbb{C}$ and $g : V \to \mathbb{C}$ of a germ $(f,p) \in \mathcal{O}_p$, let W be the connected component of $U \cap V$ that contains p. Then $f|_W = g|_W$.

c) There exists a germ $(f,p) \in \mathcal{O}_p$, such that for any open neighborhood U of p, and any representative $f : U \to \mathbb{C}$ we can find another representative of $g : V \to \mathbb{C}$ of that same germ such that $g|_{U\cap V} \neq f|_{U\cap V}$. Hint: $n = 1$ is sufficient.

Exercise* 6.1.6*: *Show that $\mathcal{O}_p$ is isomorphic to the ring of convergent power series.*

Definition 6.1.3. Let p be a point in a topological space X. We say that sets $A, B \subset X$ are equivalent if there exists a neighborhood W of p such that $A \cap W = B \cap W$. An equivalence class of sets is called a *germ of a set* at p. It is denoted by (A,p), but we may write A when the context is clear.

The concept of $(X,p) \subset (Y,p)$ is defined in an obvious manner, that is, there exist representatives X and Y, and a neighborhood W of p such that $X \cap W \subset Y \cap W$. Similarly if (X,p), (Y,p) are germs and X, Y are any representatives of these germs, then the intersection $(X,p) \cap (Y,p)$, is the germ $(X \cap Y, p)$ and the union $(X,p) \cup (Y,p)$ is the germ $(X \cup Y, p)$.

Exercise* 6.1.7*: *Check that the definition of subset, union, and intersection of germs of sets is well-defined.*

Let R be some ring of germs of complex-valued functions at $p \in X$ for some topological space X. If f is a complex-valued function, let Z_f be the zero set of f, that is $f^{-1}(0)$. When $(f,p) \in R$ is a germ of a function it makes sense to talk about the germ (Z_f, p). We take the zero set of some representative and look at its germ at p.

Exercise* 6.1.8*: *Suppose f and g are two representatives of a germ (f,p) show that the germs (Z_f,p) and (Z_g,p) are the same.*

Exercise* 6.1.9*: *Show that if (f,p) and (g,p) are in R and f and g are some representatives, then $(Z_f,p) \cup (Z_g,p) = (Z_{fg},p)$.*

6.2 Weierstrass preparation and division theorems

Suppose f is (a germ of) a holomorphic function at a point $p \in \mathbb{C}^n$. Write

$$f(z) = \sum_{k=0}^{\infty} f_k(z-p),$$

where f_k is a homogeneous polynomial of degree k, that is, $f_k(tz) = t^k f_k(z)$.

Definition 6.2.1. Let $p \in \mathbb{C}^n$ and f be a function holomorphic in a neighborhood of p. If f is not identically zero, define

$$\operatorname{ord}_p f \overset{\text{def}}{=} \min\{k \in \mathbb{N}_0 : f_k \not\equiv 0\}.$$

If $f \equiv 0$, then define $\operatorname{ord}_p f = \infty$. The number $\operatorname{ord}_p f$ is called the *order of vanishing* of f at p.

In other words, if the order of vanishing of f at p is k, then all partial derivatives of order less than k vanish at p, and there exists at least one derivative of order k that does not vanish at p.

In one complex variable, a holomorphic function vanishing to order k at 0 equals (locally) $z^k u(z)$ for a nonvanishing u. In several variables, there is a similar theorem, or in fact a pair of theorems, the so-called Weierstrass preparation and division theorems.

Definition 6.2.2. Let $U \subset \mathbb{C}^{n-1}$ be open, and let $z' \in \mathbb{C}^{n-1}$ denote the coordinates. Suppose a polynomial $P \in \mathcal{O}(U)[z_n]$ is monic of degree $k \geq 0$, that is,

$$P(z', z_n) = z_n^k + \sum_{j=0}^{k-1} c_j(z')\, z_n^j,$$

where c_j are holomorphic functions defined on U, such that $c_j(0) = 0$ for all j. Then P is called a *Weierstrass polynomial* of degree k. If the c_j are germs in $\mathcal{O}_0 = {}_{n-1}\mathcal{O}_0$, then $P \in \mathcal{O}_0[z_n]$ and P is a *germ of a Weierstrass polynomial*.

The definition (and the theorem that follows) still holds for $n = 1$. If you read the definition carefully, you will find that if $n = 1$, then the only Weierstrass polynomial of degree k is z^k. Note that for any n, if $k = 0$, then $P = 1$.

The purpose of this section is to show that every holomorphic function in $\mathcal{O}_0$ is up to a unit and a possible small rotation a Weierstrass polynomial, which carries the zeros of f. Consequently the algebraic and geometric properties of ${}_n\mathcal{O}_0$ can be understood via algebraic and geometric properties of ${}_{n-1}\mathcal{O}_0[z_n]$.

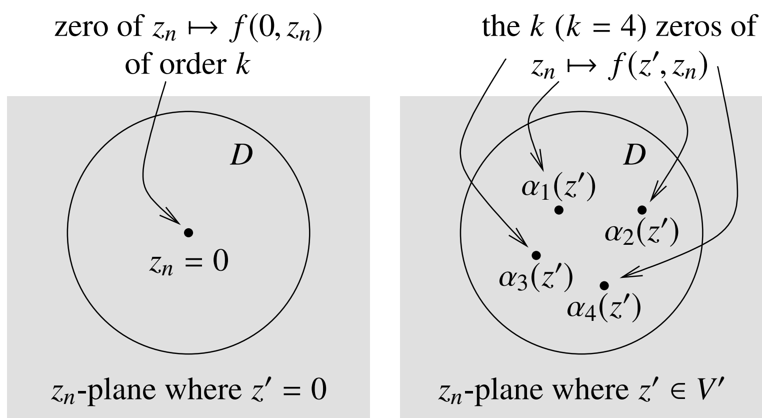
Theorem 6.2.3 (Weierstrass preparation theorem)**.** *Suppose $f \in \mathcal{O}(U)$ for an open $U \subset \mathbb{C}^{n-1} \times \mathbb{C}$, where $0 \in U$, and $f(0) = 0$. Suppose $z_n \mapsto f(0, z_n)$ is not identically zero near the origin and its order of vanishing at the origin is $k \geq 1$.*

Then there exists an open polydisc $V = V' \times D \subset \mathbb{C}^{n-1} \times \mathbb{C}$ with $0 \in V \subset U$, a unique $u \in \mathcal{O}(V)$, $u(z) \neq 0$ for all $z \in V$, and a unique Weierstrass polynomial P of degree k with coefficients holomorphic in V' such that

$$f(z', z_n) = u(z', z_n)\, P(z', z_n),$$

and such that all k roots (counting multiplicity) of $z_n \mapsto P(z', z_n)$ lie in D for all $z' \in V'$

Proof. There exists a small disc $D \subset \mathbb{C}$ centered at zero such that $\{0\} \times \overline{D} \subset U$ and such that $f(0, z_n) \neq 0$ for $z_n \in \overline{D} \setminus \{0\}$. By continuity of f we find a small polydisc $V = V' \times D$ such that $\overline{V} \subset U$ and f is not zero on $V' \times \partial D$.



By the one-variable argument principle (Theorem B.25) the number of zeros (with multiplicity) of $z_n \mapsto f(z', z_n)$ in D is

$$\frac{1}{2\pi i} \int_{\partial D} \frac{\frac{\partial f}{\partial z_n}(z', \zeta)}{f(z', \zeta)} \, d\zeta.$$

As $f(z', \zeta)$ does not vanish when $z' \in V'$ and $\zeta \in \partial D$, the above expression is a continuous integer-valued function of $z' \in V'$. The expression is equal to k when $z' = 0$, and so it is equal to k for all $z' \in V'$. Write the zeros of $z_n \mapsto f(z', z_n)$ as $\alpha_1(z'), \ldots, \alpha_k(z')$, including multiplicity. There is no particular order to the zeros, pick some order for every z'. Write

$$P(z', z_n) = \prod_{j=1}^{k} \bigl(z_n - \alpha_j(z')\bigr) = z_n^k + c_{k-1}(z')\, z_n^{k-1} + \cdots + c_0(z').$$

For a fixed z', P is uniquely defined as the order of roots does not matter in its definition (See exercise below). It is clear that u and P are unique if they exist (that is, if they exist as holomorphic functions).

The functions α_j are not even continuous in general (see Example 6.2.4). However, we will prove that the functions c_j are holomorphic. The functions c_j are (up to sign) the *elementary symmetric functions* of $\alpha_1, \ldots, \alpha_k$ (see below). It is a standard theorem in algebra (see Exercise 6.2.1) that the elementary symmetric functions are polynomials in the so-called power sum functions in the α_js:

$$s_m(z') = \sum_{j=1}^{k} \alpha_j(z')^m, \qquad m = 1, \ldots, k.$$

Therefore, if we show that the power sums s_m are holomorphic, then c_ℓ are holomorphic.

A refinement of the argument principle (see also Theorem B.25) says: If h and g are holomorphic functions on a disc D, continuous on $\overline{D}$, such that g has no zeros on ∂D, and $\alpha_1, \ldots, \alpha_k$ are the zeros of g in D, then

$$\frac{1}{2\pi i} \int_{\partial D} h(\zeta) \frac{g'(\zeta)}{g(\zeta)} \, d\zeta = \sum_{j=1}^{k} h(\alpha_j).$$

The above formula with $h(\zeta) = \zeta^m$ and $g(\zeta) = f(z', \zeta)$ says that

$$s_m(z') = \sum_{j=1}^{k} \alpha_j(z')^m = \frac{1}{2\pi i} \int_{\partial D} \zeta^m \frac{\frac{\partial f}{\partial \zeta}(z', \zeta)}{f(z', \zeta)} \, d\zeta.$$

The function s_m is clearly continuous, and if we differentiate under the integral with $\frac{\partial}{\partial \bar{z}_\ell}$ for $\ell = 1, \ldots, n-1$, we find that s_m is holomorphic.

Finally we wish to show that P divides f as claimed. For each fixed z' one variable theory says that $\frac{f(z', z_n)}{P(z', z_n)}$ has only removable singularities, and in fact, it has no zeros as we defined P to exactly cancel them all out. The Cauchy formula on f/P then says that the function

$$u(z', z_n) = \frac{1}{2\pi i} \int_{\partial D} \frac{f(z', \zeta)}{P(z', \zeta)(\zeta - z_n)} \, d\zeta$$

is equal to $\frac{f(z', z_n)}{P(z', z_n)}$. The function u is clearly continuous and holomorphic in z_n for each fixed z'. By differentiating under the integral we find that it is also holomorphic in z'. $\square$

Example 6.2.4: A useful example to keep in mind is $f(z_1, z_2) = z_2^2 - z_1$, a Weierstrass polynomial in z_2 of degree $k = 2$. So $z' = z_1$. For all z_1 except the origin there are two zeros, $\pm\sqrt{z_1}$. Call one of them $\alpha_1(z_1)$ and one of them $\alpha_2(z_1)$. Recall there is no continuous choice of a square root that works for all z_1, so no matter how you choose, α_1 and α_2 will not be continuous. At the origin there is only one zero of order two, so $\alpha_1(0) = \alpha_2(0) = 0$. On the other hand the symmetric functions $c_1(z_1) = -\alpha_1(z_1) - \alpha_2(z_1) = 0$ and $c_0(z_1') = \alpha_1(z_1)\alpha_2(z_1) = -z_1$ are holomorphic.

Also notice that the k depends on the coordinates chosen. If we do a linear change of coordinates and consider $g(z_1, z_2) = -f(z_2, z_1)$, then $g(z_1, z_2) = z_1^2 - z_2$, which is a Weierstrass polynomial in z_2 of degree $k = 1$. There is now only one zero $\alpha_1(z_1) = z_1^2$, and so $c_0(z_1) = -z_1^2$.

A function $f(z_1, \ldots, z_n)$ is *symmetric* if $f = f \circ p$ for all permutations of the variables p. The *elementary symmetric functions* of $\alpha_1, \ldots, \alpha_k$ are the coefficients σ_j of the polynomial

$$\prod_{j=1}^{k} \bigl(t + \alpha_j\bigr) = t^k + \sigma_1 t^{k-1} + \cdots + \sigma_{k-2} t^2 + \sigma_{k-1} t + \sigma_k.$$

In other words:

$$\begin{aligned}
\sigma_1 &= \alpha_1 + \alpha_2 + \cdots + \alpha_k, \\
\sigma_2 &= \alpha_1\alpha_2 \;+\; \alpha_1\alpha_3 \;+\; \cdots \;+\; \alpha_{k-1}\alpha_k, \\
&\;\;\vdots \\
\sigma_{k-1} &= \alpha_2\alpha_3\cdots\alpha_k \;+\; \alpha_1\alpha_3\alpha_4\cdots\alpha_k \;+\; \cdots \;+\; \alpha_1\alpha_2\cdots\alpha_{k-1}, \\
\sigma_k &= \alpha_1\alpha_2\cdots\alpha_k.
\end{aligned}$$

So for example when $k = 2$, then $\sigma_2 = \alpha_1\alpha_2$ and $\sigma_1 = \alpha_1 + \alpha_2$. The function σ_1 happens to already be a power sum. We can write σ_2 as a polynomial in the power sums:

$$\sigma_2 = \frac{1}{2}\bigl(\alpha_1 + \alpha_2\bigr)^2 - \bigl(\alpha_1^2 + \alpha_2^2\bigr).$$

Exercise* 6.2.1*: *Show that elementary symmetric functions are polynomials in the power sums.*

Exercise* 6.2.2*: *Prove the* fundamental theorem of symmetric polynomials*: Every symmetric polynomial can be written as a polynomial in the elementary symmetric functions. Use the following procedure. Using double induction, suppose the theorem is true if the number of variables is less than k, and the theorem is true in k variables for degree less than d. Consider a symmetric* $P(z_1, \ldots, z_k)$ *of degree d. Write* $P(z_1, \ldots, z_{k-1}, 0)$ *by induction hypothesis as a polynomial in the elementary symmetric functions of one less variable. Use the same coefficients, but plug in the elementary symmetric functions of k variables except the symmetric polynomial in k variables of degree k, that is except the* $z_1 \cdots z_k$*. You will obtain a symmetric function* $L(z_1, \ldots, z_k)$ *and you need to show* $L(z_1, \ldots, z_{k-1}, 0) = P(z_1, \ldots, z_{k-1}, 0)$*. Now use symmetry to prove that*

$$P(z_1, \ldots, z_k) = L(z_1, \ldots, z_k) + z_1 \cdots z_k Q(z_1, \ldots, z_k).$$

Then note that Q has lower degree and finish by induction.

Exercise* 6.2.3*: *Extend the previous exercise to power series. Suppose* $f(z_1, \ldots, z_k)$ *is a convergent symmetric power series at 0, show that f can be written as a convergent power series in the elementary symmetric functions.*

Exercise* 6.2.4*: *Suppose* $P(z', z_n)$ *is a Weierstrass polynomial of degree k, and write* $\alpha_1(z'), \ldots, \alpha_k(z')$ *as the roots. These are not holomorphic, but suppose that f is a symmetric convergent power series at the origin in k variables. Show that* $f\big(\alpha_1(z'), \ldots, \alpha_k(z')\big)$ *is a holomorphic function of* z' *near the origin.*

The hypotheses of the preparation theorem are not an obstacle. If a holomorphic function f is such that $z_n \mapsto f(0, z_n)$ vanishes identically, then we can make a small linear change of coordinates L (L can be a matrix arbitrarily close to the identity) such that $f \circ L$ satisfies the hypotheses of the theorem. For example, $f(z_1, z_2, z_3) = z_1 z_3 + z_2 z_3$ does not satisfy the hypotheses of the theorem as $f(0, 0, z_3) \equiv 0$. But for an arbitrarily small $\epsilon \neq 0$, replacing z_2 with $z_2 + \epsilon z_3$ leads to $\tilde{f}(z_1, z_2, z_3) = f(z_1, z_2 + \epsilon z_3, z_3) = z_1 z_3 + z_2 z_3 + \epsilon z_3^2$, and $\tilde{f}(0, 0, z_3) = \epsilon z_3^2$. Thence $\tilde{f}$ satisfies the hypotheses of the theorem.

Exercise* 6.2.5*: *Prove the above fact about the existence of L arbitrarily close to the identity.*

Exercise* 6.2.6*: *Prove that a monic polynomial* $P(\zeta)$ *of one variable is uniquely determined by its zeros up to multiplicity. That is, suppose P and Q are two monic polynomials with the same zeros up to multiplicity, then* $P = Q$*. That proves the uniqueness of the Weierstrass polynomial.*

Exercise* 6.2.7*: *Suppose* $D \subset \mathbb{C}$ *is a bounded domain,* $0 \in D$*,* $U' \subset \mathbb{C}^{n-1}$ *is a domain,* $0 \in U'$*, and* $P \in \mathcal{O}(U')[z_n]$ *is a Weierstrass polynomial such that* $P(z', z_n)$ *is not zero on* $U' \times \partial D$*. Then for any* $z' \in U$*, all roots of* $z_n \mapsto P(z', z_n)$ *are in D.*

Exercise* 6.2.8*: *Let* $D \subset \mathbb{C}$ *be a bounded domain, and* $U' \subset \mathbb{C}^{n-1}$ *a domain. Suppose f is a continuous function on* $U' \times \overline{D}$ *holomorphic on* $U' \times D$*, where f is zero on at least one point of* $U' \times D$*, and f is never zero on* $U' \times \partial D$*. Prove that* $z_n \mapsto f(z', z_n)$ *has at least one root in D for every* $z' \in U'$*.*

The order of vanishing of f at the origin is a lower bound on the number k in the theorem. The order of vanishing for a certain variable may be larger than this lower bound. If $f(z_1,z_2) = z_1^2 + z_2^3$, then the k you get is 3, but $\operatorname{ord}_0 f = 2$. One can make a small linear change of coordinates to ensure $k = \operatorname{ord}_0 f$. With the f as above for example, $f(z_1 + \epsilon z_2, z_2)$ gives $k = 2$ as expected.

The Weierstrass preparation theorem is a generalization of the implicit function theorem. When $k = 1$ in the theorem, then we obtain the Weierstrass polynomial $z_n + c_0(z')$. That is, the zero set of f is a graph of the holomorphic function $-c_0$. Therefore the Weierstrass theorem is a generalization of the implicit function theorem to the case when $\frac{\partial f}{\partial z_n}$ is zero. We can still "solve" for z_n, but we find k solutions given as the roots of the obtained Weierstrass polynomial.

There is an obvious statement of the preparation theorem for germs.

***Exercise* 6.2.9:** *State and prove a germ version of the preparation theorem.*

Theorem 6.2.5 (Weierstrass division theorem)**.** *Suppose f is holomorphic near the origin, and suppose P is a Weierstrass polynomial of degree $k \geq 1$ in z_n. Then there exists a neighborhood V of the origin and unique $q, r \in \mathcal{O}(V)$, where r is a polynomial in z_n of degree less than k, and on V,*

$$f = qP + r.$$

Do note that r need not be a Weierstrass polynomial; it need not be monic nor do the coefficients need to vanish at the origin. It is simply a polynomial in z_n with coefficients that are holomorphic functions of the first $n-1$ variables.

Proof. Uniqueness is left as an exercise. Consider a connected neighborhood $V = V' \times D$ of the origin for some disc D such that f and P are continuous in $V' \times \overline{D}$, and P is not zero on $V' \times \partial D$. Let

$$q(z', z_n) = \frac{1}{2\pi i} \int_{\partial D} \frac{f(z',\zeta)}{P(z',\zeta)(\zeta - z_n)} \, d\zeta.$$

As P is not zero on $V' \times \partial D$, the function q is holomorphic in V (differentiate under the integral). If P did divide f, then q would really be f/P. But if P does not divide f, then the Cauchy integral formula does not apply and q is not equal to f/P. Interestingly the expression does give the quotient in the division with remainder.

