Several Complex Variables

Oklahoma State University, Math 6283

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Contents

Introduction

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These notes are not meant as an exhaustive reference. They are simply a whirlwind tour of several complex variables. To find the list of the books useful for reference and further reading, see the end of the notes. Note that the sections with a star are not necessary for further reading and can be skipped at first.

Do let me know if you find any mistakes, typos, or if you have suggestions.

0.1 Motivation, single variable, and Cauchy's formula

Complex analysis is the study of holomorphic (or complex analytic) functions. So we ought to start with what these are. We will assume certain standard notation such as $\mathbb{C}$ for the complex numbers, $\mathbb{R}$ for real numbers, $\mathbb{Z}$ for integers, $\mathbb{N} = \{1,2,3,\ldots\}$ for natural numbers, $i = \sqrt{-1}$, etc... Throughout this book, we will use the standard terminology of *domain* to mean connected open set.

We will also assume the reader has basic working knowledge of real analysis and of complex analysis in one variable.

There is an awful lot you can do with polynomials, but sometimes they are just not enough. For example, there is no polynomial function that solves the simplest of differential equations $f' = f$. We need the exponential function, which is holomorphic. Holomorphic functions are a generalization of polynomials, and to get there one leaves the land of algebra to arrive in the realm of analysis.

Let us start with polynomials. In one variable a polynomial in z is an expression of the form

$$P(z) = \sum_{j=0}^{d} c_j z^j$$

where $c_j \in \mathbb{C}$. The number d is called the *degree* of the polynomial P. We can plug in some number z and simply compute $P(z)$ so we have a function $P \colon \mathbb{C} \to \mathbb{C}$.

We try to write

$$f(z) = \sum_{j=0}^{\infty} c_j z^j$$

and all is very fine, until we wish to know what $f(z)$ is for some number $z \in \mathbb{C}$. What we usually mean is

$$\sum_{j=0}^{\infty} c_j z^j = \lim_{d \to \infty} \sum_{j=0}^{d} c_j z^j.$$

As long as the limit exists, we have a function. You know all this; it is your one variable complex analysis. Let us informally review some basic results from one variable. We identify $\mathbb{C}$ with $\mathbb{R}^2$ by

$$z = x + iy,$$

where $z \in \mathbb{C}$, and $(x,y) \in \mathbb{R}^2$. The *complex conjugate* is then defined as

$$\bar{z} \overset{\text{def}}{=} x - iy.$$

A function $f \colon U \subset \mathbb{C} \to \mathbb{C}$ can be written as $f(z) = u(x,y) + iv(x,y)$ where u and v are real valued. Recall that f is *holomorphic* (or *complex analytic*) if it satisfies the *Cauchy-Riemann equations*:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

An easier way to see what these equations do is to define the following formal differential operators (the so-called *Wirtinger operators*):

$$\frac{\partial}{\partial z} \overset{\text{def}}{=} \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right), \qquad \frac{\partial}{\partial \bar{z}} \overset{\text{def}}{=} \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right).$$

The form of these operators is determined by insisting that

$$\frac{\partial}{\partial z} z = 1, \quad \frac{\partial}{\partial z} \bar{z} = 0, \quad \frac{\partial}{\partial \bar{z}} z = 0, \quad \frac{\partial}{\partial \bar{z}} \bar{z} = 1.$$

The function f is holomorphic if and only if

$$\frac{\partial f}{\partial \bar{z}} = 0.$$

Let us check:

$$\frac{\partial f}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right) = \frac{1}{2}\left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} + i\frac{\partial u}{\partial y} - \frac{\partial v}{\partial y}\right).$$

This expression is zero if and only if the real parts and the imaginary parts are zero. In other words if and only if

$$\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} = 0, \qquad \text{and} \qquad \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} = 0.$$

That is, the Cauchy-Riemann equations are satisfied.