Write f using the Cauchy integral formula in z_n and subtract qP to obtain r:

$$r(z', z_n) = f(z', z_n) - q(z', z_n)P(z', z_n) = \frac{1}{2\pi i} \int_{\partial D} \frac{f(z',\zeta)P(z',\zeta) - f(z',\zeta)P(z',z_n)}{P(z',\zeta)(\zeta - z_n)} \, d\zeta.$$

We need to show r is a polynomial in z_n of degree less than k. In the expression inside the integral, the numerator is of the form $\sum_j h_j(z',\zeta)(\zeta^j - z_n^j)$ and is therefore divisible by $(\zeta - z_n)$. The numerator is a polynomial of degree k in z_n. After dividing by $(\zeta - z_n)$, the integrand becomes a polynomial in z_n of degree $k-1$. Use linearity of the integral to integrate the coefficients of the polynomial. Each coefficient is a holomorphic function in V' and the proof is finished. Some coefficients may have integrated to zero, so we can only say that r is a polynomial of degree $k-1$ or less. $\square$

For example, let $f(z,w) = e^z + z^4 e^w + zw^2 e^w + zw$ and $P(z,w) = w^2 + z^3$. Then P is a Weierstrass polynomial in w of degree $k = 2$. A bit of computation shows

$$\frac{1}{2\pi i} \int_{\partial \mathbb{D}} \frac{e^z + z^4 e^\zeta + z\zeta^2 e^\zeta + z\zeta}{(\zeta^2 + z^3)(\zeta - w)} d\zeta = ze^w, \quad \text{so} \quad f(z,w) = \underbrace{\left(ze^w\right)}_{q} \underbrace{\left(w^2 + z^3\right)}_{P} + \underbrace{zw + e^z}_{r}.$$

Notice that r is a polynomial of degree 1 in w, but it is neither monic, nor do coefficients vanish at 0.

Exercise* 6.2.10*: *Prove the uniqueness part of the theorem.*

Exercise* 6.2.11*: *State and prove a germ version of the division theorem.*

The Weierstrass division theorem is a generalization of the division algorithm for polynomials with coefficients in a field, such as the complex numbers: If $f(\zeta)$ is a polynomial, and $P(\zeta)$ is a nonzero polynomial of degree k, then there exist polynomials $q(\zeta)$ and $r(\zeta)$ with degree of r less than k such that $f = qP + r$. If the coefficients are in a commutative ring, we can divide as long as P is monic. The Weierstrass division theorem says that in the case of the ring $\mathcal{O}_p$, we can divide by a monic $P \in {}_{n-1}\mathcal{O}_p[z_n]$, even if f is a holomorphic function (a "polynomial of infinite degree") as long as $f(0, z_n)$ has finite order.

6.3 The dependence of roots on parameters

Let us prove that the roots change holomorphically as long as they do not come together. We will prove shortly that the roots come together only on a small set; it is a zero set of a certain holomorphic function called the discriminant.

A set of roots are said to be *geometrically distinct* if they are distinct points of $\mathbb{C}$. A root is called *geometrically unique* if it is a unique complex number. For example, $(z_n - 1)^2$ has a geometrically unique root 1, and $(z_n - 1)^2(z_n + 1)$ has two geometrically distinct roots, 1 and -1.

Proposition 6.3.1. *Let $D \subset \mathbb{C}$ and $U' \subset \mathbb{C}^{n-1}$ be domains, and $f \in \mathcal{O}(U' \times D)$. Suppose that for each fixed $z' \in U'$ the function $z_n \mapsto f(z', z_n)$ has a geometrically unique root $\alpha(z') \in D$. Then α is holomorphic in U'.*

The proposition shows that the regularity conclusion of the implicit function theorem holds under the hypothesis that there exists some local solution for z_n. This result holds only for holomorphic functions and not for real-analytic functions. For example, $x^2 - y^3 = 0$ has a unique real solution $y = x^{2/3}$, but that function is not even differentiable.

Proof. We must show that α is holomorphic near any point, which, without loss of generality, is the origin. Apply the preparation theorem to find $f = uP$, where P is a Weierstrass polynomial in $\mathcal{O}(V')[z_n]$ for some $V' \subset U'$ and all roots of $z_n \mapsto P(z', z_n)$ are in D. As α is a geometrically unique root in D,

$$P(z', z_n) = \left(z_n - \alpha(z')\right)^k = z_n^k - k\alpha(z') z_n^{k-1} + \cdots$$

The coefficients of P are holomorphic, so α is holomorphic. □

Proposition 6.3.2. *Let $D \subset \mathbb{C}$ and $U' \subset \mathbb{C}^{n-1}$ be domains, and $f \in \mathcal{O}(U' \times D)$. Let $m \in \mathbb{N}$ be such that for each $z' \in U'$, the function $z_n \mapsto f(z', z_n)$ has precisely m geometrically distinct roots. Then locally near each point in U' there exist m holomorphic functions $\alpha_1(z'), \ldots, \alpha_m(z')$, positive integers $k_1, \ldots, k_m$, and a nonvanishing holomorphic function u such that*

$$f(z', z_n) = u(z', z_n) \prod_{j=1}^{m} \left(z_n - \alpha_j(z')\right)^{k_j}.$$

We can only define α_1 through α_m locally (on a smaller domain) as we do not know the order of α_1 through α_m, and the order could change as we move around U' if it is not simply connected. If U' is simply connected, then the functions can be defined globally by analytic continuation. For an example where U' is not simply connected, recall Example 6.2.4. Consider $U' = \mathbb{C} \setminus \{0\}$ and think $D = \mathbb{C}$ rather than a disc for simplicity. Then U' is not simply connected, and there do not exist continuous functions $\alpha_1(z_1)$ and $\alpha_2(z_1)$ that are roots of the Weierstrass polynomial, that is $z_2^2 - z_1 = \left(z_2 - \alpha_1(z_1)\right)\left(z_2 - \alpha_2(z_1)\right)$. These would be the two square roots of z_1, and there is no continuous (let alone holomorphic) square root defined in $\mathbb{C} \setminus \{0\}$. Such roots can be chosen to be holomorphic on any smaller simply connected open subset of U', for example, on any disc $\Delta \subset U'$.

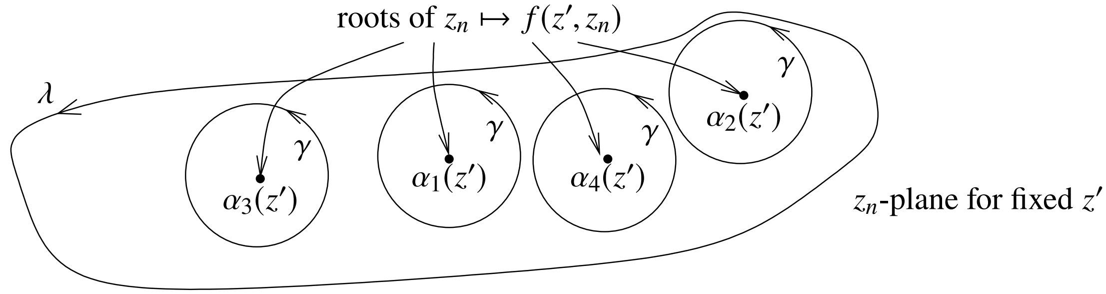
Exercise* 6.3.1: Let $D \subset \mathbb{C}$ be a bounded domain, $U' \subset \mathbb{C}^{n-1}$ a domain, f a continuous function on $U' \times \overline{D}$ holomorphic on $U' \times D$, where f is zero on at least one point of $U' \times D$, and f is never zero on $U' \times \partial D$. Suppose that for each fixed $z' \in U'$ the function $z_n \mapsto f(z', z_n)$ has at most one root in D. Prove that for each $z' \in U'$ $z_n \mapsto f(z', z_n)$ has exactly one root in D. Note: And therefore by Proposition 6.3.1, that root is a holomorphic function.*

Exercise* 6.3.2: Prove Proposition 6.3.2. See the above exercise and Proposition 6.3.1.*

Theorem 6.3.3. *Let $D \subset \mathbb{C}$ be a bounded domain, $U' \subset \mathbb{C}^{n-1}$ a domain, and $f \in \mathcal{O}(U' \times D)$. Suppose the zero set $f^{-1}(0)$ has no limit points on $U' \times \partial D$. Then there exists an $m \in \mathbb{N}$ and a holomorphic function $\Delta \colon U' \to \mathbb{C}$, not identically zero, such that for every $z' \in U' \setminus E$, where $E = \Delta^{-1}(0)$, $z_n \mapsto f(z', z_n)$ has exactly m geometrically distinct roots in D, and $z_n \mapsto f(z', z_n)$ has strictly less than m geometrically distinct roots for $z' \in E$.*

The complement of a zero set of a holomorphic function is connected, open and dense. The function Δ is called the *discriminant function* and its zero set E is called the *discriminant set*. For example for the quadratic equation, Δ is the discriminant we learned about in high school.

Proof. The roots of $z_n \mapsto f(z', z_n)$ are isolated, and there are finitely many for every z' as D is bounded and $f^{-1}(0)$ has no limit points on $U' \times \partial D$. For any $p' \in U'$ we define two useful paths. Let γ be the union of nonintersecting small simple closed curves in D, one around each geometrically distinct root of $z_n \mapsto f(p', z_n)$. Let λ be a large closed path in D going exactly once around all the roots and such that the interior of λ is in D. Suppose γ and λ intersect no roots. By continuity, the curves γ and λ do not intersect any roots for z' near p'. Since the set $f^{-1}(0)$ is closed and the roots do not accumulate on $U' \times \partial D$, then for z' near p' the roots stay a positive distance away from the boundary. So λ can be picked to go around all the roots exactly once for z' near p'.



The argument principle applied to $z_n \mapsto f(z', z_n)$ using λ for z' near p' shows that the number of roots (with multiplicity) is bounded by some $M \in \mathbb{N}$ near p'. The M is locally well-defined (it does not depend on λ as long as it contains all the roots), and it is a locally constant function of $z' \in U'$. As U' is connected, and it is a fixed number. The maximum number of geometrically distinct roots must be bounded by M. Let m be the maximal number of geometrically distinct roots and suppose that at some point in U', there are exactly m geometrically distinct roots.

Let $U'_m \subset U'$ be the set of $z' \in U'$ for which $z_n \mapsto f(z', z_n)$ has exactly m roots. Write U' as a union of disjoint sets $U' = U'_m \cup E$, where $E = U' \setminus U'_m$. By definition of m, U'_m is nonempty. Suppose $p' \in U'_m$ and γ goes around the roots as above. Let γ' be a single component curve of the path γ going around one of the roots. The argument principle with respect to γ' says that γ' must contain at least one root for all z' near p'. There are only finitely many roots, and so for z' in some neighborhood of p', $z_n \mapsto f(z', z_n)$ has at least m roots in γ, and as m is the maximum, it has exactly m roots. In other words, U'_m is open.

Locally on U'_m, there exist m holomorphic functions $\alpha_1, \ldots, \alpha_m$ giving the roots by the previous proposition. We cannot define these on all of U'_m as we do not know their order. The function

$$\Delta(z') = \prod_{j \neq k} \bigl(\alpha_j(z') - \alpha_k(z')\bigr)$$

defined for $z' \in U'_m$ does not depend on the order. That means Δ is well-defined as a function on the open set U'_m, and since α_j are locally holomorphic, Δ is holomorphic.

Let $p' \in E \cap \overline{U'_m}$, so there are fewer than m roots at p'. Suppose γ and λ are as above. In each particular component γ' of γ, there must be at least one root for all z' near p' by the same argument as above. There exist z' arbitrarily near p' where there are m roots. The region between λ and γ (including the curves) is compact, and so by continuity, if $f(p', z_n)$ was nonzero on it, so is $f(z', z_n)$ for z' near p'. As no roots of $f(z', z_n)$ lie outside λ, we have that all roots lie in one of the components of γ. So if $z' \in U'_m$ near p', there must be one component γ' that contains at least two roots. Let $\{z'_\ell\}$ be an arbitrary sequence of points in U'_m going to p'. As the number of roots is finite, $\{z'_\ell\}$ has a subsequence such that $z_n \mapsto f(z'_\ell, z_n)$ has at least two roots in some fixed component γ' of γ for z'_ℓ. Assume $\{z'_\ell\}$ is this subsequence. Label the two distinct roots at z'_ℓ as $\alpha_1(z'_\ell)$ and $\alpha_2(z'_\ell)$. At p' there is only a single (geometrically) root in γ', let us name it $\alpha_1(p')$. As $f^{-1}(0)$ is closed it must be that $\alpha_1(z'_\ell)$ and $\alpha_2(z'_\ell)$ both approach $\alpha_1(p')$ as $\ell \to \infty$. As all roots are necessarily bounded, $\lim_{\ell\to\infty} \Delta(z'_\ell) = 0$. As the limit is zero for a subsequence of an arbitrary sequence,

$$\lim_{z' \in U'_m \to p'} \Delta(z') = 0.$$

We have already defined Δ on U'_m, so set $\Delta(z') = 0$ if $z' \in E$. The function Δ is a continuous function on U' that is zero precisely on E and holomorphic on U'_m. Radó's theorem (Theorem 2.4.10) says that Δ is holomorphic in U'. □

The discriminant given above is really the discriminant of the set $f^{-1}(0)$ rather than of the corresponding Weierstrass polynomial. Often for Weierstrass polynomials the discriminant is defined as $\prod_{j \neq k} \bigl(\alpha_j(z') - \alpha_k(z')\bigr)$ taking multiple roots into account, and therefore the "discriminant" could be identically zero. It will be clear from upcoming exercises that if the Weierstrass polynomial is irreducible, then the two notions do in fact coincide.

***Exercise* 6.3.3:** *Prove that if $f \in \mathcal{O}(U)$, then $U \setminus f^{-1}(0)$ is not simply connected if $f^{-1}(0)$ is nonempty. In particular, in the theorem, $U' \setminus E$ is not simply connected if $E \neq \emptyset$.*

***Exercise* 6.3.4:** *Let $D \subset \mathbb{C}$ be a bounded domain, and $U' \subset \mathbb{C}^{n-1}$ a domain. Suppose f is a continuous function on $U' \times \overline{D}$ holomorphic on $U' \times D$, and f is never zero on $U' \times \partial D$. Suppose $\gamma \colon [0,1] \to U'$ is a continuous function such that $f\bigl(\gamma(0), c\bigr) = 0$ for some $c \in D$. Prove that there exists a continuous function $\alpha \colon [0,1] \to \mathbb{C}$ such that $\alpha(0) = c$ and $f\bigl(\gamma(t), \alpha(t)\bigr) = 0$ for all $t \in [0,1]$. Hint: Show it is possible for a path arbitrarily close to γ, but one that stays away from the discriminant.*

6.4 Properties of the ring of germs

Let us prove some basic properties of the ring of germs of holomorphic functions. First some algebra terminology. Given a commutative ring R, an *ideal* $I \subset R$ is a subset such that $fg \in I$ whenever $f \in R$ and $g \in I$ and $g + h \in I$ whenever $g, h \in I$. An intersection of ideals is again an ideal, and hence it makes sense to talk about the smallest ideal containing a set of elements. An ideal I is generated by $f_1, \ldots, f_k$ if I is the smallest ideal containing $\{f_1, \ldots, f_k\}$. We then write $I = (f_1, \ldots, f_k)$. Every element in I can be written as $c_1 f_1 + \cdots + c_k f_k$ where $c_1, \ldots, c_k \in R$. A *principal ideal* is an ideal generated by a single element, that is, (f).

For convenience, when talking about germs of functions we often identify a representative with the germ when the context is clear. So by abuse of notation, we often write $f \in \mathcal{O}_p$ instead of $(f, p) \in \mathcal{O}_p$ and $(f_1, \ldots, f_k)$ instead of $\bigl((f_1, p), \ldots, (f_k, p)\bigr)$. As in the following exercises.

***Exercise* 6.4.1:**

a) Suppose $f \in \mathcal{O}_p$ is such that $f(p) \neq 0$, and (f) is the ideal generated by f. Prove $(f) = \mathcal{O}_p$.

b) Let $\mathfrak{m}_p = (z_1 - p_1, \ldots, z_n - p_n) \subset \mathcal{O}_p$ be the ideal generated by the coordinate functions. Show that if $f(p) = 0$, then $f \in \mathfrak{m}_p$.

c) Show that if $I \subsetneq \mathcal{O}_p$ is a proper ideal *(ideal such that $I \neq \mathcal{O}_0$) then $I \subset \mathfrak{m}_p$, that is $\mathfrak{m}_p$ is a* maximal ideal.

***Exercise* 6.4.2:** *Suppose $n = 1$. Show that ${}_1\mathcal{O}_p$ is a* principal ideal domain *(PID), that is every ideal is a principal ideal. More precisely, show that given an ideal $I \subset {}_1\mathcal{O}_p$, then there exists a $k = 0, 1, 2, \ldots$, such that $I = \bigl((z - p)^k\bigr)$.*

***Exercise* 6.4.3:** *If $U, V \subset \mathbb{C}^n$ are two neighborhoods of p and $h \colon U \to V$ is a biholomorphism. First prove that it makes sense to talk about $f \circ h$ for any $(f, p) \in \mathcal{O}_p$. Then prove that $f \mapsto f \circ h$ is a ring isomorphism.*

A commutative ring R is *Noetherian* if every ideal in R is finitely generated. That is, for every ideal $I \subset R$ there exist finitely many generators $f_1, \dots, f_k \in I$: that is, every $g \in I$ can be written as $g = c_1 f_1 + \cdots + c_k f_k$, for some $c_1, \dots, c_k \in R$. In an exercise you proved ${}_1\mathcal{O}_p$ is a PID, and so it is Noetherian. For higher dimensions, the ring of germs may not be a PID, but it is Noetherian.

Theorem 6.4.1. *$\mathcal{O}_p$ is Noetherian.*

Proof. Without loss of generality $p = 0$. The proof is by induction on dimension. By an exercise above, ${}_1\mathcal{O}_0$ is Noetherian. By another exercise, we are allowed to do a change of coordinates at zero.

For induction suppose ${}_{n-1}\mathcal{O}_0$ is Noetherian, and let $I \subset {}_n\mathcal{O}_0$ be an ideal. If $I = \{0\}$ or $I = {}_n\mathcal{O}_0$, then the assertion is obvious. Therefore, assume that all elements of I vanish at the origin ($I \neq {}_n\mathcal{O}_0$), and that there exist elements that are not identically zero ($I \neq \{0\}$). Let g be such an element. After perhaps a linear change of coordinates, assume g is a Weierstrass polynomial in z_n by the preparation theorem.

The ring ${}_{n-1}\mathcal{O}_0[z_n]$ is a subring of ${}_n\mathcal{O}_0$. The set $J = {}_{n-1}\mathcal{O}_0[z_n] \cap I$ is an ideal in the ring ${}_{n-1}\mathcal{O}_0[z_n]$. By the Hilbert basis theorem (see Theorem D.4 in the appendix for a proof), as ${}_{n-1}\mathcal{O}_0$ is Noetherian, the ring ${}_{n-1}\mathcal{O}_0[z_n]$ is also Noetherian. Thus J has finitely many generators, that is $J = (h_1, \dots, h_k)$, in the ring ${}_{n-1}\mathcal{O}_0[z_n]$.

By the division theorem, every $f \in I$ is of the form $f = qg + r$, where $r \in {}_{n-1}\mathcal{O}_0[z_n]$ and $q \in {}_n\mathcal{O}_0$. As f and g are in I, so is r. As g and r are in ${}_{n-1}\mathcal{O}_0[z_n]$, they are both in J. Write $g = c_1 h_1 + \cdots + c_k h_k$ and $r = d_1 h_1 + \cdots + d_k h_k$. Then $f = (qc_1 + d_1)h_1 + \cdots + (qc_k + d_k)h_k$. So $h_1, \dots, h_k$ also generate I in ${}_n\mathcal{O}_0$. □

***Exercise* 6.4.4:** *Prove that any proper ideal $I \subset \mathcal{O}_0$ where $I \neq \{0\}$ is generated by Weierstrass polynomials. As a technicality, note that a Weierstrass polynomial of degree 0 is just 1, so it works for $I = \mathcal{O}_0$.*

***Exercise* 6.4.5:** *We saw above that ${}_1\mathcal{O}_p$ is a PID. Prove that if $n > 1$, then ${}_n\mathcal{O}_p$ is not a PID.*

Theorem 6.4.2. *$\mathcal{O}_p$ is a* unique factorization domain *(UFD). That is, up to a multiplication by a unit, every element has a unique factorization into irreducible elements of $\mathcal{O}_p$.*

Proof. Again assume $p = 0$ and induct on the dimension. The one-dimensional statement is an exercise below. If ${}_{n-1}\mathcal{O}_0$ is a UFD, then ${}_{n-1}\mathcal{O}_0[z_n]$ is a UFD by the Gauss lemma (see Theorem D.6).

Take $f \in {}_n\mathcal{O}_0$. After perhaps a linear change of coordinates $f = qP$, for q a unit in ${}_n\mathcal{O}_0$, and P a Weierstrass polynomial in z_n. As ${}_{n-1}\mathcal{O}_0[z_n]$ is a UFD, P has a unique factorization in ${}_{n-1}\mathcal{O}_0[z_n]$ into $P = P_1 P_2 \cdots P_k$. So $f = qP_1 P_2 \cdots P_k$. That P_j are irreducible in ${}_n\mathcal{O}_0$ is left as an exercise.

Suppose $f = \tilde{q} g_1 g_2 \cdots g_m$ is another factorization. The preparation theorem applies to each g_j. Therefore write $g_j = u_j \widetilde{P}_j$ for a unit u_j and a Weierstrass polynomial $\widetilde{P}_j$. We obtain $f = u\widetilde{P}_1 \widetilde{P}_2 \cdots \widetilde{P}_m$ for a unit u. By uniqueness part of the preparation theorem we obtain $P = \widetilde{P}_1 \widetilde{P}_2 \cdots \widetilde{P}_m$. Conclusion is then obtained by noting that ${}_{n-1}\mathcal{O}_0[z_n]$ is a UFD. □

***Exercise* 6.4.6:** *Finish the proof of the theorem by proving ${}_1\mathcal{O}_p$ is a unique factorization domain.*

***Exercise* 6.4.7:** *Show that if an element is irreducible in ${}_{n-1}\mathcal{O}_0[z_n]$, then it is irreducible in ${}_n\mathcal{O}_0$.*

6.5 Varieties

As before, if $f \colon U \to \mathbb{C}$ is a function, let $Z_f = f^{-1}(0) \subset U$ denote the zero set of f.

Definition 6.5.1. Let $U \subset \mathbb{C}^n$ be an open set. Let $X \subset U$ be a set such that near each point $p \in U$, there exists a neighborhood W of p and a family of holomorphic functions $\mathcal{F}$ defined on W such that

$$W \cap X = \left\{ z \in W : f(z) = 0 \text{ for all } f \in \mathcal{F} \right\} = \bigcap_{f \in \mathcal{F}} Z_f .$$

Then X is called a (*complex* or *complex-analytic*) *variety* or a *subvariety* of U. Sometimes X is called an *analytic set*. We say $X \subset U$ is a proper subvariety if $\emptyset \neq X \subsetneq U$.

We generally leave out the "complex" from "complex subvariety" as it is clear from context. But you should know that there are other types of subvarieties, namely real subvarieties given by real-analytic functions. We will not cover those in this book.

Example 6.5.2: The set $X = \{0\} \subset \mathbb{C}^n$ is a subvariety as it is the only common vanishing point of functions $\mathcal{F} = \{z_1, \ldots, z_n\}$. Similarly $X = \mathbb{C}^n$ is a subvariety of $\mathbb{C}^n$, where we let $\mathcal{F} = \emptyset$.

Example 6.5.3: The set defined by $z_2 = e^{1/z_1}$ is a subvariety of $U = \left\{ z \in \mathbb{C}^2 : z_1 \neq 0 \right\}$. It is not a subvariety of any open set larger than U.

It is useful to note what happens when we replace "near each point $p \in U$" with "near each point $p \in X$." We get a slightly different concept, and X is said to be a *local variety*. A local variety X is a subvariety of some neighborhood of X, but it is not necessarily closed in U. As a simple example, the set $X = \left\{ z \in \mathbb{C}^2 : z_1 = 0, |z_2| < 1 \right\}$ is a local variety, but not a subvariety of $\mathbb{C}^2$. On the other hand X is a subvariety of the unit ball $\left\{ z \in \mathbb{C}^2 : \|z\| < 1 \right\}$.

Note that $\mathcal{F}$ depends on p and near each point may have a different set of functions. Clearly the family $\mathcal{F}$ is not unique. We will prove below that we would obtain the same definition if we restricted to finite families $\mathcal{F}$.

We work with germs of functions. Recall, that when (f, p) is a germ of a function the germ (Z_f, p) is the germ of the zero set of some representative. Let

$$I_p(X) \overset{\text{def}}{=} \left\{ (f, p) \in \mathcal{O}_p : (X, p) \subset (Z_f, p) \right\}.$$

That is, $I_p(X)$ is the set of germs of holomorphic functions vanishing on X near p. If a function vanishes on X, then any multiple of it also vanishes on X, so $I_p(X)$ is an ideal. Really $I_p(X)$ depends only on the germ of X at p, so define $I_p\big((X,p)\big) = I_p(X)$.

Every ideal in $\mathcal{O}_p$ is finitely generated. Let $I \subset \mathcal{O}_p$ be an ideal generated by $f_1, f_2, \ldots, f_k$. Write

$$V(I) \overset{\text{def}}{=} (Z_{f_1}, p) \cap (Z_{f_2}, p) \cap \cdots \cap (Z_{f_k}, p).$$

That is, $V(I)$ is the germ of the subvariety "cut out" by the elements of I, since every element of I vanishes on the points where all the generators vanish. Suppose representatives $f_1, \ldots, f_k$ of the generators are defined in some neighborhood W of p, and a germ $(g, p) \in I$ has a representative g defined in W such that $g = c_1 f_1 + \cdots + c_k f_k$, where c_k are also holomorphic functions on W. If $q \in Z_{f_1} \cap \cdots \cap Z_{f_k}$, then $g(q) = 0$. Thus, $Z_{f_1} \cap \cdots \cap Z_{f_k} \subset Z_g$, or in terms of germs, $V(I) \subset (Z_g, p)$. The reason why we did not define $V(I)$ to be the intersection of zero sets of all germs in I is that this would be an infinite intersection, and we did not define such an object for germs.

Exercise* 6.5.1*: *Show that $V(I)$ is independent of the choice of generators.*

Exercise* 6.5.2*: *Suppose $I_p(X)$ is generated by the functions $f_1, f_2, \ldots, f_k$. Prove*

$$(X, p) = (Z_{f_1}, p) \cap (Z_{f_2}, p) \cap \cdots \cap (Z_{f_k}, p).$$

Exercise* 6.5.3*: *Show that given a germ (X, p) of a subvariety at p then $V\big(I_p(X)\big) = (X, p)$ (see above), and given an ideal $I \subset \mathcal{O}_p$, then $I_p\big(V(I)\big) \supset I$.*

As $\mathcal{O}_p$ is Noetherian, $I_p(X)$ must be finitely generated. Near each point p only finitely many functions are necessary to define a subvariety, that is, by an exercise above, those functions "cut out" the subvariety. When one says *defining functions* for a germ of a subvariety, one generally means that those functions generate the ideal, not just that their common zero set happens to be the subvariety. A theorem that we will not prove here in full generality, the *Nullstellensatz*, says that if we take the germ of a subvariety defined by functions in an ideal $I \subset \mathcal{O}_p$, and look at the ideal given by that subvariety, we obtain the radical* of I. In more concise language the Nullstellensatz says $I_p\big(V(I)\big) = \sqrt{I}$. Germs of subvarieties are in one-to-one correspondence with radical ideals of $\mathcal{O}_p$.

Example 6.5.4: The subvariety $X = \{0\} \subset \mathbb{C}^2$ can be given by $\mathcal{F} = \{z_1^2, z_2^2\}$. If $I = (z_1^2, z_2^2) \subset \mathcal{O}_0$ is the ideal of germs generated by these two functions then $I_0(X) \neq I$. We have seen that the ideal $I_0(X)$ is the maximal ideal $\mathfrak{m}_0 = (z_1, z_2)$. If we prove that all the nonconstant monomials are in $\sqrt{I}$, then $\sqrt{I} = (z_1, z_2) = \mathfrak{m}_0$. The only nonconstant monomials that are not in I are z_1, z_2, and $z_1 z_2$, but the square of each of these is in I, so $\sqrt{I} = \mathfrak{m}_0$.

The local properties of a subvariety at p are encoded in the properties of the ideal $I_p(X)$. Therefore, the study of subvarieties often involves the study of the various algebraic properties of the ideals of $\mathcal{O}_p$. Let us also mention in passing that the other object that is studied is the so-called *coordinate ring* $\mathcal{O}_p / I_p(X)$, which represents the functions on (X, p). That is, we identify two functions if they differ by something in the ideal, since then they are equal on X.

Most points of a subvariety behave like a piece of $\mathbb{C}^k$, more precisely like a graph over $\mathbb{C}^k$. A graph of a mapping $f : U' \subset \mathbb{C}^k \to \mathbb{C}^{n-k}$ is the set $\Gamma_f \subset U' \times \mathbb{C}^{n-k} \subset \mathbb{C}^k \times \mathbb{C}^{n-k}$ defined by

$$\Gamma_f \overset{\text{def}}{=} \big\{(z, w) \in U' \times \mathbb{C}^{n-k} : w = f(z)\big\}.$$

*The radical of I is defined as $\sqrt{I} \overset{\text{def}}{=} \{f : f^m \in I, \text{ for some } m\}$.

Definition 6.5.5. Let $X \subset U \subset \mathbb{C}^n$ be a subvariety of an open set U. Let $p \in X$ be a point. Suppose that after a permutation of coordinates, near p the set X can be written as a graph of a holomorphic mapping. That is, after relabeling coordinates, there is a neighborhood $U' \times U'' \subset \mathbb{C}^k \times \mathbb{C}^{n-k}$ of p, where $k \in \mathbb{N}_0$, such that

$$X \cap (U' \times U'') = \Gamma_f$$

for a holomorphic mapping $f : U' \to \mathbb{C}^{n-k}$. Then p is a *regular point* (or *simple point*) of X and the *dimension* of X at p is k. We write $\dim_p X = k$. If all points of X are regular points of dimension k, then X is called a *complex manifold*, or *complex submanifold*, of (complex) dimension k.

As the ambient[*] dimension is n we say X is of *codimension* $n - k$ at p.

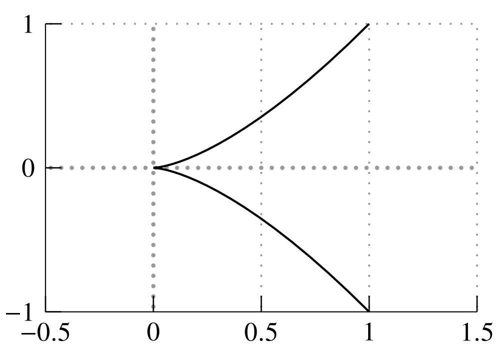
The set of regular points of X is denoted by X_{reg}. Any point that is not regular is *singular*. The set of singular points of X is denoted by X_{sing}.

A couple of remarks are in order, a subvariety X can have regular points of several different dimensions, although if a point is a regular point of dimension k, then all nearby points are regular points of dimension k as the same U' and U'' works. Any isolated point of X is automatically a regular point of dimension 0. Final remark is that dimension is well-defined. We leave it as an exercise. Sometimes the empty set is considered a complex manifold of dimension -1 (or $-\infty$).

Example 6.5.6: The set $X = \mathbb{C}^n$ is a complex submanifold of dimension n (codimension 0). In particular, $X_{reg} = X$ and $X_{sing} = \emptyset$.

The set $Y = \left\{ z \in \mathbb{C}^3 : z_3 = z_1^2 - z_2^2 \right\}$ is a complex submanifold of dimension 2 (codimension 1). Again, $Y_{reg} = Y$ and $Y_{sing} = \emptyset$.

On the other hand, the so-called *cusp*, $C = \left\{ z \in \mathbb{C}^2 : z_1^3 - z_2^2 = 0 \right\}$ is not a complex submanifold. The origin is a singular point of C (see exercise below). At every other point we can write $z_2 = \pm z_1^{3/2}$, so $C_{reg} = C \setminus \{0\}$, and so $C_{sing} = \{0\}$. The dimension at every regular point is 1. Here is a plot of C in two real dimensions:



Exercise* 6.5.4*: *Prove that if p is a regular point of a subvariety $X \subset U \subset \mathbb{C}^n$ of a domain U, then the dimension is well-defined. Hint: If there were two possible U' of different dimension (possibly different affine coordinates), construct a map from one such U' to another such U' with nonvanishing derivative.*

Exercise* 6.5.5*: *Consider the cusp $C = \left\{ z \in \mathbb{C}^2 : z_1^3 - z_2^2 = 0 \right\}$. Prove that the origin is not a regular point of C.*

[*]The word *ambient* is used often to mean the set that contains whatever object we are talking about.

***Exercise* 6.5.6:** *Show that p is a regular point of dimension k of a subvariety X if and only if there exists a local biholomorphic change of coordinates that puts p to the origin and near 0, X is given by $w = 0$, where $(z,w) \in \mathbb{C}^k \times \mathbb{C}^{n-k}$. In other words, if we allow a biholomorphic change of coordinates, we can let $f = 0$ in the definition.*

We also define dimension at a singular point. A fact that we will not prove in general is that the set of regular points of a subvariety is open and dense in the subvariety; a subvariety is regular at most points. Therefore, the following definition makes sense without resorting to the convention that $\max \emptyset = -\infty$.

Definition 6.5.7. Let $X \subset U \subset \mathbb{C}^n$ be a (complex) subvariety of U. Let $p \in X$ be a point. We define the *dimension* of X at p to be

$$\dim_p X \overset{\text{def}}{=} \max\bigl\{k \in \mathbb{N}_0 : \forall \text{ neighborhoods } W \text{ of } p, \exists\, q \in W \cap X_{reg} \text{ with } \dim_q X = k\bigr\}.$$

If (X,p) is a germ and X a representative, we say the dimension of (X,p) is the dimension of X at p.

The dimension of the entire subvariety X is defined to be

$$\dim X \overset{\text{def}}{=} \max_{p \in X} \dim_p X.$$

We say that X is of *pure dimension* k if at all points p, dimension of X at p is k. We say a germ (X,p) is of pure dimension k if there exists a representative of X that is of pure dimension k. We define the word *codimension* as before, that is, the ambient dimension minus the dimension of X.

Example 6.5.8: We saw that $C = \bigl\{z \in \mathbb{C}^2 : z_1^3 - z_2^2 = 0\bigr\}$ is of dimension 1 at all the regular points, and the only singular point is the origin. Hence $\dim_0 C = 1$, and so $\dim C = 1$. The subvariety C is of pure dimension 1.

We have the following theorem, which we state without proof, at least in the general setting.

Theorem 6.5.9. *Let $U \subset \mathbb{C}^n$ be open and connected and let $X \subset U$ be a subvariety, then the set of regular points X_{reg} is open and dense in X. In fact, $X_{sing} \subset X$ is a subvariety.*

***Exercise* 6.5.7:** *Suppose that $X \subset U \subset \mathbb{C}^n$ is a subvariety of a domain U, such that X_{reg} is connected. Show that X is of pure dimension. Feel free to assume X_{reg} is dense in X.*

6.6 Hypervarieties

Pure codimension 1 subvarieties are particularly nice. Sometimes pure codimension 1 subvarieties are called *hypervarieties*. We have already proved, back in section 1.6, the following theorem (Theorem 1.6.2). We restate it in the language of varieties.

Theorem 6.6.1. *Let $U \subset \mathbb{C}^n$ be a domain and $f \in \mathcal{O}(U)$. Then Z_f is empty, a subvariety of pure codimension 1, or $Z_f = U$. Furthermore, if Z_f is a pure codimension 1 subvariety, then $(Z_f)_{reg}$ is open and dense in Z_f.*

We can improve on that slightly. Let us prove that $(Z_f)_{sing}$ is (locally) contained in the zero set of some holomorphic function that is not zero on any nonempty open subset of Z_f. It suffices to assume $0 \in Z_f$ and to prove this locally near the origin. After possibly a linear change of coordinates, we as usual assume that near the origin Z_f is contained in some $U' \times D \subset \mathbb{C}^{n-1} \times \mathbb{C}$ for a disc D and Z_f does not intersect $U' \times \partial D$. Apply Theorem 6.3.3 to find the discriminant function Δ and the discriminant $E = \Delta^{-1}(0)$. Above each point in $U' \setminus E$, the set Z_f is union of m distinct graphs of holomorphic functions (see Proposition 6.3.2), so those points are regular. Above each point of E there are only finitely many points of Z_f, and E is nowhere dense in U'. So the function $(z', z_n) \mapsto \Delta(z')$ does not vanish on any nonempty open subset of Z_f.

Theorem 6.6.2. *If (X,p) is a germ of a pure codimension 1 subvariety, then there is a germ holomorphic function f at p such that $(Z_f,p) = (X,p)$. Furthermore $I_p(X)$ is generated by (f,p).*

Proof. We already proved all the relevant pieces of the first part of this theorem.

For the second part there has to exist a germ of a function that vanishes on (X,p). Assume $p = 0$, and after a linear change of coordinates assume we can apply the Weierstrass preparation theorem to the function. Taking representatives of the germs, we assume X is a pure codimension 1 subvariety of a small enough neighborhood $U' \times D \subset \mathbb{C}^{n-1} \times \mathbb{C}$ of the origin, where D is a disc, and the function that vanishes on X is a Weierstrass polynomial $P(z', z_n)$ defined for $z' \in U'$, and all zeros of $z_n \mapsto P(z', z_n)$ are in D for $z' \in U$.

Theorem 6.3.3 applies. Let $E \subset U'$ be the discriminant set, a zero set of a holomorphic function. On $U' \setminus E$, there are a certain number of geometrically distinct roots of $z_n \mapsto P(z', z_n)$.

Let X' be a topological component of $X \setminus (E \times D)$. Above each point $z' \in U' \setminus E$, let $\alpha_1(z'), \ldots, \alpha_k(z')$ denote the distinct roots that are in X', that is $\bigl(z', \alpha_j(z')\bigr) \in X'$. If α_j is a holomorphic function in some small neighborhood and $\bigl(z', \alpha_j(z')\bigr) \in X'$ at one point, then $\bigl(z', \alpha_j(z')\bigr) \in X'$ for all nearby points too. Furthermore this means that the set X' contains only regular points of X of dimension $n - 1$.

The number of such geometrically distinct roots above each point in $U' \setminus E$ is locally constant, and as $U' \setminus E$ is connected there exists a unique k. Take

$$F(z', z_n) = \prod_{j=1}^{k} \bigl(z_n - \alpha_j(z')\bigr) = z_n^k + \sum_{j=0}^{k-1} g_j(z') z_n^j.$$

The coefficients g_j are well-defined for $z \in U' \setminus E$ as they are independent of the order of $\alpha_1, \ldots, \alpha_k$. The g_j are holomorphic for $z \in U' \setminus E$ as locally we can choose the order so that each α_j is holomorphic. The coefficients g_j are bounded on U' and therefore extend to holomorphic functions of U'. Hence, the polynomial F is a polynomial in $\mathcal{O}(U')[z_n]$. The roots of F above $z' \in U' \setminus E$ are simple and give precisely X'. By using the argument principle again, we find that all roots above points of E are limits of roots above points in $U' \setminus E$. Consequently, the zero set of F is the closure of X' in $U' \times D$ by continuity. It is left to the reader to check that all the functions g_j vanish at the origin and F is a Weierstrass polynomial, a fact that will be useful in the exercises below.

If the polynomial $P(z', z_n)$ is of degree m, then $z' \mapsto P(z', z_n)$ has at most m roots. Together with the fact that $U' \setminus E$ is connected, this means that $X \setminus (E \times D)$ has at most finitely many components (at most m). So we can find an F for every topological component of $X \setminus (E \times D)$. Then we multiply those functions together to get f.

The fact that this f will generate $I_p(X)$ is left as an exercise below. $\square$

In other words, local properties of a codimension 1 subvariety can be studied by studying the zero set of a single Weierstrass polynomial.

Example 6.6.3: It is not true that if a dimension of a subvariety in $\mathbb{C}^n$ is $n-k$ (codimension k), there are k holomorphic functions that "cut it out." That only works for $k = 1$. The set defined by

$$\operatorname{rank} \begin{bmatrix} z_1 & z_2 & z_3 \\ z_4 & z_5 & z_6 \end{bmatrix} < 2$$

is a pure 4-dimensional subvariety of $\mathbb{C}^6$, so of codimension 2, and the defining equations are $z_1z_5 - z_2z_4 = 0$, $z_1z_6 - z_3z_4 = 0$, and $z_2z_6 - z_3z_5 = 0$. Let us state without proof that the unique singular point is the origin and there exist no 2 holomorphic functions near the origin that define this subvariety. In more technical language, the subvariety is not a *complete intersection*.

Example 6.6.4: If X is a hypervariety and E the corresponding discriminant set, it is tempting to say that the singular set of X is the set $X \cap (E \times \mathbb{C})$, which is a codimension 2 subvariety. It is true that $X \cap (E \times \mathbb{C})$ will contain the singular set, but in general the singular set is smaller. A very simple example of this behavior is the set defined by $z_2^2 - z_1 = 0$. The defining function is a Weierstrass polynomial in z_2 and the discriminant set is given by $z_1 = 0$. However, the subvariety has no singular points. A less trivial example is given in an exercise below.

Interestingly we also proved the following theorem. Same theorem is true for higher codimension, but it is harder to prove.