The derivative in z is the standard complex derivative you know and love. We can compute it as

$$\frac{\partial f}{\partial z} = \frac{1}{2}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + \frac{i}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right).$$

And recall that for a holomorphic function

$$\frac{\partial f}{\partial z}(z_0) = f'(z_0) = \lim_{w \to z_0} \frac{f(w) - f(z_0)}{w - z_0}.$$

A function on $\mathbb{C}$ is really a function defined on $\mathbb{R}^2$ as identified above and hence it is a function of x and y. Writing $x = \frac{z+\bar{z}}{2}$ and $y = \frac{z-\bar{z}}{2i}$, we think of it as a function of two complex variables z and $\bar{z}$. Pretend for a moment as if $\bar{z}$ did not depend on z. The Wirtinger operators work as if z and $\bar{z}$ really were independent variables. For example:

$$\frac{\partial}{\partial z}\left[z^2\bar{z}^3 + z^{10}\right] = 2z\bar{z}^3 + 10z^9 \qquad \text{and} \qquad \frac{\partial}{\partial \bar{z}}\left[z^2\bar{z}^3 + z^{10}\right] = z^2(3\bar{z}^2) + 0.$$

So a holomorphic function is a function not depending on $\bar{z}$.

One of the most important theorems in one variable is the *Cauchy integral formula*.

Theorem 0.1.1 (Cauchy integral formula)**.** *Let $U \subset \mathbb{C}$ be a domain where ∂U is a piecewise smooth path. Let $f \colon \overline{U} \to \mathbb{C}$ be a continuous function that is holomorphic in U. Orient ∂U positively (going around counter clockwise). Then for $z \in U$:*

$$f(z) = \frac{1}{2\pi i} \int_{\partial U} \frac{f(\zeta)}{\zeta - z} d\zeta.$$

Notice that as a differential form $dz = dx + i dy$. If you are uneasy about differential forms you have probably defined this integral using the Riemann-Stieltjes integral. Let us write down what this means in terms of the standard Riemann integral in a special case. Let

$$\mathbb{D} \overset{\text{def}}{=} \{z : |z| < 1\},$$

be the *unit disc*. The boundary is the unit circle $\partial\mathbb{D} = \{z : |z| = 1\}$ and we orient it positively. We parametrize $\partial\mathbb{D}$ by e^{it}, where t goes from 0 to 2π. If $\zeta = e^{it}$ then $d\zeta = ie^{it}dt$ and

$$f(z) = \frac{1}{2\pi i} \int_{\partial\mathbb{D}} \frac{f(\zeta)}{\zeta - z} d\zeta = \frac{1}{2\pi} \int_0^{2\pi} \frac{f(e^{it})e^{it}}{e^{it} - z} dt.$$

It is useful to keep this in mind. If you are not completely comfortable with path or surface integrals try to think about how you would parametrize the path and write the integral as an integral any calculus student would recognize.

I venture a guess that 90% of what you have learned in a complex analysis course (depending on who has taught it) is more or less a straightforward consequence of having Cauchy integral formula. An important theorem from one variable that follows from the Cauchy formula is the *maximum principle*, which has several versions, let us give the simplest one.

Theorem 0.1.2 (Maximum principle)**.** *Suppose $U \subset \mathbb{C}$ is a domain and $f \colon U \to \mathbb{C}$ holomorphic function. If*

$$\sup_{z \in U} |f(z)| = f(z_0)$$

for some $z_0 \in U$, then f is constant ($f \equiv f(z_0)$).

That is if the supremum is attained in the interior of the domain, then the function must be constant. Another way to state the maximum principle is to say that if f extends continuously to the boundary of a domain then the supremum of $|f(z)|$ is attained on the boundary. In one variable you learned that the maximum principle is really a property of harmonic functions.

Theorem 0.1.3 (Maximum principle)**.** *Let $U \subset \mathbb{C}$ be a domain and $h \colon U \to \mathbb{R}$ harmonic, that is,*

$$\nabla^2 h = \frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} = 0.$$

If

$$\sup_{z \in U} h(z) = h(z_0)$$

for some $z_0 \in U$, then h is constant ($h \equiv h(z_0)$).