Corollary 6.6.5. *Let (X, p) is a germ of a subvariety of pure codimension 1. Then there exists a neighborhood U of p, a representative $X \subset U$ of (X, p) and subvarieties $X_1, \ldots, X_k \subset U$ of pure codimension 1 such that $(X_j)_{reg}$ is connected for every j, and $X = X_1 \cup \cdots \cup X_k$.*

Proof. A particular X_j is defined by considering a topological component of $X \setminus (E \times D)$ as in the proof of Theorem 6.6.2, getting the F, and setting $X_j = Z_F$. The topological component is of course a connected set and it is dense in $(X_j)_{reg}$, which proves the corollary. ◻

Exercise* 6.6.1*:

a) Prove that the hypervariety in $\mathbb{C}^n$, $n \geq 2$, given by $z_1^2 + z_2^2 + \cdots + z_n^2 = 0$ has an isolated singularity at the origin (that is, the origin is the only singular point).

b) For any $0 \leq k \leq n-2$, find a hypervariety X of $\mathbb{C}^n$ whose set of singular points is a subvariety of dimension k.

Exercise* 6.6.2*: *Suppose $p(z', z_n)$ is a Weierstrass polynomial of degree k such that for an open dense set of z' near the origin $z_n \mapsto p(z', z_n)$ has geometrically k roots, and such that the regular points of Z_p are connected. Show that p is irreducible in the sense that if $p = rs$ for two Weierstrass polynomials r and s, then either $r = 1$ or $s = 1$.*

Exercise* 6.6.3*: *Suppose f is a function holomorphic in a neighborhood of the origin with $z_n \mapsto f(0, z_n)$ being of finite order. Show that*

$$f = u p_1^{d_1} p_2^{d_2} \cdots p_\ell^{d_\ell},$$

where p_j are Weierstrass polynomials of degree k_j that have generically (that is, on an open dense set) k_j distinct roots (no multiple roots), the regular points of Z_{p_j} are connected, and u is a nonzero holomorphic function in a neighborhood of the origin. See also the next section, these polynomials will be the irreducible factors in the factorization of f.

***Exercise* 6.6.4:** *Suppose (X, p) is a germ of a pure codimension 1 subvariety. Show that the ideal $I_p(X)$ is a principal ideal (has a single generator).*

***Exercise* 6.6.5:** *Suppose $I \subset \mathcal{O}_p$ is an ideal such that $V(I)$ is a germ of a pure codimension 1 subvariety. Show that the ideal I is principal.*

***Exercise* 6.6.6:** *Let $I \subset \mathcal{O}_p$ be a principal ideal. Prove the* Nullstellensatz *for hypervarieties: $I_p\big(V(I)\big) = \sqrt{I}$. That is, show that if $(f, p) \in I_p\big(V(I)\big)$, then $(f^k, p) \in I$ for some integer k.*

***Exercise* 6.6.7:** *Suppose $X \subset U$ is a subvariety of pure codimension 1 for an open set $U \subset \mathbb{C}^n$. Let X' be a topological component of X_{reg}. Prove that the closure $\overline{X'}$ is a subvariety of U of pure codimension 1.*

6.7 Irreducibility, local parametrization, and Puiseux theorem

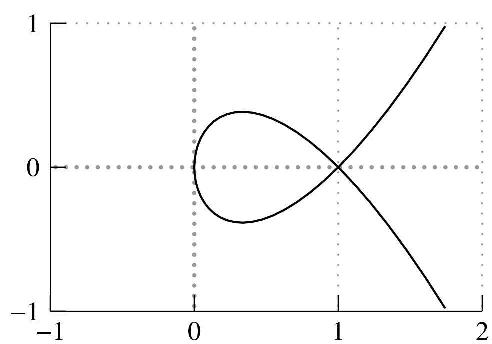
Definition 6.7.1. A germ of a subvariety $(X, p) \subset (\mathbb{C}^n, p)$ is said to be *reducible* at p if there exist two germs (X_1, p) and (X_2, p) with $(X_1, p) \not\subset (X_2, p)$ and $(X_2, p) \not\subset (X_1, p)$ such that $(X, p) = (X_1, p) \cup (X_2, p)$. Else the germ (X, p) is *irreducible* at p.

Similarly globally, a subvariety $X \subset U$ is *reducible* in U if there exist two subvarieties X_1 and X_2 of U with $X_1 \not\subset X_2$ and $X_2 \not\subset X_1$ such that $X = X_1 \cup X_2$. Otherwise, the subvariety X is *irreducible* in U.

Example 6.7.2: Local and global reducibility are different. The subvariety given by

$$z_2^2 = z_1(z_1 - 1)^2$$

is irreducible in $\mathbb{C}^2$ (the regular points are connected), but locally at the point $(1, 0)$ it is reducible. There, the subvariety is a union of two graphs: $z_2 = \pm\sqrt{z_1}(z_1 - 1)$. Here is a plot in two real dimensions:



***Exercise* 6.7.1:** *Prove a germ of a subvariety (X, p) is irreducible if and only if $I_p(X)$ is a prime ideal. Recall an ideal I is* prime *if $ab \in I$ implies either $a \in I$ or $b \in I$.*

***Exercise* 6.7.2:** *Suppose a germ of a subvariety (X, p) is of pure codimension 1. Prove (X, p) is irreducible if and only if there exists a representative of X, such that X_{reg} is connected.*

***Exercise* 6.7.3:** *Let $X \subset U$ be a subvariety of pure codimension 1 of a domain $U \subset \mathbb{C}^n$. Prove X is irreducible if and only if the set of regular points is connected. Hint: See previous exercise.*

For complex subvarieties, a subvariety is irreducible if and only if the set of regular points is connected. We omit the proof in the general case, and for hypervarieties it is an exercise above. It makes sense then, that we can split a subvariety into its irreducible parts.

Proposition 6.7.3. *Let $(X, p) \subset (\mathbb{C}^n, p)$ be a germ of a subvariety. Then there exist finitely many irreducible subvarieties $(X_1, p), \ldots, (X_k, p)$ such that $(X_1, p) \cup \ldots \cup (X_k, p) = (X, p)$ and such that $(X_j, p) \not\subset (X_\ell, p)$ for all j and ℓ.*

Proof. Suppose (X, p) is reducible, so find $(Y_1, p) \not\subset (Y_2, p)$ and $(Y_2, p) \not\subset (Y_1, p)$ such that $(Y_1, p) \cup (Y_2, p) = (X, p)$. As $(Y_j, p) \subsetneq (X, p)$, then $I_p(Y_j) \supsetneq I_p(X)$ for both j. If both (Y_1, p) and (Y_2, p) are irreducible, then stop, we are done. Otherwise apply the same reasoning to whichever (or both) (Y_j, p) that was reducible. After finitely many steps you must come to a stop as you cannot have an infinite ascending chain of ideals since $\mathcal{O}_p$ is Noetherian. ◻

These $(X_1, p), \ldots, (X_k, p)$ are called *irreducible components*, we omit the proof in general that they are unique. For a germ of a hypervariety the UFD property of ${}_n\mathcal{O}_p$ gives the irreducible components. You found this factorization in an exercise above, and so this factorization is unique.

For each irreducible component, we have the following structure. We give the theorem without proof in the general case, although we have essentially proved it already for pure codimension 1 (to put it together is left as an exercise).

Theorem 6.7.4 (Local parametrization theorem)**.** *Let $(X, 0)$ be an irreducible germ of a subvariety of dimension k in $\mathbb{C}^n$. Let X denote a representative of the germ. Then after a linear change of coordinates, we let $\pi : \mathbb{C}^n \to \mathbb{C}^k$ be the projection onto the first k components, and obtain that there exists a neighborhood $U \subset \mathbb{C}^n$ of the origin, and a proper subvariety $E \subset \pi(U)$ such that*

(i) $X' = X \cap U \setminus \pi^{-1}(E)$ is a connected k-dimensional complex manifold that is dense in $X \cap U$.

(ii) $\pi : X' \to \pi(U) \setminus E$ is an m-sheeted covering map for some integer m.

(iii) $\pi : X \cap U \to \pi(U)$ is a proper mapping.

The m-sheeted covering map in this case is a local biholomorphism that is an m-to-1 map.

> ***Exercise* 6.7.4*:*** *Use Theorem 6.3.3 to prove the parametrization theorem if $(X, 0)$ is of pure codimension 1.*

Let $(z_1, \ldots, z_n)$ be the coordinates. The linear change of coordinates needed in the theorem is to ensure that the set defined by $z_1 = z_2 = \cdots = z_k = 0$ intersected with X is an isolated point at the origin. This is precisely the same condition needed to apply Weierstrass preparation theorem in the case when X is the zero set of a single function.

We saw hypersurfaces are the simpler cases of complex-analytic subvarieties. At the other end of the spectrum, complex-analytic subvarieties of dimension 1 are also reasonably simple for different reasons. Locally, complex-analytic subvarieties of dimension 1 are analytic discs. Moreover, they are locally the one-to-one holomorphic images of discs, and so they have a natural topological manifold structure even at singular points.

Example 6.7.5: The image of the analytic disc $\xi \mapsto (\xi^2, \xi^3)$ is the cusp subvariety defined by $z_1^3 - z_2^2 = 0$ in $\mathbb{C}^2$.

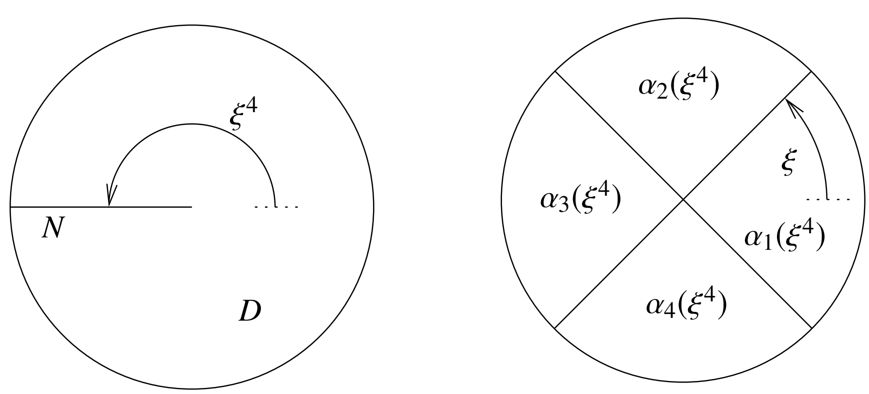
The following theorem is often stated only in $\mathbb{C}^2$ for zero sets of a single function although it follows in the same way from the local parametrization theorem in higher-dimensional spaces. Of course, we only proved that theorem (or in fact you the reader did so in an exercise), for codimension 1 subvarieties, and therefore, we also only have a complete proof of the following in $\mathbb{C}^2$.

Theorem 6.7.6 (Puiseux)**.** *Let $(z, w) \in \mathbb{C} \times \mathbb{C}^{n-1}$ be coordinates. Suppose $f : U \subset \mathbb{C} \times \mathbb{C}^{n-1} \to \mathbb{C}^\ell$ is a holomorphic map such that $f(z, w) = 0$ defines a dimension 1 subvariety X of U, $0 \in X$, and $w \mapsto f(0, w)$ has an isolated zero at the origin.*

Then there exists an integer k and a holomorphic function g defined near the origin in $\mathbb{C}$ such that for all ξ near the origin

$$f\big(\xi^k, g(\xi)\big) = 0.$$

Proof. Without loss of generality assume $(X, 0)$ is irreducible so that the local parametrization theorem applies. We work in a small disc $D \subset \mathbb{C}$ centered at the origin, such that the origin is the unique point of the discriminant set (the subvariety E). Let $N = \{z \in D : \operatorname{Im} z = 0, \operatorname{Re} z \leq 0\}$. As $D \setminus N$ is simply connected we have the well-defined functions $\alpha_1(z), \ldots, \alpha_m(z)$ holomorphic on $D \setminus N$ that are solutions to $f\big(z, \alpha_j(z)\big) = 0$. These functions continue analytically across N away from the origin. The continuation equals one of the roots, e.g. $\alpha_j(z)$ becomes $\alpha_\ell(z)$ (and by continuity it is the same root along the entire N). So there is a permutation σ on m elements such that as z moves counter-clockwise around the origin from the upper half plane across N to the lower half plane, $\alpha_j(z)$ is continued as $\alpha_{\sigma(j)}(z)$. There exists some number k (e.g. $k = m!$) such that σ^k is the identity. As ξ goes around a circle around the origin, ξ^k goes around the origin k times. Start at a positive real ξ and start defining a function $g(\xi)$ as $\alpha_1(\xi^k)$. Move ξ around the origin counter-clockwise continuing g analytically. Divide the disc into sectors of angle $2\pi/k$, whose boundaries are where $\xi^k \in N$. Transition to $\alpha_{\sigma(1)}(\xi^k)$ after we reach the boundary of the first sector, then to $\alpha_{\sigma(\sigma(1))}(\xi^k)$ after we reach the boundary of the next sector, and so on. After k steps, that is as ξ moved all the way around the circle, we are back at $\alpha_1(\xi^k)$, because σ^k is the identity. So $g(\xi)$ is a well-defined holomorphic function outside the origin. Let $g(0) = 0$, and g is holomorphic at 0 by the Riemann extension theorem. See the following figure for an example:



In the figure, $m = k = 4$, and the permutation σ takes 1 to 2, 2 to 3, 3 to 4, and 4 to 1. As ξ moves along the short circular arrow on the right, ξ^4 moves along the long circular arrow on the left. The definition of g is then given in the right hand diagram. $\square$

Exercise* 6.7.5*: *Suppose $(X,0) \subset (\mathbb{C}^2,0)$ is an irreducible germ defined by an irreducible Weierstrass polynomial $f(z,w) = 0$ (polynomial in w) of degree k. Prove that there exists a holomorphic g such that $f\big(z^k, g(z)\big) = 0$ and $z \mapsto \big(z^k, g(z)\big)$ is one-to-one and onto a neighborhood of 0 in X.*

Exercise* 6.7.6*: *Suppose $(X,0) \subset (\mathbb{C}^2,0)$ is a germ of a dimension 1 subvariety. Show that after perhaps a linear change of coordinates, there are natural numbers $d_1, \ldots, d_k$ and holomorphic functions $c_1(z), \ldots, c_k(z)$ vanishing at 0, such that X can be defined near 0 by*

$$\prod_{j=1}^{k} \big(w^{d_j} - c_j(z)\big) = 0.$$

Exercise* 6.7.7*: *Using the local parametrization theorem, prove that if (X,p) is an irreducible germ of a subvariety of dimension greater than 1, then there exists a neighborhood U of p and a closed subvariety $X \subset U$ (whose germ at p is (X,p)), such that for every $q \in X$ there exists an irreducible subvariety $Y \subset X$ of dimension 1 such that $p \in Y$ and $q \in Y$.*

Exercise* 6.7.8*: *Prove a stronger version of the above exercise. Show that not only is there a Y, but an analytic disc $\varphi \colon \mathbb{D} \to U$ such that $\varphi(\mathbb{D}) \subset X$, $\varphi(0) = p$ and $\varphi(1/2) = q$.*

Exercise* 6.7.9*: *Suppose $X \subset U$ is a subvariety of a domain $U \subset \mathbb{C}^n$. Suppose that for any two points p and q on X there exists a finite sequence of points $p_0 = p, p_1, \ldots, p_k = q$ in X, and a sequence of analytic discs $\Delta_j \subset X$ such that p_j and p_{j-1} are in Δ_j.*

Exercise* 6.7.10*: *Prove a* maximum principle for subvarieties *using the above exercises: Suppose $X \subset U$ is an irreducible subvariety of an open set U, and suppose $f \colon U \to \mathbb{R} \cup \{-\infty\}$ is a plurisubharmonic function. If the modulus of the restriction $f|_X$ achieves a maximum at some point $p \in X$, then the restriction $f|_X$ is constant.*

Exercise* 6.7.11*: *Prove that an analytic disc (namely the image of the disc) in $\mathbb{C}^2$ is a one-dimensional local variety (that is, a subvariety of some open subset of $\mathbb{C}^2$).*

Using the Puiseux theorem, we often simply parametrize germs of complex one-dimensional subvarieties. And for larger-dimensional varieties, we can find enough one-dimensional curves through any point and parametrize those.

6.8 Segre varieties and CR geometry

The existence of analytic discs (or subvarieties) in boundaries of domains says a lot about the geometry of the boundary.

Example 6.8.1: Let $M \subset \mathbb{C}^n$ be a smooth real hypersurface containing a complex hypersurface X (zero set of a holomorphic function with nonzero derivative), at some $p \in X \subset M$. Apply a local biholomorphic change of coordinates at p, so that in the new coordinates $(z,w) \in \mathbb{C}^{n-1} \times \mathbb{C}$, X is given by $w = 0$, and p is the origin. The tangent hyperplane to M at 0 must contain $w = 0$, and after rotating the w coordinate (multiplying it by $e^{i\theta}$), we can assume that M is tangent to the set $\{(z,w) : \operatorname{Im} w = 0\}$. In other words, M is given by

$$\operatorname{Im} w = \rho(z,\bar{z},\operatorname{Re} w),$$

where $d\rho = 0$. As $w = 0$ on M, then $\rho = 0$ when $\operatorname{Re} w = 0$. That is, ρ is divisible by $\operatorname{Re} w$, that is M is defined by,

$$\operatorname{Im} w = (\operatorname{Re} w)\widetilde{\rho}(z,\bar{z},\operatorname{Re} w),$$

for a smooth function $\widetilde{\rho}$. So the Levi form at the origin vanishes. As $p = 0$ was an arbitrary point on $M \cap X$, the Levi form of M vanishes on $M \cap X$.

Example 6.8.2: The vanishing of the Levi form is not necessary if the complex varieties in M are small. Consider $M \subset \mathbb{C}^3$ with a nondegenerate (but not definite) Levi form:

$$\operatorname{Im} w = |z_1|^2 - |z_2|^2 .$$

For any $\theta \in \mathbb{R}$, M contains the complex line L_θ, given by $z_1 = e^{i\theta} z_2$ and $w = 0$. In this case, the union $\bigcup_\theta L_\theta$ of those complex lines is not contained in some unique complex subvariety inside M. Any complex subvariety that contains all L_θ must contain the hypersurface given by $w = 0$, which is not contained in M.

Exercise* 6.8.1*: *Let $M \subset \mathbb{C}^n$ be a real hypersurface. Show that if M at p contains a complex submanifold of (complex) dimension more than $\frac{n-1}{2}$, then the Levi form must be degenerate, that is, it must have at least one zero eigenvalue.*

Exercise* 6.8.2*: *Let $M \subset \mathbb{C}^n$ be a pseudoconvex real hypersurface (one side of M is pseudoconvex). Suppose M at p contains a dimension k complex submanifold X. Show that the Levi form has at least k zero eigenvalues.*

Exercise* 6.8.3*: *Find an example of a real hypersurface $M \subset \mathbb{C}^n$ that contains a unique complex-analytic subvariety through a point p and this subvariety is singular.*

Let us discuss a tool, *Segre variety*, that allows us to find such complex subvarieties inside M, and much more. Segre varieties only work in the real-analytic setting and rely on complexification.

Let $M \subset \mathbb{C}^n$ be a real-analytic hypersurface and $p \in M$. Suppose $M \subset U$, where $U \subset \mathbb{C}^n$ is a small domain with a defining function $r : U \to \mathbb{R}$ for M. That is, r is a real-analytic function in U such that $M = r^{-1}(0)$, but $dr \neq 0$ on M. Define

$$U^* = \{ z \in \mathbb{C}^n : \bar{z} \in U \}.$$

Suppose U is small enough so that the Taylor series for r converges in $U \times U^*$ when treating z and $\bar{z}$ as separate variables. That is, $r(z,\zeta)$ is a well-defined function on $U \times U^*$, and $r(z,\zeta) = 0$ defines a complexification $\mathcal{M}$ in $U \times U^*$. Assume also that U is small enough that the complexified dr does not vanish on $\mathcal{M}$ and that $\mathcal{M}$ is connected. See also Proposition 3.2.8.

Given $q \in U$, define the *Segre variety*

$$\Sigma_q(U,r) = \{ z \in U : r(z,\bar{q}) = 0 \} = \{ z \in U : (z,\bar{q}) \in \mathcal{M} \}.$$

A priory, the subvariety Σ_p depends on U and r. However, if $\widetilde{r}$ is a real-analytic function that complexifies to $U \times U^*$ and vanishes on M, it must also vanish on the complexification $\mathcal{M}$. If $\widetilde{r}$ is a defining function as above, that is, $d\widetilde{r}$ does not vanish on its zero set and the zero set of the complexified $\widetilde{r}$ is connected in $U \times U^*$, then $\widetilde{r}(z,\zeta) = 0$ also defines $\mathcal{M}$. Hence the actual r does not matter. As long as $q \in M$, then $q \in \Sigma_q(U,r)$, and furthermore the Segre variety is a complex hypersurface for every q. It is not hard to see that if $\widetilde{U}$ is a small neighborhood of q, the same r is a defining function in $\widetilde{U}$ and we get the same complexification in $\widetilde{U} \times \widetilde{U}^*$. So the germ at $q \in U$ is well-defined and we write

$$\Sigma_q = \bigl(\Sigma_q(U,r), q\bigr).$$

The Segre variety is well-defined as a germ, and so often when one talks about Σ_q without mentioning the U or r, then one means some small enough representative of a Segre variety or the germ itself.

Exercise* 6.8.4*: *Let $r \colon U \to \mathbb{R}$ be a real-valued real-analytic function that complexifies to $U \times U^*$. Show that $r(z,\bar{\zeta}) = 0$ if and only if $r(w,\bar{\zeta}) = 0$. In other words, $z \in \Sigma_\zeta(U,r)$ if and only if $\zeta \in \Sigma_z(U,r)$.*

Example 6.8.3: Suppose we start with the real-analytic hypersurface M given by

$$\operatorname{Im} w = (\operatorname{Re} w)\rho(z,\bar{z},\operatorname{Re} w),$$

with ρ vanishing at the origin. Rewriting in terms of w and $\bar{w}$ we find

$$\frac{w-\bar{w}}{2i} = \left(\frac{w+\bar{w}}{2}\right)\rho\left(z,\bar{z},\frac{w+\bar{w}}{2}\right).$$

Setting $\bar{z} = \bar{w} = 0$ we find

$$\frac{w}{2i} = \left(\frac{w}{2}\right)\rho\left(z,0,\frac{w}{2}\right).$$

As ρ vanishes at the origin, then near the origin the equation defines the complex hypersurface given by $w = 0$. So Σ_0 is defined by $w = 0$. This is precisely the complex hypersurface that lies inside M.

The last example is not a fluke. The most important property of Segre varieties is that it locates complex subvarieties in a real-analytic submanifold. We will phrase it in terms of analytic discs, which is enough as complex subvarieties can be filled with analytic discs, as we have seen.

Proposition 6.8.4. *Let $M \subset \mathbb{C}^n$ be a real-analytic hypersurface and $p \in M$. Suppose $\Delta \subset M$ is an analytic disc through p. Then as germs $(\Delta, p) \subset \Sigma_p$.*

Proof. Let U be a neighborhood of p where a representative of Σ_p is defined, that is, we assume that Σ_p is a closed subset of U, and suppose $r(z,\bar{z})$ is the corresponding defining function. Let $\varphi \colon \mathbb{D} \to \mathbb{C}^n$ be the parametrization of Δ with $\varphi(0) = p$. We can restrict φ to a smaller disc around the origin, and since we are only interested in the germ of Δ at p this is sufficient (if there are multiple points of $\mathbb{D}$ that go to p, we repeat the argument for each one). So let us assume without loss of generality that $\varphi(\mathbb{D}) = \Delta \subset U$. Since $\Delta \subset M$ we have

$$r\bigl(\varphi(\xi), \overline{\varphi(\xi)}\bigr) = r\bigl(\varphi(\xi), \bar{\varphi}(\bar{\xi})\bigr) = 0.$$

The function $\xi \mapsto r\bigl(\varphi(\xi), \bar{\varphi}(\bar{\xi})\bigr)$ is a real-analytic function of ξ, and therefore for some small neighborhood of the origin, it complexifies. In fact, it complexifies to $\mathbb{D} \times \mathbb{D}$ as $\varphi(\xi) \in U$ for all $\xi \in \mathbb{D}$. So we can treat ξ and $\bar{\xi}$ as separate variables. By complexification, the equation holds for all such independent ξ and $\bar{\xi}$. Set $\bar{\xi} = 0$ to obtain

$$0 = r\bigl(\varphi(\xi), \bar{\varphi}(0)\bigr) = r\bigl(\varphi(\xi), \bar{p}\bigr) \qquad \text{for all } \xi \in \mathbb{D}.$$

In particular, $\varphi(\mathbb{D}) \subset \Sigma_p$ and the result follows. □

***Exercise* 6.8.5:** *Show that if a real-analytic real hypersurface $M \subset \mathbb{C}^n$ is strongly pseudoconvex at $p \in M$ (one side of M is strongly pseudoconvex at p), then $\Sigma_p \cap (M, p) = \{p\}$ (as germs).*

***Exercise* 6.8.6:** *Use the proposition and the above exercise to show that if a real-analytic real hypersurface M is strongly pseudoconvex, then M contains no analytic discs.*

Let us end our discussion of Segre varieties by its perhaps most well-known application, the so-called Diederich–Fornæss lemma. Although we state it only for real-analytic hypersurfaces it works in greater generality. There are two parts to it, although it is generally the corollary that is called the *Diederich–Fornæss lemma*.

First, for real-analytic hypersurfaces each point has a fixed neighborhood such that germs of complex subvarieties contained in the hypersurface extend to said fixed neighborhood.

Theorem 6.8.5 (Diederich–Fornæss)**.** *Suppose $M \subset \mathbb{C}^n$ is a real-analytic hypersurface. For every $p \in M$ there exists a neighborhood U of p with the following property: If $q \in M \cap U$ and (X, q) is a germ of a complex subvariety such that $(X, q) \subset (M, q)$, then there exists a complex subvariety $Y \subset U$ (in particular a closed subset of U) such that $Y \subset M$ and $(X, q) \subset (Y, q)$.*

Proof. Suppose U is a polydisc centered at p, small enough so that the defining function r of M complexifies to $U \times U^*$ as above. Suppose $q \in M \cap U$ is a point such that (X, q) is a germ of a positive-dimensional complex subvariety with $(X, q) \subset (M, q)$. Most points of a subvariety are regular, so without loss of generality assume q is a regular point, that is, (X, q) is a germ of a complex submanifold. Let X be a representative of the germ (X, q) such that $X \subset M$, and $X \subset U$, although we do not assume it is closed.

Assume X is a holomorphic image of an open subset $V \subset \mathbb{C}^k$ via a holomorphic surjective mapping $\varphi\colon V \to X$. Since $r\bigl(\varphi(\xi), \overline{\varphi(\xi)}\bigr) = 0$ for all $\xi \in V$, then we may treat ξ and $\bar{\xi}$ separately. In particular, we find $r(z, \bar{\zeta}) = 0$ for all $z, \zeta \in X$.

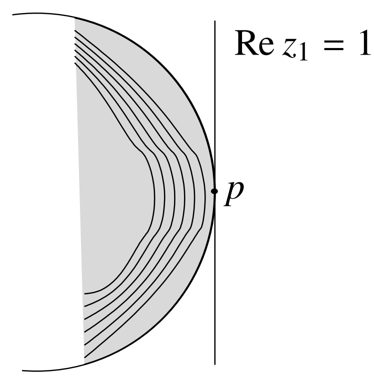
Define complex subvarieties $Y', Y \subset U$ (closed in U) by

$$Y' = \bigcap_{a \in X} \Sigma_a(U, r) \qquad \text{and} \qquad Y = \bigcap_{a \in Y'} \Sigma_a(U, r).$$

If $a \in Y'$ and $b \in X$, then $r(a, \bar{b}) = 0$. Because r is real-valued, $r(b, \bar{a}) = 0$. Therefore, $X \subset Y \subset Y'$. Furthermore $r(z, \bar{z}) = 0$ for all $z \in Y$ and so $Y \subset M$. □

Corollary 6.8.6 (Diederich–Fornæss)**.** *Suppose $M \subset \mathbb{C}^n$ is a compact real-analytic hypersurface. Then there does not exist any point $q \in M$ such that there exists a germ of a positive-dimensional complex subvariety (X, q) such that $(X, q) \subset (M, q)$.*

Proof. Let $S \subset M$ be the set of points through which there exists a germ of a positive-dimensional complex subvariety contained in M. As M, and hence $\overline{S}$, is compact, there must exist a point $p \in \overline{S}$ that is furthest from the origin. After a rotation by a unitary and rescaling assume $p = (1, 0, \ldots, 0)$. Let U be the neighborhood from the previous theorem around p. There exist germs of varieties in M through points arbitrarily close to p. So for any distance $\epsilon > 0$, there exists a subvariety $Y \subset U$ (in particular, Y closed in U) of positive dimension with $Y \subset M$ that contains points ϵ close to p. Consider the function $\operatorname{Re} z_1$, whose modulus attains a strict maximum on $\overline{S}$ at p. Because $\operatorname{Re} z_1$ achieves a maximum strictly smaller than 1 on $\partial U \cap \overline{S}$, for a small enough ϵ, we would obtain a pluriharmonic function with a strict maximum on Y, which is impossible by the maximum principle for varieties that you proved in Exercise 6.7.10. The picture would look as follows:



□

Example 6.8.7: The above results do not work in the smooth setting. First let us disprove the theorem in the smooth setting, and the corollary is left as an exercise. Let $g\colon \mathbb{R} \to \mathbb{R}$ be a smooth function that is strictly positive for $|t| > 1$ and $g(t) = 0$ for all $|t| \leq 1$. Then let M in $(z, w) \in \mathbb{C}^{n-1} \times \mathbb{C}$ be defined by

$$\operatorname{Im} w = g\big(\|z\|^2\big).$$

M is a smooth hypersurface. Consider $p = (1, 0, \ldots, 0) \in M$. For every $0 < s < 1$, let $q_s = (s, 0, \ldots, 0) \in M$ and $X_s = \big\{(z, w) \in M : w = s\big\}$. Each X_s is a local complex subvariety of dimension $n - 1$ and $(X_s, q_s) \subset (M, q_s)$. The size of X_s goes to zero as $s \to 1$ and X_s cannot extend to a larger complex subvariety inside M. So, no neighborhood U at p (as in the theorem) exists.

***Exercise* 6.8.7:** *Find a smooth compact hypersurface $M \subset \mathbb{C}^n$ that contains a germ of a positive dimensional complex subvariety.*

…and that is how using sheep's bladders can prevent earthquakes!

Appendix A

Basic notation and terminology

Let us just quickly review some of the basic notation that we use in this book, that is perhaps not described elsewhere. We use $\mathbb{C}$, $\mathbb{R}$ for complex and real numbers, and i for imaginary unit (a square root of -1). We use $\mathbb{N} = \{1, 2, 3, \ldots\}$ for the natural numbers, $\mathbb{N}_0 = \{0, 1, 2, 3, \ldots\}$ for the zero-based natural numbers, and $\mathbb{Z}$ for all integers. When we write $\mathbb{C}^n$ or $\mathbb{R}^n$ we implicitly mean that $n \geq 1$, unless otherwise stated.

The notation $f \colon X \to Y$ means a function with domain X and codomain Y. By $f(S)$ we mean the direct image of S by f. We use the notation f^{-1} for the inverse image of sets and single points, and when the mapping is bijective (one-to-one and onto) then we use it for the inverse mapping. So for example $f^{-1}(T)$ for a set $T \subset Y$ denotes all the points of X that f maps to T. Similarly $f^{-1}(q)$ could denote all the points that map to q, but if the mapping is bijective then it would just mean the unique point mapping to q.

If we wish to define a function without necessarily giving it a name, we use the notation

$$x \mapsto F(x)$$

where $F(x)$ would generally be some formula giving the output.

The notation $f|_S$ means the restriction of f to S, that is, it is a function $f|_S \colon S \to Y$ such that $f|_S(x) = f(x)$ for all $x \in S$.

We denote the set subtraction by $A \setminus B$, meaning all elements of A which are not in B. We denote the complement of a set by X^c. In this case, the ambient set should be clear. So, for example, if $X \subset \mathbb{C}$ naturally, then $X^c = \mathbb{C} \setminus X$.

The topological closure of a set S is denoted by $\overline{S}$, its boundary is denoted by ∂S. If S is relatively compact subset of X (its closure in X is compact) or compact we write $S \subset\subset X$.

A function $f \colon U \to \mathbb{C}$ is said to be *compactly supported* if the *support*, that is the set $\overline{\{p \in U : f(p) \neq 0\}}$, is compact.

When we wish to say that two functions are identically equal, we write $f \equiv g$. This means that $f(x) = g(x)$ for all x in the domain.

The notation $f \circ g$ denotes the composition defined by $x \mapsto f\bigl(g(x)\bigr)$.

We write

$$X \overset{\text{def}}{=} Y$$

to define X to be Y rather than just show equality.

Appendix B

Results from one complex variable

Let us review some results from one complex variable that may be useful for reading this book. The reader should first look through section 0.1 for basic notation and motivation, although we will review some of the results again here. We start with holomorphic functions. Let $U \subset \mathbb{C}$ be open. A function $f \colon U \to \mathbb{C}$ is *holomorphic* if it is complex differentiable at every point, that is

$$f'(z) = \lim_{h \in \mathbb{C} \to 0} \frac{f(z+h) - f(z)}{h}$$

exists for all $z \in U$. For example, polynomials and rational functions in z are holomorphic. One of the most important examples is the solution to the differential equation $f'(z) = f(z)$, $f(0) = 1$, that is the complex exponential, defined in the entire complex plane:

$$f(z) = e^z = e^{x+iy} = e^x \bigl(\cos y + i \sin(y)\bigr).$$

A piecewise-C^1 path (or curve) in $\mathbb{C}$ is a continuous function $\gamma \colon [a,b] \to \mathbb{C}$, continuously differentiable except at finitely many points, such that one-sided limits of γ' exist at all points of $[a,b]$. By abuse of notation, when γ is used as a set, then we mean the image $\gamma\bigl([a,b]\bigr)$. For a continuous function $f \colon \gamma \to \mathbb{C}$, define

$$\int_\gamma f(z)\, dz \overset{\text{def}}{=} \int_a^b f\bigl(\gamma(t)\bigr)\gamma'(t)\, dt.$$

As γ' is defined at all but finitely many points and is otherwise continuous, the integral is well-defined. Similarly one defines the more general path integral in $dz = dx + i\,dy$ and $d\bar{z} = dx - i\,dy$. Let $z = \gamma(t) = \gamma_1(t) + i\,\gamma_2(t) = x + i\,y$ parametrize the path. Then

$$\begin{aligned}
\int_\gamma f(z)\, dz + g(z)\, d\bar{z} &= \int_\gamma \Bigl(f\bigl(x+i\,y\bigr) + g\bigl(x+i\,y\bigr) \Bigr)\, dx + i \Bigl(f\bigl(x+i\,y\bigr) - g\bigl(x+i\,y\bigr) \Bigr)\, dy \\
&= \int_a^b \Bigl(f\bigl(\gamma(t)\bigr)\gamma'(t) + f\bigl(\gamma(t)\bigr)\overline{\gamma'(t)} \Bigr)\, dt \\
&= \int_a^b \Bigl(\Bigl(f\bigl(\gamma(t)\bigr) + g\bigl(\gamma(t)\bigr) \Bigr) \gamma_1'(t) + i \Bigl(f\bigl(\gamma(t)\bigr) - g\bigl(\gamma(t)\bigr) \Bigr) \gamma_2'(t) \Bigr)\, dt.
\end{aligned}$$

A path is *closed* if $\gamma(a) = \gamma(b)$, and a path is *simple* if $\gamma|_{[a,b]}$ is one-to-one with the possible exception of $\gamma(a) = \gamma(b)$.

As in the introduction, we have the following version of Cauchy integral formula. An open set has piecewise-C^1 boundary if ∂U is composed of piecewise-C^1 paths, and at point $p \in \partial U$, every disc $\Delta_\rho(p)$ contains points from U and points from the complement of $\overline{U}$. If at each point where the parametrization of ∂U is differentiable, the domain is on the left ($\gamma'(t)$ rotated by $\frac{\pi}{2}$ points into the domain), then the boundary is *positively oriented*.

Theorem B.1 (Cauchy integral formula)**.** *Let $U \subset \mathbb{C}$ be a bounded open set with piecewise-C^1 boundary ∂U oriented positively, and let $f \colon \overline{U} \to \mathbb{C}$ be a continuous function holomorphic in U. Then for $z \in U$,*

$$f(z) = \frac{1}{2\pi i} \int_{\partial U} \frac{f(\zeta)}{\zeta - z}\, d\zeta .$$

The theorem follows from Green's theorem, which is the Stokes' theorem in two dimensions. In the versions we state, one needs to approximate the open set by smaller open sets from the inside to insure the partial derivatives are bounded. See Theorem 4.1.1. Let us state Green's theorem using the dz and $d\bar{z}$ for completeness. See appendix C for an overview of differential forms.

Theorem B.2 (Green's theorem)**.** *Let $U \subset \mathbb{C}$ be a bounded open set with piecewise-C^1 boundary ∂U oriented positively, and let $f \colon \overline{U} \to \mathbb{C}$ be a continuous function with bounded continuous partial derivatives in U. Then*

$$\begin{aligned} \int_{\partial U} f(z)\, dz + g(z)\, d\bar{z} &= \int_U d\Big(f(z)\, dz + g(z)\, d\bar{z}\Big) = \int_U \left(\frac{\partial g}{\partial z} - \frac{\partial f}{\partial \bar{z}}\right) dz \wedge d\bar{z} \\ &= (-2i) \int_U \left(\frac{\partial g}{\partial z} - \frac{\partial f}{\partial \bar{z}}\right) dx \wedge dy = (-2i) \int_U \left(\frac{\partial g}{\partial z} - \frac{\partial f}{\partial \bar{z}}\right) dA. \end{aligned}$$

The Cauchy integral formula is equivalent to what's usually called just Cauchy's theorem:

Theorem B.3 (Cauchy)**.** *Let $U \subset \mathbb{C}$ be a bounded open set with piecewise-C^1 boundary ∂U oriented positively, and let $f \colon \overline{U} \to \mathbb{C}$ be a continuous function holomorphic in U. Then*

$$\int_{\partial U} f(z)\, dz = 0.$$

There is a converse to Cauchy as well. A triangle $T \subset \mathbb{C}$ is the convex hull of the three vertices (so we include the inside of the triangle), and ∂T is the boundary of the triangle oriented counter-clockwise. Let us state the following theorem as an if and only if, even though, usually it is only the reverse direction that is called Morera's theorem.

Theorem B.4 (Morera)**.** *Suppose $U \subset \mathbb{C}$ is an open set, and $f \colon U \to \mathbb{C}$ is continuous. Then f is holomorphic if and only if*

$$\int_{\partial T} f(z)\, dz = 0$$

for all triangles $T \subset U$.

As we saw in the introduction, a holomorphic function has a power series.

Proposition B.5. *If $U \subset \mathbb{C}$ is an open set, and $f : U \to \mathbb{C}$ holomorphic, then f is infinitely differentiable, and if $\Delta_\rho(p) \subset \mathbb{C}$ is a disc, then f has a power series that converges absolutely uniformly on compact subsets of $\Delta_\rho(p)$:*

$$f(z) = \sum_{k=0}^{\infty} c_k (z-p)^k,$$

where given a γ that is a simple closed (piecewise-C^1) path once counterclockwise around p inside $\Delta_\rho(p)$,

$$c_k = \frac{f^{(k)}(p)}{k!} = \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{(\zeta - z)^{k+1}}\, d\zeta.$$

Cauchy estimates follow: If M is the maximum of $|f|$ on the circle $\partial\Delta_r(p)$, then

$$|c_k| \leq \frac{M}{r^k}.$$

Conversely, if a power series satisfies the Cauchy estimates, it converges on $\Delta_r(p)$.

A function $f : \mathbb{C} \to \mathbb{C}$ that is holomorphic is *entire*. An immediate application of Cauchy estimates is Liouville's theorem:

Theorem B.6 (Liouville)**.** *If f is entire and bounded, then f is constant.*

And as a holomorphic function has a power series it satisfies the identity theorem:

Theorem B.7 (Identity)**.** *If $U \subset \mathbb{C}$ is a domain, and $f : U \to \mathbb{C}$ holomorphic such that the zero set $f^{-1}(0)$ has a limit point in U, then $f \equiv 0$.*

Another consequence of the Cauchy integral formula is that there is a differential equation characterizing holomorphic functions.

Proposition B.8 (Cauchy–Riemann equations)**.** *Given an open set $U \subset \mathbb{C}$, a function $f : U \to \mathbb{C}$ is holomorphic if and only if f is continuously differentiable and*

$$\frac{\partial f}{\partial \bar{z}} = 0 \qquad \text{on } U.$$

Yet another consequence of the Cauchy formula (and one can make an argument that everything in this appendix is a consequence of the Cauchy formula) is the open mapping theorem.