In one variable if $f = u + iv$ is holomorphic then u and v are harmonic. And in fact, locally, any harmonic function is the real (or imaginary) part of a holomorphic function, so studying harmonic functions is almost equivalent to studying holomorphic functions in one complex variable. Things will be decidedly different two or more variables.

Holomorphic functions have a power series representation in z at each point a:

$$f(z) = \sum_{j=0}^{\infty} c_j (z-a)^j.$$

Notice there is no $\bar{z}$ necessary there since $\frac{\partial f}{\partial \bar{z}} = 0$.

Let us see how this follows from the Cauchy integral formula as we will require this computation in several variables as well. Given $a \in \mathbb{C}$ and $\rho > 0$ define the disc of radius ρ around a

$$\Delta_\rho(a) \overset{\text{def}}{=} \{z : |z-a| < \rho\}.$$

Suppose $U \subset \mathbb{C}$ is a domain, $f \colon U \to \mathbb{C}$ is holomorphic, $a \in U$, and $\overline{\Delta_\rho(a)} \subset U$ (that is, the closure of the disc is in U, and so its boundary is also in U).

Suppose $z \in \Delta_\rho(a)$ and $\zeta \in \partial \Delta_\rho(a)$. Then

$$\left| \frac{z-a}{\zeta-a} \right| = \frac{|z-a|}{\rho} < 1.$$

In fact, if $|z-a| \leq \rho' < \rho$, then $\left|\frac{z-a}{\zeta-a}\right| \leq \frac{\rho'}{\rho}$. Therefore the geometric series

$$\sum_{j=0}^{\infty} \left(\frac{z-a}{\zeta-a}\right)^j = \frac{1}{1-\frac{z-a}{\zeta-a}} = \frac{\zeta-a}{\zeta-z}$$

converges uniformly in z for $z \in \overline{\Delta_{\rho'}(a)}$. So the series converges uniformly on compact subsets of $\Delta_\rho(a)$.

Let γ be the curve going around $\partial\Delta_\rho(a)$ once in the positive direction. Compute

$$\begin{aligned}
f(z) &= \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{\zeta-z}d\zeta \\
&= \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{\zeta-a}\frac{\zeta-a}{\zeta-z}d\zeta \\
&= \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{\zeta-a}\sum_{j=0}^{\infty}\left(\frac{z-a}{\zeta-a}\right)^j d\zeta \\
&= \sum_{j=0}^{\infty}\left(\frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{(\zeta-a)^{j+1}}d\zeta\right)(z-a)^j.
\end{aligned}$$

The last equality follows from the fact that we can interchange the limit on the sum and the integral as the convergence is uniform.

The key point is writing the *Cauchy kernel* $\frac{1}{\zeta-z}$ as

$$\frac{1}{\zeta-z} = \frac{1}{\zeta-a}\frac{\zeta-a}{\zeta-z}$$

and then using the geometric series.

Not only have we computed that f has a power series, but we computed that the radius of convergence is at least R where R is the maximum R such that $\Delta_R(a) \subset U$. We also obtained a formula for the coefficients

$$c_j = \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{(\zeta-a)^{j+1}}d\zeta.$$

For a set K denote the supremum norm

$$\|f\|_K \overset{\text{def}}{=} \sup_{z\in K}|f(z)|.$$

By a brute force estimation we obtain the very useful *Cauchy estimates*

$$|c_j| = \left|\frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{(\zeta-a)^{j+1}}d\zeta\right| \leq \frac{1}{2\pi}\int_\gamma \frac{\|f\|_\gamma}{\rho^{j+1}}|d\zeta| = \frac{\|f\|_\gamma}{\rho^j}.$$

We differentiate Cauchy's formula j times,

$$\frac{\partial^j f}{\partial z^j} = \frac{1}{2\pi i} \int_\gamma \frac{j! f(\zeta)}{(\zeta - z)^{j+1}} d\zeta,$$

and therefore

$$j! c_j = \frac{\partial^j f}{\partial z^j}(a).$$

Consequently,

$$\left| \frac{\partial^j f}{\partial z^j}(a) \right| \leq \frac{j! \|f\|_\gamma}{\rho^j}.$$