Theorem B.9 (Open mapping theorem)**.** *Suppose $U \subset \mathbb{C}$ is open and $f : U \to \mathbb{C}$ is holomorphic. Then f is an open mapping, that is, $f(V)$ is open whenever V is open.*

The real and imaginary parts u and v of a holomorphic function $f = u + iv$ are harmonic, that is $\nabla^2 u = \nabla^2 v = 0$, where ∇^2 is the Laplacian. A domain U is *simply connected* if every simple closed path is homotopic in U to a constant, in other words, if the domain has no holes. For example a disc is simply connected.

Proposition B.10. *If $U \subset \mathbb{C}$ is a simply connected domain and $u : U \to \mathbb{R}$ is harmonic, then there exists a harmonic function $v : U \to \mathbb{R}$ such that $f = u + iv$ is holomorphic.*

The function v is called the *harmonic conjugate* of u. For further review of harmonic functions see section 2.4 on harmonic functions. We have the following versions of the maximum principle.

Theorem B.11 (Maximum principles)**.** *Suppose $U \subset \mathbb{C}$ is a domain.*

(i) If $f : U \to \mathbb{C}$ is holomorphic and $|f|$ achieves a local maximum in U, then f is constant.

(ii) If U is bounded and $f : \overline{U} \to \mathbb{C}$ is holomorphic in U and continuous, then $|f|$ achieves its maximum on ∂U.

(iii) If $f : U \to \mathbb{R}$ is harmonic achieves a local maximum or a minimum in U, then f is constant.

(iv) If U is bounded and $f : \overline{U} \to \mathbb{R}$ is harmonic in U and continuous, then f achieves its maximum and minimum on ∂U.

The maximum principle immediately implies the following lemma.

Lemma B.12 (Schwarz's lemma)**.** *Suppose $f : \mathbb{D} \to \mathbb{C}$ is holomorphic and $f(0) = 0$, then*

(i) $|f(z)| \leq |z|$, and

(ii) $|f'(0)| \leq 1$.

Furthermore, if $|f(z)| = |z|$ for any $z \in \mathbb{D}$ or $|f'(0)| = 1$, then $f(z) = e^{i\theta} z$ for some $\theta \in \mathbb{R}$.

The above theorem is actually quite general.

Theorem B.13 (Riemann mapping theorem)**.** *If $U \subset \mathbb{C}$ is a nonempty simply connected domain such that $U \neq \mathbb{C}$, then U is biholomorphic to $\mathbb{D}$. Given $z_0 \in U$ there exists a unique biholomorphic $f : U \to \mathbb{D}$ such that $f'(z_0) > 0$, and f maximizes $|f'(z_0)|$ among all holomorphic maps to $\mathbb{D}$.*

Schwarz's Lemma can also be used to classify the automorphisms of the disc (and hence any simply connected domain). Let $\operatorname{Aut}(\mathbb{D})$ denote the group of biholomorphic (both f and f^{-1} are holomorphic) self maps of the disc to itself.

Proposition B.14. *If $f \in \operatorname{Aut}(\mathbb{D})$ then there exists an $a \in \mathbb{D}$ and $\theta \in \mathbb{R}$ such that*

$$f(z) = e^{i\theta} \frac{z - a}{1 - \bar{a} z}.$$

Speaking of automorphisms. We have the following version of inverse function theorem.

Theorem B.15. *Suppose U and V are open subsets of $\mathbb{C}$.*

(i) If $f : U \to V$ is holomorphic and bijective (one-to-one and onto), then $f'(z) \neq 0$ for all $z \in V$, and $f^{-1} : V \to U$ is holomorphic. If $f(p) = q$, then

$$\left(f^{-1}\right)(q) = \frac{1}{f'(p)}.$$

(ii) If $f : U \to V$ is holomorphic, $f(p) = q$, and $f'(p) \neq 0$, then there exists a neighborhood W of q and a holomorphic function $g : W \to U$ that is one-to-one and $f\big(g(z)\big) = z$ for all $z \in W$.

The Riemann mapping theorem actually follows from the following theorem about existence of branches of the logarithm.

Theorem B.16. *Suppose $U \subset \mathbb{C}$ is a simply connected domain, and $f: U \to \mathbb{C}$ is a holomorphic function without zeros in U. Then there exists a holomorphic function $L: U \to \mathbb{C}$ such that*

$$e^L = f.$$

In particular, we can take roots: for any $k \in \mathbb{N}$, there exists a holomorphic function $g: U \to \mathbb{C}$ such that

$$g^k = f.$$

In one complex variable, zeros of holomorphic functions can be divided out. Moreover, zeros of holomorphic functions are of finite order unless the function is identically zero.

Proposition B.17. *Suppose $U \subset \mathbb{C}$ is a domain and $f: U \to \mathbb{C}$ is holomorphic, not identically zero, and $f(p) = 0$ for some $p \in U$. There exists a $k \in \mathbb{N}$ and a holomorphic function $g: U \to \mathbb{C}$, such that $g(p) \neq 0$ and*

$$f(z) = (z-p)^k g(z) \qquad \text{for all } z \in U.$$

The number k above is called the *order* or *multiplicity* of the zero at p. We can use this fact and the existence of roots to show that every holomorphic function is locally like z^k. The function φ below can be thought of as a local change of coordinates.

Proposition B.18. *Suppose $U \subset \mathbb{C}$ is a domain and $f: U \to \mathbb{C}$ is holomorphic, not identically zero, and $p \in U$. Then there exists a $k \in \mathbb{N}$, a neighborhood $V \subset U$ of p, and a holomorphic function $\varphi: V \to \mathbb{C}$ with $\varphi'(p) \neq 0$, such that*

$$\bigl(\varphi(z)\bigr)^k = f(z) - f(p) \qquad \text{for all } z \in V.$$

Convergence of holomorphic functions is the same as for continuous functions: uniform convergence on compact subsets. Sometimes this is called *normal convergence*.

Proposition B.19. *Suppose $U \subset \mathbb{C}$ is an open set, and $f_k: U \to \mathbb{C}$ a sequence of holomorphic functions which converge uniformly on compact subsets of U to $f: U \to \mathbb{C}$. Then f is holomorphic, and every derivative $f_k^{(\ell)}$ converges uniformly on compact subsets to the derivative $f^{(\ell)}$.*

Holomorphic functions satisfy a Heine–Borel-like property:

Theorem B.20 (Montel)**.** *If $U \subset \mathbb{C}$ is an open set and $f_n \subset U \to \mathbb{C}$ is a sequence of holomorphic functions that is uniformly bounded on compact subsets of U, then there exists a subsequence converging uniformly on compact subsets of U.*

A sequence of holomorphic functions cannot create or delete zeros out of thin air:

Theorem B.21 (Hurwitz)**.** *Suppose $U \subset \mathbb{C}$ is a domain and $f_n \subset U \to \mathbb{C}$ is a sequence of holomorphic functions converging uniformly on compact subsets of U to $f: U \to \mathbb{C}$. If f is not identically zero and z_0 is a zero of f, then there exists a disc $\Delta_r(z_0)$ and an N, such that for all $n \geq N$, f_n has the same number of zeros (counting multiplicity) in $\Delta_r(z_0)$ as f (counting multiplicity).*

A common application, and sometimes the way the theorem is stated, is that if f_n have no zeros in U, then either the limit f is identically zero, or it also has no zeros.

If $U \subset \mathbb{C}$ is open, $p \in U$, and $f : U \setminus \{p\} \to \mathbb{C}$ is holomorphic we say that f has an *isolated singularity* at p. An isolated singularity is *removable* if there exists a holomorphic function $F : U \to \mathbb{C}$ such that $f(z) = F(z)$ for all $z \in U \setminus \{p\}$. An isolated singularity is a *pole* if

$$\lim_{z \to p} f(z) = \infty \qquad \text{(that is } |f(z)| \to \infty \text{ as } |z| \to \infty).$$

An isolated singularity that is neither removable nor a pole is said to be *essential*.

At non-essential isolated singularities the function blows up to a finite integral order. The first part of the following proposition is usually called the *Riemann extension theorem*.

Proposition B.22. *Suppose $U \subset \mathbb{C}$ is an open set, $p \in U$, and $f : U \setminus \{p\} \to \mathbb{C}$ holomorphic.*

(i) If f is bounded (near p is enough), then p is a removable singularity.

(ii) If p is a pole, there exists a $k \in \mathbb{N}$ such that

$$g(z) = (z - p)^k f(z)$$

is bounded near p and hence g has a removable singularity at p.

The number k above is called the *order* of the pole. There is a symmetry between zeros and poles: If f has a zero of order k, then $\frac{1}{f}$ has a pole of order k. If f has a pole of order k, then $\frac{1}{f}$ has a removable singularity, and the extended function has a zero of order k.

Let $\mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$ be the *Riemann sphere*. The topology on $\mathbb{P}^1$ is given by insisting that the function $\frac{1}{z}$ is a homeomorphism of $\mathbb{P}^1$ to itself, where $\frac{1}{\infty} = 0$ and $\frac{1}{0} = \infty$. A function $f : U \to \mathbb{P}^1$ is called *meromorphic*, if it is not identically ∞, is holomorphic on $U \setminus f^{-1}(\infty)$, and has poles at $f^{-1}(\infty)$. A holomorphic function with poles is meromorphic by setting the value to be ∞ at the poles. A meromorphic function is one that can locally be written as a quotient of holomorphic functions.

At an isolated singularity we can expand a holomorphic function via the so-called *Laurent series* by adding all negative powers. The Laurent series also characterizes the type of the singularity.

Proposition B.23. *If $\Delta \subset \mathbb{C}$ is a disc centered at $p \in \mathbb{C}$, and $f : \Delta \setminus \{p\} \to \mathbb{C}$ holomorphic, then there exists a double sequence $\{c_k\}_{k=-\infty}^{\infty}$ such that*

$$f(z) = \sum_{k=-\infty}^{\infty} c_k (z - p)^k,$$

converges absolutely uniformly on compact subsets of Δ. If γ is a simple closed piecewise-C^1 path going once counterclockwise around p in Δ, then

$$c_k = \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{(\zeta - z)^{k+1}} \, d\zeta.$$

The singularity at p is

(i) removable *if $c_k = 0$ for all $k < 0$.*

(ii) pole *of order $\ell \in \mathbb{N}$ if $c_k = 0$ for all $k < -\ell$ and $c_{-\ell} \neq 0$.*

(iii) essential *if for every exist infinitely negative k such that $c_k \neq 0$.*

If p is an isolated singularity of f, then call the corresponding c_{-1} the *residue* of f at p, and write it as $\operatorname{Res}(f,p)$. The proposition says that for a small γ around p in the positive direction,

$$\operatorname{Res}(f,p) = c_{-1} = \frac{1}{2\pi i} \int_\gamma f(z)\,dz.$$

Combining with Cauchy's theorem tells us that to compute integrals of functions with isolated singularities we simply need to find the residues, which tend to be simpler to compute. For example, if p is a simple pole (of order 1), then

$$\operatorname{Res}(f,p) = \lim_{z\to p} (z-p) f(z).$$

Theorem B.24 (Residue theorem)**.** *Suppose $U \subset \mathbb{C}$ is an open set, and γ is a piecewise-C^1 simple closed path in U such that the interior of γ is in U. Suppose that $f \colon U \setminus S \to \mathbb{C}$ is a holomorphic function with isolated singularities in a finite set S, and suppose S lies in the interior of γ. Then*

$$\int_\gamma f(z)\,dz = 2\pi i \sum_{p\in S} \operatorname{Res}(f,p).$$

The identity theorem says that zeros of a nonconstant holomorphic f have no limit points, and so are isolated points. Since $\frac{1}{f}$ is a meromorphic function with zeros at the poles of f, poles are also isolated. Zeros and poles of can be counted fairly easily.

Theorem B.25 (Argument principle)**.** *Suppose $U \subset \mathbb{C}$ is an open set, and γ is a piecewise-C^1 simple closed path in U such that the interior of γ is in U. Suppose that $f \colon U \to \mathbb{P}^1$ is a meromorphic function with no zeros or poles on γ. Then*

$$\frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z)}\,dz = N - P,$$

where N is the number of zeros of f inside γ and P is the number of poles, both counted with multiplicity.

Furthermore, if $h \colon U \to \mathbb{C}$ is holomorphic, and $\#(p)$ is the order (multiplicity) of a zero or pole at p, then

$$\frac{1}{2\pi i} \int_\gamma h(z) \frac{f'(z)}{f(z)}\,dz = \sum_{p\in f^{-1}(0)} \#(p)\, h(p) \quad - \quad \sum_{p\in f^{-1}(\infty)} \#(p)\, h(p).$$

The proof is an immediate application of the residue theorem. Simply compute the residues at the zeros and poles of f. In particular, if f has a zero at p, then $h(z)\frac{f'(z)}{f(z)}$ has a simple pole at p with residue $\#(p)h(p)$. Similarly, if f has a pole at p, then $h(z)\frac{f'(z)}{f(z)}$ has residue $-\#(p)h(p)$ at p.

In the above couple of theorems we avoided introducing winding numbers by making γ a simple closed curve, so the above statements may be slightly different than what you have seen in your one-variable course.

Another useful way to count zeros is Rouchè's theorem.

Theorem B.26 (Rouchè)**.** *Suppose $U \subset \mathbb{C}$ is an open set, and γ is a piecewise-C^1 simple closed path in U such that the interior of γ is in U. Suppose that $f : U \to \mathbb{C}$ and $g : U \to \mathbb{C}$ are holomorphic functions such that*

$$\lvert f(z) - g(z) \rvert < \lvert f(z) \rvert + \lvert g(z) \rvert$$

for all $z \in \gamma$. Then f and g have the same number of zeros inside γ (up to multiplicity).

In the classical statement of the theorem the weaker inequality $\lvert f(z) - g(z) \rvert < \lvert f(z) \rvert$ is used. Notice that either inequality precludes any zeros on γ itself.

An function with an essential singularity actually achieves essentially every value. A weak version of this result (and an easy to prove one) is the *Casorati–Weierstrass theorem*: If f has an essential singularity at p, then for any neighborhood W of p, $f\big(W \setminus \{p\}\big)$ is dense in $\mathbb{C}$. Let us state the much stronger theorem of Picard: A function with an essential singularity is very wild. It achieves every value (except possibly one) infinitely often.

Theorem B.27 (Picard's big theorem)**.** *Suppose $U \subset \mathbb{C}$ is open, $f : U \setminus \{p\} \to \mathbb{C}$ is holomorphic, and f has an essential singularity at p. Then for any neighborhood W of p, $f\big(W \setminus \{p\}\big)$ is either $\mathbb{C}$ or $\mathbb{C}$ minus a point.*

For example, $e^{1/z}$ has an essential singularity at the origin and the function is never 0. Since we stated the big theorem, let us also state the little theorem.

Theorem B.28 (Picard's little theorem)**.** *If $f : \mathbb{C} \to \mathbb{C}$ is holomorphic then $f(\mathbb{C})$ is either $\mathbb{C}$ or $\mathbb{C}$ minus a point.*

One theorem from algebra that is important in complex analysis, and becomes perhaps even more important in several variables is the fundamental theorem of algebra. It really is a theorem of complex analysis and its standard proof is via the maximum principle.

Theorem B.29 (Fundamental theorem of algebra)**.** *If $P : \mathbb{C} \to \mathbb{C}$ is a nonzero polynomial of degree k, then P has exactly k roots (zeros) in $\mathbb{C}$ counted with multiplicity.*

The set of rational functions is dense in the space of holomorphic functions, and we even have control over where the poles need to be. Note that a nonconstant polynomial has a "pole at infinity" meaning $P(z) \to \infty$ as $z \to \infty$. Letting $\mathbb{P}^1$ again be the Riemann sphere, we have Runge's approximation theorem.

Theorem B.30 (Runge)**.** *Suppose $U \subset \mathbb{C}$ is an open set and $A \subset \mathbb{P}^1 \setminus U$ is a set containing at least one point from each component of $\mathbb{P}^1 \setminus U$. Suppose $f : U \to \mathbb{C}$ is holomorphic. Then for any $\epsilon > 0$ and any compact $K \subset\subset U$, there exists a rational function R with poles in A such that*

$$\lvert R(z) - f(z) \rvert < \epsilon \qquad \text{for all } z \in K.$$

Perhaps a surprising generalization of the classical Weierstrass approximation theorem, and one of my favorite one-variable theorems, is Mergelyan's theorem. It may be good to note that Mergelyan does not follow from Runge.

Theorem B.31 (Mergelyan)**.** *Suppose $K \subset\subset \mathbb{C}$ is a compact set such that $\mathbb{C} \setminus K$ is connected and $f : K \to \mathbb{C}$ is a continuous function that is holomorphic in the interior K°. Then for any $\epsilon > 0$ and any compact $K \subset\subset U$, there exists a polynomial P such that*

$$|P(z) - f(z)| < \epsilon \qquad \text{for all } z \in K.$$

The reason why the theorem is perhaps surprising is that K may have only a small or no interior. Using a closed interval $K = [a, b]$ of the real line we recover the Weierstrass approximation theorem.

Given an open set $U \subset \mathbb{C}$, we say U is *symmetric with respect to the real axis* if $z \in U$ implies $\bar{z} \in U$. We divide U into three parts

$$U_+ = \{z \in U : \operatorname{Im} z > 0\}, \qquad U_0 = \{z \in U : \operatorname{Im} z = 0\}, \qquad U_- = \{z \in U : \operatorname{Im} z < 0\}.$$

We have the following theorem for extending (reflecting) holomorphic functions past boundaries.

Theorem B.32 (Schwarz Reflection Principle)**.** *Suppose $U \subset \mathbb{C}$ is a domain symmetric with respect to the real axis, $f : U_+ \cup U_0 \to \mathbb{C}$ a continuous function holomorphic on U_+ and real valued on U_0. Then the function $g : U \to \mathbb{C}$ defined by*

$$g(z) = f(z) \quad \text{if } z \in U_+ \cup U_0, \qquad g(z) = \overline{f(\bar{z})} \quad \text{if } z \in U_-,$$

is holomorphic on U.

In fact, the extension is really about harmonic functions.

Theorem B.33 (Schwarz reflection principle for harmonic functions)**.** *Suppose $U \subset \mathbb{C}$ is a domain symmetric with respect to the real axis, $f : U_+ \cup U_0 \to \mathbb{R}$ a continuous function harmonic on U_+ and zero on U_0. Then the function $g : U \to \mathbb{R}$ defined by*

$$g(z) = f(z) \quad \text{if } z \in U_+ \cup U_0, \qquad g(z) = -f(\bar{z}) \quad \text{if } z \in U_-,$$

is harmonic on U.

Functions may be defined locally, and continued along paths. Suppose p is a point and D is a disc with $p \in D$. A holomorphic function $f : D \to \mathbb{C}$ can be *analytically continued* along a path $\gamma : [0, 1] \to \mathbb{C}$ if for every $t \in [0, 1]$ there exists a disc D_t centered at $\gamma(t)$ and a holomorphic function $f_t : D_t \to \mathbb{C}$ and for each $t_0 \in [0, 1]$ there is an $\epsilon > 0$ such that if $|t - t_0| < \epsilon$, then $f_t = f_{t_0}$ in $D_t \cap D_{t_0}$. The monodromy theorem says that as long as there are no holes, analytic continuation defines a function uniquely.

Theorem B.34 (Monodromy theorem)**.** *If $U \subset \mathbb{C}$ is a simply connected domain, $D \subset U$ a disc and $f : D \to \mathbb{C}$ a holomorphic function that can be analytically continued from $p \in D$ to any $q \in U$, then there exists a unique holomorphic function $F : U \to \mathbb{C}$ such that $F|_D = f$.*

An interesting and useful theorem getting an inequality in the opposite direction from Schwarz's lemma, and one which is often not covered in a one-variable course is the Koebe $\frac{1}{4}$-theorem. Think of why no such theorem could possibly hold for just smooth functions. At first glance the theorem should seem quite counterintuitive, and at second glance, it should seem outright outrageous.

Theorem B.35 (Koebe quarter theorem)**.** *Suppose $f: \mathbb{D} \to \mathbb{C}$ is holomorphic and injective. Then $f(\mathbb{D})$ contains a disc centered at $f(0)$ and radius $\frac{|f'(0)|}{4}$.*

The $\frac{1}{4}$ is sharp, that is, it is the best it can be.

Finally, it is useful to factor out all the zeros of a holomorphic function, not just finitely many. Similarly we can work with poles.

Theorem B.36 (Weierstrass product theorem)**.** *Suppose $U \subset \mathbb{C}$ is a domain, $\{a_k\}$, $\{b_k\}$ are countable sets in U with no limit points in U, and $\{n_k\}$, $\{m_k\}$ any countable sets of natural numbers. Then there exists a meromorphic function f of U whose zeros are exactly at a_k, with orders given by n_k, and poles are exactly at b_k, with orders given by m_k.*

For a more explicit statement, we need infinite products. The product $\prod_{j=1}^{\infty}(1 + a_j)$ converges if the sequence partial products $\prod_{j=1}^{n}(1 + a_j)$ converges. We say that it the product converges absolutely if $\prod_{j=1}^{\infty}(1 + |a_j|)$ converges, which is equivalent to $\sum_{j=1}^{\infty}|a_j|$ converging.

Define

$$E_0(z) = (1 - z), \qquad E_m(z) = (1 - z) \exp\left(z + \frac{z^2}{2} + \cdots + \frac{z^m}{m}\right).$$

The function $E_m\left(z/a\right)$ has a zero of order 1 at a.

Theorem B.37 (Weierstrass factorization theorem)**.** *Let f be an entire function with zeros (with multiplicity) at points of the sequence $\{a_k\}$ except the zero at the origin, whose order is m (possibly $m = 0$). Then there exists an entire function g and a sequence p_k such that*

$$f(z) = z^m e^{g(z)} \prod_{k=1}^{\infty} E_{p_k}\left(\frac{z}{a_k}\right),$$

converges absolutely and uniformly on all compact subsets.

The p_k are chosen such that

$$\sum_{j=1}^{\infty} \left|\frac{r}{a_k}\right|^{1+p_k}$$

converges for all $r > 0$.

* * *

There are many other useful theorems in one complex variable, and we could spend a lot of time listing them all. However, hopefully the above listing is useful as a refresher for the reader of the most common results, some of which are used in this book, some of which are useful in the exercises, and some of which are just too interesting not to mention.

Appendix C

Differential forms and Stokes' theorem

Differential forms come up quite a bit in this book, especially in chapter 4 and chapter 5. Let us overview their definition and state the general Stokes' theorem. No proofs are given, this appendix is just a bare bones guide. For a more complete introduction to differential forms, see Rudin [R2].

The short story about differential forms is that a k-form is an object that can be integrated (summed) over a k-dimensional object, taking orientation into account. For simplicity, as in most of this book, everything in this appendix is stated for smooth (C^∞) objects to avoid worrying about how much regularity is needed.

The main point of differential forms is to find the proper context for the Fundamental Theorem of Calculus,

$$\int_a^b f'(x)\,dx = f(b) - f(a).$$

We interpret both sides as integration. The left hand side is an integral of the 1-form $f'\,dx$ over the 1-dimensional interval $[a,b]$ and the right hand side is an integral of the 0-form, that is a function, f over the 0-dimensional (two-point) set $\{a,b\}$. Both sides consider orientation, $[a,b]$ is integrated from a to b, $\{a\}$ is oriented negatively and $\{b\}$ is oriented positively. The two-point set $\{a,b\}$ is the boundary of $[a,b]$, and the orientation of $\{a,b\}$ is induced by $[a,b]$.

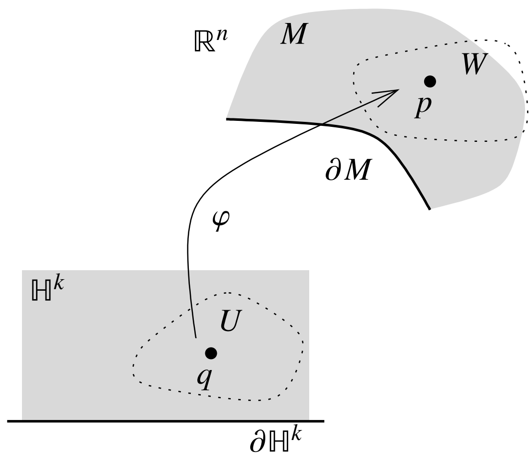
Let us define the objects over which we integrate, that is, smooth submanifolds of $\mathbb{R}^n$. Our model for a k-dimensional submanifold-with-boundary is the upper-half-space and its boundary:

$$\mathbb{H}^k \overset{\text{def}}{=} \{x \in \mathbb{R}^k : x_k \geq 0\}, \qquad \partial\mathbb{H}^k \overset{\text{def}}{=} \{x \in \mathbb{R}^k : x_k = 0\},$$

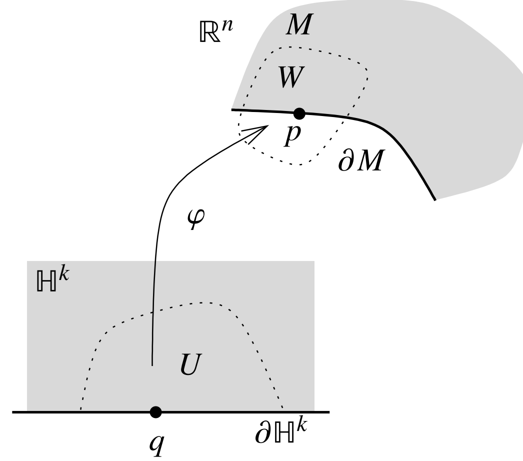
Definition C.1. Let $M \subset \mathbb{R}^n$ have the induced subspace topology. Let $k \in \mathbb{N}_0$. Let M have the property that for each $p \in M$, there exists a neighborhood $W \subset \mathbb{R}^n$ of p, a point $q \in \mathbb{H}^k$, a neighborhood $U \subset \mathbb{H}^k$ of q, and a smooth one-to-one open* mapping $\varphi\colon U \to M$ such that $\varphi(q) = p$, the derivative $D\varphi$ has rank k at all points, and $\varphi(U) = M \cap W$. Then M is an *embedded submanifold-with-boundary* of dimension k. The map φ is called a *local parametrization*. If q is such that $q_k = 0$ (the last component is zero), then $p = \varphi(q)$ is a *boundary point*. Let ∂M denote the set of boundary points. If $\partial M = \emptyset$, then we say M is simply an *embedded submanifold*.

The situation for a boundary point and an interior point is depicted in the following diagram. Note that W is a bigger neighborhood in $\mathbb{R}^n$ than the image $\varphi(U)$.

*By open, we mean that $\varphi(V)$ is a relatively open set of M for every open set $V \subset U$.



Interior point



Boundary point

Completely correctly, we should say *submanifold of* $\mathbb{R}^k$. Sometimes people (including me) say *manifold* when they mean *submanifold*. A manifold is a more abstract concept, but all submanifolds are manifolds. The word *embedded* has to do with the topology on M, and this has to do with the condition $\varphi(U) = M \cap W$ and φ being open. The condition means that φ is a homeomorphism onto $M \cap W$. It is important that W is an open set in $\mathbb{R}^n$. For our purposes here, all submanifolds will be embedded. We have also made some economy in the definition. If q is not on the boundary of $\mathbb{H}^k$, then we might as well have used $\mathbb{R}^k$ instead of $\mathbb{H}^k$. A submanifold is something that is locally like $\mathbb{R}^k$, and if it has a boundary, then near the boundary it is locally like $\mathbb{H}^k$ near a point of $\partial\mathbb{H}^k$.

We also remark that submanifolds are often defined in reverse rather than by parametrizations, that is, by starting with the (relatively) open sets $M \cap W$, and the maps φ^{-1}, calling those *charts*, and calling the entire set of charts an *atlas*. The particular version of the definition we have chosen makes it easy to evaluate integrals in the same way that parametrizing curves makes it easy to evaluate integrals.

Examples of such submanifolds are domains with smooth boundaries as in Definition 2.2.1, we can take the inclusion map $x \mapsto x$ as our parametrization. The domain is then the submanifold M and ∂M is the boundary of the domain. Domains are the key application for our purposes. Another example are smooth curves.

If M is an embedded submanifold-with-boundary of dimension k, then ∂M is also an embedded submanifold of dimension $k-1$. Simply restrict the parametrizations to the boundary of $\mathbb{H}^k$.

We also need to define an orientation.

Definition C.2. Let $M \subset \mathbb{R}^n$ be an embedded submanifold-with-boundary of dimension $k \geq 2$. Suppose a set of parametrizations can be chosen such that each point of M is in the image of one of the parametrizations, and if $\varphi\colon U \to M$ and $\widetilde{\varphi}\colon \widetilde{U} \to M$ are two parametrizations such that $\varphi(U) \cap \widetilde{\varphi}(\widetilde{U}) \neq \emptyset$, then the *transition map* (automatically smooth) defined by

$$\widetilde{\varphi}^{-1} \circ \varphi$$

on $\varphi^{-1}\big(\varphi(U) \cap \widetilde{\varphi}(\widetilde{U})\big)$ (in other words, wherever it makes sense) is orientation preserving, that is

$$\det D\big(\widetilde{\varphi}^{-1} \circ \varphi\big) > 0$$

at all points. The set of such parametrizations is the *orientation* on M, and we usually take the maximal set of such parametrizations.

If M is oriented, then the restrictions of the parametrizations to $\partial \mathbb{H}^k$ give an orientation on ∂M. We say this is the *induced orientation* on ∂M.

For dimensions $k = 0$ (isolated points) and $k = 1$ (curves) we must define orientation differently. For $k = 0$, we simply give a each point an orientation of $+$ or $-$. For $k = 1$, we need to allow parametrization by open subsets not only of $\mathbb{H}^1 = [0, \infty)$, but also $-\mathbb{H}^1 = (-\infty, 0]$. The definition is the same otherwise. To define the orientation of the boundary, if the boundary point corresponds to the 0 in $[0, \infty)$ we give this boundary point the orientation $-$, and if it corresponds to the 0 in $(-\infty, 0]$, then we give this point the orientation $+$. The reason for this complication is that unlike in $\mathbb{R}^k$ for $k \geq 2$, the set $\mathbb{H}^1 = [0, \infty)$ cannot be "rotated" (in $\mathbb{R}^1$) or mapped via an orientation preserving map onto $-\mathbb{H}^1 = (-\infty, 0]$, but in $\mathbb{R}^2$ the upper-half-plane $\mathbb{H}^2$ can be rotated to the lower-half-plane $-\mathbb{H}^2 = \{x \in \mathbb{R}^2 : x_2 \leq 0\}$. For computations, it is often useful for compact curves with endpoints (boundary) to just give one parametrization from $[0, 1]$ or perhaps $[a, b]$, then a corresponds to the $-$ and b corresponds to the $+$.

The fact that the transition map is smooth does require a proof, which is a good exercise in basic analysis. It requires a bit of care at boundary points.

An orientation allows us to have a well-defined integral on M, just like a curve needs to be oriented in order to define a line integral. However, unlike for curves, not every submanifold of dimension more than one is *orientable*, that is, admits an orientation. A classical non-orientable example is the Möbius strip.

Now that we know "on" what we integrate, let us figure out what "it" is that we integrate. Let us start with 0-forms. We define 0-forms as smooth functions (possibly complex-valued). Sometimes we need a function defined just on a submanifold. A function f defined on a submanifold M is smooth when $f \circ \varphi$ is smooth on U for every parametrization $\varphi \colon U \to M$. Equivalently, one can prove that f is the restriction of some smooth function defined on some neighborhood of M in $\mathbb{R}^n$.

A 0-form ω defined on a 0-dimensional oriented submanifold M is integrated as

$$\int_M \omega \overset{\text{def}}{=} \sum_{p \in M} \epsilon_p \omega(p),$$

where ϵ_p is the orientation of p given as $+1$ or -1. To avoid problems of integrability, one can assume that ω is compactly supported (it is nonzero on at most finitely many points of M) or that M is compact (it is a finite set).

The correct definition of a 1-form is that it is a "smooth section" of the dual of the vector bundle $T\mathbb{R}^n$. That is, it is something that eats a vector field, and spits out a function. The 1-form dx_k is supposed to be the object that does

$$dx_k \left(\frac{\partial}{\partial x_k} \right) = 1, \qquad dx_k \left(\frac{\partial}{\partial x_j} \right) = 0 \quad \text{if } j \neq k.$$

For our purposes here, just suppose that a 1-form in $\mathbb{R}^n$ is an object of the form

$$\omega = g_1 dx_1 + g_2 dx_2 + \cdots + g_n dx_n,$$

where $g_1, g_2, \ldots, g_n$ are smooth functions. That is, a 1-form is at each point a linear combination of $dx_1, dx_2, \ldots, dx_n$ that varies smoothly from point to point. Suppose M is a one-dimensional submanifold (possibly with boundary), $\varphi \colon U \to M$ is a parametrization compatible with the orientation of M, and g_j is supported in $\varphi(U)$. Define

$$\int_M \omega \overset{\text{def}}{=} \sum_{j=1}^{n} \int_U g_j\big(\varphi(t)\big)\varphi_j'(t)\,dt,$$

where the integral $\int_U \cdots\, dt$ is evaluated with the usual positive orientation (left to right) as $U \subset \mathbb{R}$, and φ_j is the jth component of φ.

Generally, a 1-form has support bigger than just $\varphi(U)$. In this case, one needs to use a so-called partition of unity to write ω as a locally finite sum

$$\omega = \sum_{\ell} \omega_\ell,$$

where each ω_ℓ has support in the image of a single parametrization. By locally finite, we mean that on each compact neighborhood only finitely many ω_ℓ are nonzero. Define

$$\int_M \omega \overset{\text{def}}{=} \sum_{\ell} \int_M \omega_\ell.$$

The definition makes sense only if this sum actually exists. For example, if ω is compactly supported then this sum is only finite, and so it exists.

Higher degree forms are constructed out of 1-forms and 0-forms by the so-called wedge product. Given a k-form ω and an ℓ-form η,

$$\omega \wedge \eta$$

is a $(k+\ell)$-form. We require the wedge product to be bilinear at each point: if f and g are smooth functions, then

$$(f\omega + g\eta) \wedge \xi = f(\omega \wedge \xi) + g(\eta \wedge \xi), \qquad \text{and} \qquad \omega \wedge (f\eta + g\xi) = f(\omega \wedge \eta) + g(\omega \wedge \xi).$$

The wedge product is not commutative; we require it to be anticommutative on 1-forms. If ω and η are 1-forms, then

$$\omega \wedge \eta = -\eta \wedge \omega.$$

The negative keeps track of orientation for us. When ω is a k-form and η is an ℓ-form then we get

$$\omega \wedge \eta = (-1)^{k\ell} \eta \wedge \omega.$$

We wedge together the basis 1-forms to get all k-forms. A k-form is then an expression

$$\omega = \sum_{j_1=1}^{n} \sum_{j_2=1}^{n} \cdots \sum_{j_k=1}^{n} g_{j_1,\ldots,j_k}\, dx_{j_1} \wedge dx_{j_2} \wedge \cdots \wedge dx_{j_k},$$

where $g_{j_1,\ldots,j_k}$ are smooth functions. But we can simplify even more. Since the wedge is anticommutative on 1-forms,

$$dx_j \wedge dx_m = -dx_m \wedge dx_j, \qquad \text{and} \qquad dx_j \wedge dx_j = 0.$$

In other words, every form $dx_{j_1} \wedge dx_{j_2} \wedge \cdots \wedge dx_{j_k}$ is either zero, if any two indices from $j_1, \ldots, j_k$ are equal, or can be put into the form $\pm dx_{j_1} \wedge dx_{j_2} \wedge \cdots \wedge dx_{j_k}$, where $j_1 < j_2 < \cdots < j_k$. Thus, a k-form can always be written as

$$\omega = \sum_{1 \leq j_1 < j_2 < \cdots < j_k \leq n} g_{j_1,\ldots,j_k} \, dx_{j_1} \wedge dx_{j_2} \wedge \cdots \wedge dx_{j_k}.$$

Consider an oriented k-dimensional submanifold M (possibly with boundary), a parametrization $\varphi \colon U \to M$ from the orientation, and a k-form ω supported in $\varphi(U)$ (that is each $g_{j_1,\ldots,j_k}$ is supported in $\varphi(U)$). Denote by $t \in U \subset \mathbb{R}^k$ the coordinates on U. Define

$$\int_M \omega \overset{\text{def}}{=} \sum_{1 \leq j_1 < j_2 < \cdots < j_k \leq n} \int_U g_{j_1,\ldots,j_k}(t) \det D(\varphi_{j_1}, \varphi_{j_2}, \ldots, \varphi_{j_k}) \, dt$$

where the integral $\int_U \cdots \, dt$ is evaluated in the usual orientation on $\mathbb{R}^k$ with dt the standard measure on $\mathbb{R}^k$ (think $dt = dt_1 dt_2 \cdots dt_n$), and $D(\varphi_{j_1}, \varphi_{j_2}, \ldots, \varphi_{j_k})$ denotes the derivative of the mapping whose ℓth component is φ_{j_ℓ}.

Similarly as before, if ω is not supported in the image of a single parametrization, write

$$\omega = \sum_{\ell} \omega_\ell$$

as a locally finite sum, where each ω_ℓ has support in the image of a single parametrization of the orientation. Then

$$\int_M \omega \overset{\text{def}}{=} \sum_{\ell} \int_M \omega_\ell.$$

Again, the sum has to exist, such as when ω is compactly supported and the sum is finite.

Notice that the only nontrivial differential forms on $\mathbb{R}^n$ are $0, 1, 2, \ldots, n$ forms. The only n-forms are object of the form

$$f(x) \, dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n.$$

The form $dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n$ is called the volume form. Integrating it over a domain (an n-dimensional submanifold) gives the standard volume integral.

More generally one defines integration of k-forms over k-chains, which are just linear combinations of smooth submanifolds, but we do not need that level of generality.

In computations, we can avoid sets of zero measure (k-dimensional), so we can ignore the boundary of the submanifold. Similarly if we parametrize several subsets we can leave out a measure zero subset. Let us give a couple of examples of computations.

Example C.3: Consider the circle $S^1 \subset \mathbb{R}^2$. We use a parametrization $\varphi \colon (-\pi, \pi) \to S^1$ where $\varphi(t) = \big(\cos(t), \sin(t)\big)$, so the circle is oriented counterclockwise. Let $\omega(x_1, x_2) = P(x_1, x_2) \, dx_1 + Q(x_1, x_2) \, dx_2$, then

$$\int_{S^1} \omega = \int_{-\pi}^{\pi} \Big(P\big(\cos(t), \sin(t)\big) \big(-\sin(t)\big) + Q\big(\cos(t), \sin(t)\big) \cos(t) \Big) \, dt.$$

We can ignore the point $(-1, 0)$ as a single point is of 1-dimensional measure zero.

Example C.4: Consider a domain $U \subset \mathbb{R}^n$, then U is an oriented submanifold. We use the parametrization $\varphi\colon U \to U$, where $\varphi(x) = x$. Then

$$\int_U f(x)\, dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n = \int_U f(x)\, dx_1\, dx_2 \,\cdots\, dx_n = \int_U f(x)\, dV,$$

where dV is the standard volume measure.

Example C.5: Finally consider M the upper hemisphere of the unit sphere $S^2 \subset \mathbb{R}^3$ as a submanifold with boundary. That is consider

$$M = \left\{ x \in \mathbb{R}^3 : x_1^2 + x_2^2 + x_3^2 = 1, x_3 \geq 0 \right\}.$$

The boundary is the circle in the (x_1, x_2)-plane:

$$\partial M = \left\{ x \in \mathbb{R}^3 : x_1^2 + x_2^2 = 1, x_3 = 0 \right\}.$$

Consider the parametrization of M using the spherical coordinates

$$\varphi(\theta, \psi) = \bigl(\cos(\theta)\sin(\psi), \sin(\theta)\sin(\psi), \cos(\psi)\bigr)$$

for U given by $-\pi < \theta < \pi$, $0 < \psi \leq \pi/2$. After a rotation this is a subset of a half-plane with the points corresponding to $\psi = \pi/2$ corresponding to boundary points. We miss the points where $\theta = \pi$, including the point $(0,0,1)$, but the set of those points is a 1-dimensional curve, and so a set of 2-dimensional measure zero. For the purposes of integration we can ignore it. Let

$$\omega(x_1, x_2, x_3) = P(x_1, x_2, x_3)\, dx_1 \wedge dx_2 + Q(x_1, x_2, x_3)\, dx_1 \wedge dx_3 + R(x_1, x_2, x_3)\, dx_2 \wedge dx_3 .$$

Then

$$\begin{aligned}
\int_M \omega = \int_{-\pi}^{\pi} \int_0^{\pi/2} \Biggl[& P\bigl(\varphi(\theta,\psi)\bigr) \left(\frac{\partial \varphi_1}{\partial \theta} \frac{\partial \varphi_2}{\partial \psi} - \frac{\partial \varphi_2}{\partial \theta} \frac{\partial \varphi_1}{\partial \psi} \right) + Q\bigl(\varphi(\theta,\psi)\bigr) \left(\frac{\partial \varphi_1}{\partial \theta} \frac{\partial \varphi_3}{\partial \psi} - \frac{\partial \varphi_3}{\partial \theta} \frac{\partial \varphi_1}{\partial \psi} \right) \\
& + R\bigl(\varphi(\theta,\psi)\bigr) \left(\frac{\partial \varphi_2}{\partial \theta} \frac{\partial \varphi_3}{\partial \psi} - \frac{\partial \varphi_3}{\partial \theta} \frac{\partial \varphi_2}{\partial \psi} \right) \Biggr]\, d\theta\, d\psi \\
= \int_{-\pi}^{\pi} \int_0^{\pi/2} \Bigl[& P\bigl(\cos(\theta)\sin(\psi), \sin(\theta)\sin(\psi), \cos(\psi)\bigr) \bigl(-\cos(\psi)\sin(\psi) \bigr) \\
& + Q\bigl(\cos(\theta)\sin(\psi), \sin(\theta)\sin(\psi), \cos(\psi)\bigr) \sin(\theta)\sin^2(\psi) \\
& + R\bigl(\cos(\theta)\sin(\psi), \sin(\theta)\sin(\psi), \cos(\psi)\bigr) \bigl(-\cos(\theta)\sin^2(\psi) \bigr) \Bigr]\, d\theta\, d\psi .
\end{aligned}$$

The induced orientation on the boundary ∂M is the counterclockwise orientation used in Example C.3, because that is the parametrization we get when we restrict to the boundary, $\varphi(\theta, \pi/2) = \bigl(\cos(\theta), \sin(\theta), 0\bigr)$.