Chapter 1

Holomorphic functions in several variables

1.1 Onto several variables

Let $\mathbb{C}^n$ denote the complex Euclidean space. We denote by $z = (z_1, z_2, \dots, z_n)$ the coordinates of $\mathbb{C}^n$. Let $x = (x_1, x_2, \dots, x_n)$ and $y = (y_1, y_2, \dots, y_n)$ denote the coordinates in $\mathbb{R}^n$. We identify $\mathbb{C}^n$ with $\mathbb{R}^n \times \mathbb{R}^n = \mathbb{R}^{2n}$ by letting $z = x + iy$. Just as in one complex variable we write $\bar{z} = x - iy$. We call z the *holomorphic coordinates* and $\bar{z}$ the *antiholomorphic coordinates*.

Definition 1.1.1. For $\rho = (\rho_1, \dots, \rho_n)$ where $\rho_j > 0$ and $a \in \mathbb{C}^n$ define a *polydisc*

$$\Delta_\rho(a) \overset{\text{def}}{=} \{z \in \mathbb{C}^n : |z_j - a_j| < \rho_j\}.$$

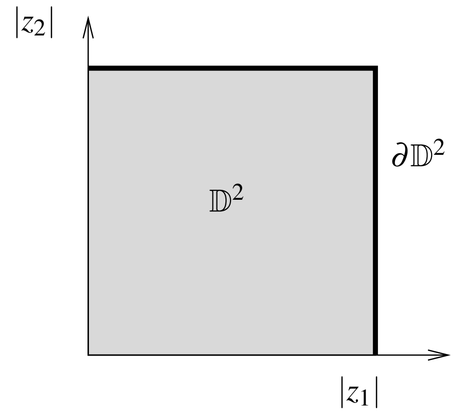
We call a the *center* and ρ the *polyradius* or simply the *radius* of the polydisc $\Delta_\rho(a)$. If $\rho > 0$ is a number then

$$\Delta_\rho(a) \overset{\text{def}}{=} \{z \in \mathbb{C}^n : |z_j - a_j| < \rho\}.$$

As there is the unit disc $\mathbb{D}$ in one variable, so is there the *unit polydisc* in several variables:

$$\mathbb{D}^n = \mathbb{D} \times \mathbb{D} \times \cdots \times \mathbb{D} = \Delta_1(0) = \{z \in \mathbb{C}^n : |z_j| < 1\}.$$

In more than one complex dimension, it is difficult to draw exact pictures for lack of real dimensions on our paper. We can visualize a polydisc in two variables (a *bidisc*) by drawing the following picture by plotting just against the modulus of the variables:



Recall the *Euclidean inner product* on $\mathbb{C}^n$

$$\langle z, w\rangle \overset{\text{def}}{=} z_1\bar{w}_1 + \cdots + z_n\bar{w}_n.$$

Using the inner product we obtain the standard *Euclidean norm* on $\mathbb{C}^n$

$$\|z\| \overset{\text{def}}{=} \sqrt{\langle z, z\rangle} = \sqrt{|z_1|^2 + \cdots + |z_n|^2}.$$

This norm agrees with the standard Euclidean norm on $\mathbb{R}^{2n}$. We define balls as in $\mathbb{R}^{2n}$:

$$B_\rho(a) \overset{\text{def}}{=} \{z \in \mathbb{C}^n : \|z - a\| < \rho\},$$

And we define the unit ball as

$$\mathbb{B}_n = B_1(0) = \{z \in \mathbb{C}^n : \|z\| < 1\}.$$

To define holomorphic functions, as in one variable we define the Wirtinger operators

$$\frac{\partial}{\partial z_j} \overset{\text{def}}{=} \frac{1}{2}\left(\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j}\right),$$
$$\frac{\partial}{\partial \bar{z}_j} \overset{\text{def}}{=} \frac{1}{2}\left(\frac{\partial}{\partial x_j} + i\frac{\partial}{\partial y_j}\right).$$

Definition 1.1.2. Let $U \subset \mathbb{C}^n$ be an open set, and let $f\colon U \to \mathbb{C}$ be a locally bounded function*. If the first partial derivatives exist and f satisfies the *Cauchy-Riemann equations*

$$\frac{\partial f}{\partial \bar{z}_j} = 0 \qquad \text{for } j = 1, 2, \ldots, n.$$

We then say f is *holomorphic*.

*For every $p \in U$, there is a neighborhood N of p such that $f|_N$ is bounded.

In other words, f is holomorphic if it is holomorphic in each variable separately as a function of one variable. Let us first prove that we may as well have assumed differentiability in the definition.

Proposition 1.1.3. *Let $U \subset \mathbb{C}^n$ be a domain and suppose $f \colon U \to \mathbb{C}$ is holomorphic. Then f is infinitely differentiable.*

Proof. Suppose $\Delta = \Delta_\rho(a) = \Delta_1 \times \cdots \times \Delta_n$ is a polydisc centered at a, where each Δ_j is a disc and suppose $\overline{\Delta} \subset U$, that is, f is holomorphic in the closure of Δ. Orient $\partial\Delta_1$ positively and apply the Cauchy formula:

$$f(z) = \frac{1}{2\pi i} \int_{\partial\Delta_1} \frac{f(\zeta_1, z_2, \ldots, z_n)}{\zeta_1 - z_1} d\zeta_1.$$

Apply it again on the second factor, again orienting $\partial\Delta_2$ positively:

$$f(z) = \frac{1}{(2\pi i)^2} \int_{\partial\Delta_1} \int_{\partial\Delta_2} \frac{f(\zeta_1, \zeta_2, z_3, \ldots, z_n)}{(\zeta_1 - z_1)(\zeta_2 - z_2)} d\zeta_2 \, d\zeta_1.$$

Applying the formula n times we obtain

$$f(z) = \frac{1}{(2\pi i)^n} \int_{\partial\Delta_1} \int_{\partial\Delta_2} \cdots \int_{\partial\Delta_n} \frac{f(\zeta_1, \zeta_2, \ldots, \zeta_n)}{(\zeta_1 - z_1)(\zeta_2 - z_2)\cdots(\zeta_n - z_n)} d\zeta_n \cdots d\zeta_2 \, d\zeta_1.$$

At this point we notice that we can simply differentiate underneath the integral. We are really differentiating only in the real and imaginary parts of the z_j variables, and the function underneath the integral is infinitely differentiable in those variables. □

In the definition of holomorphicity, we could have very well assumed that f was smooth and satisfies the Cauchy-Riemann equations. However, the way that we stated the definition makes it easier to apply.

Above, we have really derived the Cauchy integral formula in several variables. To write the formula more concisely we apply the Fubini theorem to write it as a single integral. We will write it down using differential forms. If you are unfamiliar with differential forms, feel free to think of the integral simply as the iterated integral above. If you are familiar with differential forms for real variables, then note that if $z_j = x_j + iy_j$ then $dz_j = dx_j + i dy_j$.

Theorem 1.1.4 (Cauchy integral formula)**.** *Let Δ be a polydisc centered at $a \in \mathbb{C}^n$. Suppose $f \colon \overline{\Delta} \to \mathbb{C}$ is a continuous function holomorphic in Δ. Write $\Gamma = \partial\Delta_1 \times \cdots \times \partial\Delta_n$ oriented appropriately (each $\partial\Delta_j$ has positive orientation). Then for $z \in \Delta$*

$$f(z) = \frac{1}{(2\pi i)^n} \int_{\Gamma} \frac{f(\zeta_1, \zeta_2, \ldots, \zeta_n)}{(\zeta_1 - z_1)(\zeta_2 - z_2)\cdots(\zeta_n - z_n)} d\zeta_1 \wedge d\zeta_2 \wedge \cdots \wedge d\zeta_n.$$

Note that we have stated a more general result where f is only continuous on $\overline{\Delta}$ and holomorphic in Δ. The proof of this slight generalization is contained within the next two exercises.