The derivative on k-forms is the *exterior derivative*, which is a linear operator that eats k-forms and spits out $(k+1)$-forms. For a k-form

$$\omega = g_{j_1,\ldots,j_k}\, dx_{j_1} \wedge dx_{j_2} \wedge \cdots \wedge dx_{j_k},$$

define the exterior derivative $d\omega$ as

$$d\omega \overset{\text{def}}{=} dg_{j_1,\ldots,j_k} \wedge dx_{j_1} \wedge dx_{j_2} \wedge \cdots \wedge dx_{j_k} = \sum_{\ell=1}^{n} \frac{\partial g_{j_1,\ldots,j_k}}{\partial x_\ell} dx_\ell \wedge dx_{j_1} \wedge dx_{j_2} \wedge \cdots \wedge dx_{j_k}.$$

Then define d on every k-form by extending it linearly.

For example,

$$\begin{aligned} d\,(P\,dx_2 \wedge dx_3 + Q\,dx_3 \wedge dx_1 + R\,dx_1 \wedge dx_2) \\ = \frac{\partial P}{\partial x_1} dx_1 \wedge dx_2 \wedge dx_3 + \frac{\partial Q}{\partial x_2} dx_2 \wedge dx_3 \wedge dx_1 + \frac{\partial R}{\partial x_3} dx_3 \wedge dx_1 \wedge dx_2 \\ = \left(\frac{\partial P}{\partial x_1} + \frac{\partial Q}{\partial x_2} + \frac{\partial R}{\partial x_3} \right) dx_1 \wedge dx_2 \wedge dx_3. \end{aligned}$$

You should recognize the divergence of the vector field (P,Q,R) from vector calculus. All the various derivative operations in $\mathbb{R}^3$ from vector calculus make an appearance. If ω is a 0-form in $\mathbb{R}^3$, then $d\omega$ is like the gradient. If ω is a 1-form in $\mathbb{R}^3$, then $d\omega$ is like the curl. If ω is a 2-form in $\mathbb{R}^3$, then $d\omega$ is like the divergence.

Something to notice is that

$$d(d\omega) = 0$$

for every ω, which follows because partial derivatives commute. In particular, we get a so-called complex: If $\Lambda^k(M)$ denotes the k-forms on an n-dimensional submanifold M, then we get the complex

$$\Lambda^0(M) \xrightarrow{d} \Lambda^1(M) \xrightarrow{d} \Lambda^2(M) \xrightarrow{d} \cdots \xrightarrow{d} \Lambda^n(M) \xrightarrow{d} 0.$$

We remark that one can study the topology of M by computing from this complex the cohomology groups, $\frac{\ker(d\colon \Lambda^k \to \Lambda^{k+1})}{\operatorname{im}(d\colon \Lambda^{k-1} \to \Lambda^k)}$, which is really about global solvability of the differential equation $d\omega = \eta$ for an unknown ω. There are variations on this idea and one appears in chapter 4, but we digress.

Let us now state *Stokes' theorem*, sometimes called the *generalized Stokes' theorem* to distinguish it from the classical Stokes' theorem you know from vector calculus, which is a special case.

Theorem C.6 (Stokes)**.** *Suppose $M \subset \mathbb{R}^n$ is an embedded compact smooth oriented $(k+1)$-dimensional submanifold-with-boundary, ∂M has the induced orientation, and ω is a smooth k-form defined on M. Then*

$$\int_{\partial M} \omega = \int_M d\omega.$$

One can get away with less regularity, both on ω and M (and ∂M) including "corners." In $\mathbb{R}^2$, it is easy to state in more generality, see Theorem B.2.

A final note is that the classical Stokes' theorem is just the generalized Stokes' theorem with $n = 3$, $k = 2$. Classically instead of using differential forms, the line integral is an integral of a vector field instead of a 1-form ω, and its derivative $d\omega$ is the curl operator.

As to at least get a flavor of the theorem, let us prove it in a simpler setting, which however is often almost good enough, and it is the key idea in the proof. Suppose that $U \subset \mathbb{R}^n$ is a domain such that for any $k = 1, \ldots, n$ there exist two smooth functions α_k and β_k such that for any k, the set U is given by

$$
\begin{aligned}
&(x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_n) \in \pi_k(U), \\
&\alpha_k(x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_n) \leq x_k \leq \beta_k(x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_n),
\end{aligned}
$$

where $\pi_k(U)$ is the projection of U onto the $(x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_n)$ components. Orient ∂U as usual.

Write $x' = (x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_n)$, and let dV_{n-1} be the volume form for $\mathbb{R}^{n-1}$. Consider the $(n-1)$-form

$$\omega = f \, dx_1 \wedge \cdots \wedge dx_{k-1} \wedge dx_{k+1} \wedge \cdots \wedge dx_n.$$

Then $d\omega = \frac{\partial f}{\partial x_k} dx_1 \wedge \cdots \wedge dx_n$. By the fundamental theorem of calculus,

$$
\begin{aligned}
\int_U d\omega &= \int_U \frac{\partial f}{\partial x_k} \, dV_n \\
&= \int_{\pi_k(U)} \int_{\alpha_k(x')}^{\beta_k(x')} \frac{\partial f}{\partial x_k} \, dx_k \, dV_{n-1} \\
&= \int_{\pi_k(U)} f(x_1, \ldots, x_{k-1}, \beta_k(x'), x_{k+1}, \ldots, x_n) \, dV_{n-1} \\
&\qquad - \int_{\pi_k(U)} f(x_1, \ldots, x_{k-1}, \alpha_k(x'), x_{k+1}, \ldots, x_n) \, dV_{n-1} \\
&= \int_{\partial U} \omega.
\end{aligned}
$$

Any $(n-1)$-form can be written as a sum of forms like ω for various k. Integrating each one of them in the correct direction provides the result.

Appendix D

Basic terminology and results from algebra

Let us quickly go over some of the basic definitions from commutative algebra for reference, and a result or two that we will need in chapter 6. See a book such as Zariski–Samuel [ZS] for a full reference.

Definition D.1. A set G is called a *group* if it has an operation $x * y$ defined on it and it satisfies the following axioms:

(G1) If $x \in G$ and $y \in G$, then $x * y \in G$.

(G2) *(associativity)* $(x * y) * z = x * (y * z)$ for all $x, y, z \in G$.

(G3) *(identity)* There exists an element $1 \in G$ such that $1 * x = x$ for all $x \in G$.

(G4) *(inverse)* For every element $x \in G$ there exists an element $x^{-1} \in G$ such that $x * x^{-1} = 0$.

A group G is called *abelian* if it also satisfies:

(G5) *(commutativity)* $x * y = y * x$ for all $x, y \in G$.

A subset $K \subset G$ is called a *subgroup* if K is a group with the same operation as the group G.
If G and H are groups, a function $f : G \to H$ is a *group homomorphism* if it respects the group law, that is, $f(a * b) = f(a) * f(b)$. If f is bijective, then it is called a *group isomorphism*.

An example of a group is a group of automorphisms. For example, let $U \subset \mathbb{C}$ be an open set, and suppose that G is the set of bijective holomorphic functions $f : U \to U$. Then G is a group under composition, but G is not necessarily abelian. For example, if $U = \mathbb{C}$, then $f(z) = z + 1$ and $g(z) = -z$ are members of G, but $f \circ g(z) = -z + 1$ but $g \circ f(z) = -z - 1$.

Definition D.2. A set R is called a *commutative ring* if it has two operations defined on it, addition $x + y$ and multiplication xy, and if it satisfies the following axioms:

(A1) If $x \in R$ and $y \in R$, then $x + y \in R$.

(A2) *(commutativity of addition)* $x + y = y + x$ for all $x, y \in R$.

(A3) *(associativity of addition)* $(x + y) + z = x + (y + z)$ for all $x, y, z \in R$.

(A4) There exists an element $0 \in R$ such that $0 + x = x$ for all $x \in R$.

(A5) For every element $x \in R$ there exists an element $-x \in R$ such that $x + (-x) = 0$.

(M1) If $x \in R$ and $y \in R$, then $xy \in R$.

(M2) *(commutativity of multiplication)* $xy = yx$ for all $x, y \in R$.

(M3) *(associativity of multiplication)* $(xy)z = x(yz)$ for all $x, y, z \in R$.

(M4) There exists an element $1 \in R$ (and $1 \neq 0$) such that $1x = x$ for all $x \in R$.

(D) *(distributive law)* $x(y + z) = xy + xz$ for all $x, y, z \in R$.

The ring R is called a *field* if furthermore:

(F) For every $x \in R$ such that $x \neq 0$ there exists an element $1/x \in R$ such that $x(1/x) = 1$.

In a commutative ring R, the elements $u \in R$ for which there exists an inverse $1/u$ as above are called *units*.

If R and S are rings, a function $f : R \to S$ is a *ring homomorphism* if it respects the ring operations, that is, $f(a + b) = f(a) + f(b)$ and $f(ab) = f(a)f(b)$, and such that $f(1) = 1$. If f is bijective, then it is called a *ring isomorphism*.

That is a commutative ring is an abelian additive group (by additive group we just mean we use $+$ for the operation and 0 for the respective identity), with multiplication thrown in. If the multiplication also defines a group on the set of nonzero elements then the ring is a field. A ring that is not commutative, is one that doesn't satisfy commutativity of multiplication. Some authors also define ring without asking for an existence of 1.

A ring that often comes up in this book is the ring of holomorphic functions. Let $\mathcal{O}(U)$ be the set of holomorphic functions defined on an open set U. Then pointwise addition and multiplication give a ring structure on $\mathcal{O}(U)$. The set of units are the set of functions that never vanish in U. The set of units is a multiplicative group.

Given a commutative ring R, let $R[x]$ be the set of polynomials

$$P(x) = c_k x^k + c_{k-1} x^{k-1} + \cdots + c_1 x + c_0,$$

where $c_0, \ldots, c_k \in R$. The integer k is called the *degree* of the polynomial and c_k is called the *leading coefficient* of $P(x)$. If the leading coefficient is 1, then P is said to be *monic*. If R is a commutative ring, then so is $R[x]$. Similarly, we define the commutative ring $R[x_1, \ldots, x_n]$ of polynomials in n indeterminates.

The most basic result about polynomials, Theorem B.29 the fundamental theorem of algebra, which states that every nonconstant polynomial over $R = \mathbb{C}$ has a root, is really a theorem in one complex variable.

Definition D.3. Let R be a commutative ring. A subset $I \subset R$ is an *ideal* if $f \in R$ and $g, h \in I$ implies that $fg \in I$ and $g + h \in I$. In short, $I \subset R$ is an additive subgroup such that $RI = I$.

Given a set of elements $S \subset R$, the intersection I of all ideals containing S is called the *ideal generated by* S. If $S = \{f_1, \ldots, f_k\}$ is a finite set, we say I is *finitely generated*, and we write $I = (f_1, \ldots, f_k)$.

A *principal ideal* is an ideal generated by a single element. A commutative ring where every ideal is a principal ideal is called a *principal ideal domain* or a PID.

A commutative ring R is *Noetherian* if every ideal in R is finitely generated.

It is not difficult to prove that "an ideal generated by S" really is an ideal, that is, intersection of ideals is an ideal. If an ideal I is generated by $f_1, \ldots, f_k$, then every $g \in I$ can be written as

$$g = c_1 f_1 + \cdots + c_k f_k,$$

for some $c_1, \ldots, c_k \in R$. Clearly the set of such elements is the smallest ideal containing $f_1, \ldots, f_k$.

Theorem D.4 (Hilbert basis theorem)**.** *If R is a Noetherian commutative ring, then $R[x]$ is Noetherian.*

As the proof is rather short, we include it here.

Proof. Suppose R is Noetherian, and $I \subset R[x]$ is an ideal. Starting with the polynomial f_1 of minimal degree in I, construct a (possibly finite) sequence of polynomials $f_1, f_2, \ldots$ such that f_k is the polynomial of minimal degree from the set $I \setminus (f_1, \ldots, f_{k-1})$. The sequence of degrees $\deg(f_1), \deg(f_2), \ldots$ is by construction nondecreasing. Let c_k be the leading coefficient of f_k.

As R is Noetherian, then there exists a finite k such that $(c_1, c_2, \ldots, c_m) \subset (c_1, c_2, \ldots, c_k)$ for all m. Suppose for contradiction there exists a f_{k+1}, that is, the sequence of polynomials did not end at k. In particular $(c_1, \ldots, c_{k+1}) \subset (c_1, \ldots, c_k)$ or

$$c_{k+1} = a_1 c_1 + \cdots a_k c_k.$$

As degree of f_{k+1} is at least the degree of f_1, through f_k, we can define the polynomial

$$g = a_1 x^{\deg(f_{k+1}) - \deg(f_1)} f_1 + a_2 x^{\deg(f_{k+1}) - \deg(f_2)} f_2 + \cdots + a_k x^{\deg(f_{k+1}) - \deg(f_k)} f_k.$$

The polynomial g has the same degree as f_{k+1}, and in fact it also has the same leading term, c_{k+1}. On the other hand $g \in (f_1, \ldots, f_k)$ while $f_{k+1} \notin (f_1, \ldots, f_k)$ by construction. The polynomial $g - f_{k+1}$ is also not in $(f_1, \ldots, f_k)$, but as the leading terms canceled, $\deg(g - f_{k+1}) < \deg(f_{k+1})$, but that is a contradiction, so f_{k+1} does not exist and $I = (f_1, \ldots, f_k)$. □

Definition D.5. An element $f \in R$ is said to be *irreducible* if whenever $f = gh$ for two elements $g, h \in R$, then either g or h is a unit. A commutative ring R is a *unique factorization domain* (UFD) if up to multiplication by a unit, every element has a unique factorization into irreducible elements of R.

One version of a result called the Gauss lemma says that just like the property of being Noetherian, the property of being a UFD is retained when we take polynomials.

Theorem D.6 (Gauss lemma)**.** *If R is a commutative ring that is a UFD, then $R[x]$ is a UFD.*

The proof is not difficult, but it is perhaps beyond the scope of this book.

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List of Notation

Notation	Description	Page
z^α	$z_1^{\alpha_1} z_2^{\alpha_2} \cdots z_n^{\alpha_n}$	16
dz	$dz_1 \wedge dz_2 \wedge \cdots \wedge dz_n$	16
$\lvert\alpha\rvert$	$\alpha_1 + \alpha_2 + \cdots + \alpha_n$	16
$\alpha!$	$\alpha_1!\alpha_2!\cdots\alpha_n!$	16
$\frac{\partial^{\lvert\alpha\rvert}}{\partial z^\alpha}$	$\frac{\partial^{\alpha_1}}{\partial z_1^{\alpha_1}} \frac{\partial^{\alpha_2}}{\partial z_2^{\alpha_2}} \cdots \frac{\partial^{\alpha_n}}{\partial z_n^{\alpha_n}}$	16
$\sum_\alpha c_\alpha z^\alpha$	multiindex power series	16
$\mathcal{O}(U)$	the set/ring of holomorphic functions on U	23
$f \circ g$	composition, $p \mapsto f\big(g(p)\big)$	25, 151
Df, $Df(a)$	Jacobian matrix / derivative (at a)	26, 47, 52
$D_{\mathbb{R}} f$, $D_{\mathbb{R}} f(a)$	real Jacobian matrix / real derivative (at a)	26, 47
f^{-1}	function inverse	28
$\subset\subset$	compact or relatively compact subset	28
$f^{-1}(S)$	pullback of a set, $\{q : f(q) \in S\}$	28
S^{2n-1}	unit sphere in $\mathbb{C}^n$, $S^{2n-1} = \partial\mathbb{B}_n$	29
$T_p\mathbb{R}^n$, T_pM	tangent space	45
$\frac{\partial}{\partial x_j}\Big\rvert_p$	tangent vector	45
$T\mathbb{R}^n$, TM	tangent bundle	46, 47
∇r	gradient of r	48
$O(\ell)$	big-oh notation	49
$\mathbb{C} \otimes T_p\mathbb{R}^n$, $\mathbb{C} \otimes T_pM$	complexified tangent space	51
$T_p^{(1,0)}\mathbb{R}^n$, $T_p^{(1,0)}M$	holomorphic tangent vectors	51
$T_p^{(0,1)}\mathbb{R}^n$, $T_p^{(0,1)}M$	antiholomorphic tangent vectors	51
$D_{\mathbb{C}} f$, $D_{\mathbb{C}} f(a)$	complexified real derivative	51
$\mathcal{L}(X_p, X_p)$	Levi form	54
v^*, A^*	conjugate transpose	55
$P_r(\theta)$	Poisson kernel for the unit disc	65
dV, $dV(w)$	volume measure, euclidean volume form	70, 116
$f * g$	convolution	70
$\widehat{K}$, $\widehat{K}_{\mathcal{F}}$	hull of K with respect to $\mathcal{F}$	73

Notation	**Description**	**Page**
$\operatorname{dist}(x,y)$	distance between points, point and a set, or two sets	78
$\widehat{K}_U$	holomorphic hull of K	80
C^ω	real-analytic	86
$[v]^2$	$v_1^2 + \cdots + v_n^2$	99
dA, $dA(w)$	area form, $dA = dx \wedge dy$	109
$d\psi$	exterior derivative	111, 168
$\partial\psi$	holomorphic part of exterior derivative	111, 117
$\bar{\partial}\psi$	antiholomorphic part of exterior derivative	111, 117
$\widehat{dx_j}$	removed one-form, $dx_1 \wedge \widehat{dx_2} \wedge dx_3 = dx_1 \wedge dx_2$	116
$A^2(U)$	Bergman space, square integrable holomorphic functions	119
$L^2(U)$	square integrable functions	119
$\lVert f\rVert_{A^2(U)}$, $\lVert f\rVert_{L^2(U)}$	L^2 norm	119
$\langle f,g\rangle$	L^2 inner product	119
$K_U(z,\bar{\zeta})$	Bergman kernel of U	120
$H^2(\partial U)$	Hardy space	123
$S_U(z,\bar{\zeta})$	Szegö kernel of U	124
(f,p)	germ of f at p	125
${}_n\mathcal{O}_p$, $\mathcal{O}_p$	ring of germs of holomorphic functions at p	125
(A,p)	germ of a set A at p	126
Z_f	zero set of f, $f^{-1}(0)$	126, 137
$\operatorname{ord}_a f$	order of vanishing of f at a	127
$\mathcal{O}(U)[z_n]$	polynomial in z_n with coefficients in $\mathcal{O}(U)$	127
$\mathcal{O}_0[z_n]$	polynomial in z_n with coefficients in $\mathcal{O}_0$	127
(f), $(f_1,\ldots,f_n)$	ideal generated by f or by $f_1,\ldots,f_n$	135, 172
$I_p(X)$	ideal of germs at p vanishing on X	137
$V(I)$	the common zero set of germs in I	137
Γ_f	graph of f	138
X_{reg}	regular points of X	139
X_{sing}	singular points of X	139
$\dim_p X$	dimension of X at p	140
$\dim X$	dimension of X	140