Exercise* 1.1.1*: *Suppose $f\colon \overline{\mathbb{D}^2} \to \mathbb{C}$ is continuous and holomorphic on $\mathbb{D}^2$. For any $\theta \in \mathbb{R}$, prove*

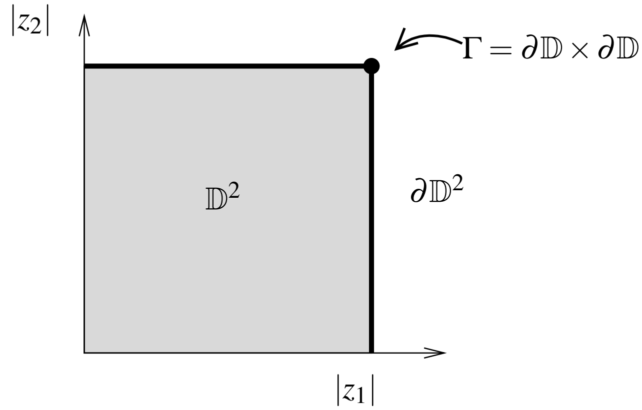
$$g_1(\xi) = f(\xi, e^{i\alpha}) \qquad \text{and} \qquad g_2(\xi) = f(e^{i\alpha}, \xi)$$

are holomorphic in $\mathbb{D}$.

Exercise* 1.1.2*: *Prove the theorem above, that is, the slightly more general Cauchy integral formula given f is only continuous on $\overline{\Delta}$ and holomorphic in Δ.*

The Cauchy integral formula shows an important and subtle point about holomorphic functions in several variables: the function in Δ is actually determined by the values of f on the set Γ, which is much smaller than the boundary of the polydisc $\partial\Delta$. In fact, the Γ is of real dimension n, while the boundary of the polydisc has dimension $2n-1$.

The set $\Gamma = \partial\Delta_1 \times \cdots \times \partial\Delta_n$ is called the *distinguished boundary*. For the unit bidisc we have:



The set Γ is a 2-dimensional torus, like the surface of a donut. Whereas the set $\partial\mathbb{D}^2 = (\partial\mathbb{D} \times \mathbb{D}) \cup (\mathbb{D} \times \partial\mathbb{D})$ is the union of two filled donuts, or more precisely it is both the inside and the outside of the donut put together and these two things meet on the surface of the donut. So you can see the set Γ is quite small in comparison to the entire boundary.

Exercise* 1.1.3*: *Suppose Δ is a polydisc, Γ its distinguished boundary, and $f\colon \overline{\Delta} \to \mathbb{C}$ is continuous and holomorphic on Δ. Prove: a) $|f(z)|$ achieves its maximum on Γ. b) If $|f(z)|$ achieves its maximum on $\partial\Delta \setminus \Gamma$, then f is constant.*

Exercise* 1.1.4*: *Show that differentiable in each variable separately does not imply differentiable even in the case where the function is locally bounded. Show that $\frac{xy}{x^2+y^2}$ is a locally bounded function in $\mathbb{R}^2$, that is differentiable in each variable separately (all partial derivatives exist), but the function is not even continuous. There is something very special about the holomorphic category.*

1.2 Power series representation

As you notice writing out all the components can be a pain. It would become even more painful later on. Just as we have been doing with writing vectors z instead of $(z_1, z_2, \ldots, z_n)$ we similarly define notation to deal with formulas as above.

We will often use the so-called *multi-index notation*. Let $\alpha \in \mathbb{Z}^n$ be a vector of integers. We write

$$\begin{aligned}
z^\alpha &\overset{\text{def}}{=} z_1^{\alpha_1} z_2^{\alpha_2} \cdots z_n^{\alpha_n} \\
\frac{1}{z} &\overset{\text{def}}{=} \frac{1}{z_1 z_2 \cdots z_n} \\
dz &\overset{\text{def}}{=} dz_1 \wedge dz_2 \wedge \cdots \wedge dz_n \\
|\alpha| &\overset{\text{def}}{=} \alpha_1 + \alpha_2 + \cdots + \alpha_n \\
\alpha! &\overset{\text{def}}{=} \alpha_1! \alpha_2! \cdots \alpha_n! \\
\frac{\partial^{|\alpha|}}{\partial z^\alpha} &\overset{\text{def}}{=} \frac{\partial^{\alpha_1}}{\partial z_1^{\alpha_1}} \frac{\partial^{\alpha_2}}{\partial z_2^{\alpha_2}} \cdots \frac{\partial^{\alpha_n}}{\partial z_n^{\alpha_n}}
\end{aligned}$$

Usually the exponent α will be in $\mathbb{N}_0^n$, where $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, but in general this notation is used even with negative powers. Furthermore, when we use 1 as a vector it will mean $(1, 1, \ldots, 1)$. For example if $z \in \mathbb{C}^n$ then,

$$1 - z = (1 - z_1, 1 - z_2, \ldots, 1 - z_n).$$

In this notation, the Cauchy formula becomes the perhaps deceptively simple

$$f(z) = \frac{1}{(2\pi i)^n} \int_\Gamma \frac{f(\zeta)}{(\zeta - z)} \, d\zeta.$$

It goes without saying that when using this notation it is important to be careful to always realize which symbol lives where.

Let us move to power series. For simplicity let us first start with power series at the origin. Using the multinomial notation we write such a series as

$$\sum_{\alpha \in \mathbb{N}_0^n} c_\alpha z^\alpha.$$

It is important to note what this means. Firstly the sum does not have some natural ordering. We are summing over $\alpha \in \mathbb{N}_0$ and there just is not any natural ordering. So it does not make sense to talk about conditional convergence. When we will say the series converges, we will mean absolutely. Fortunately power series converge absolutely, and so the ordering does not matter. You have to admit that the above is far nicer to write than for example for 3 variables writing

$$\sum_{j=0}^\infty \sum_{k=0}^\infty \sum_{\ell=0}^\infty c_{jk\ell} z_1^j z_2^k z_3^\ell.$$

We will often write just

$$\sum_{\alpha} c_\alpha z^\alpha,$$

when it is clear from context that we are talking about a power series and therefore all the powers are nonnegative.

To begin, we need the *geometric series in several variables*. If $z \in \mathbb{D}^n$ (unit polydisc) then

$$\begin{aligned}\frac{1}{1-z} = \frac{1}{(1-z_1)(1-z_2)\cdots(1-z_n)} &= \left(\sum_{j=0}^{\infty} {z_1}^j\right)\left(\sum_{j=0}^{\infty} {z_2}^j\right)\cdots\left(\sum_{j=0}^{\infty} {z_n}^j\right) \\ &= \sum_{j_1=0}^{\infty}\sum_{j_2=0}^{\infty}\cdots\sum_{j_n=0}^{\infty}\left({z_1}^{j_1}{z_n}^{j_2}\cdots{z_n}^{j_n}\right) = \sum_{\alpha} z^\alpha.\end{aligned}$$

The convergence is uniform on compact subsets of the unit polydisc. In fact any compact set in the unit polydisc is contained in a polydisc Δ centered at 0 of radius $1-\varepsilon$ for some $\varepsilon > 0$. Then the convergence is uniform on Δ (or in fact on the closure of Δ). This claim follows by simply noting the same fact for each factor is true in one dimension.

We now prove that holomorphic functions are precisely those having a power series expansion.

Theorem 1.2.1. *Let $\Delta = \Delta_\rho(a)$. Suppose $f \colon \overline{\Delta} \to \mathbb{C}$ is continuous and holomorphic on Δ. Then on Δ, f is equal to a series converging uniformly on compact subsets of Δ:*

$$f(z) = \sum_{\alpha} c_\alpha (z-a)^\alpha. \tag{1.1}$$

Conversely, if f is defined by (1.1) *converging uniformly on compact subsets of Δ, then f is holomorphic on Δ.*

Proof. First assume that f is holomorphic. As in one variable we write the kernel of the Cauchy formula as

$$\frac{1}{\zeta - z} = \frac{1}{\zeta - a}\frac{1}{\left(1 - \frac{z-a}{\zeta-a}\right)} = \frac{1}{\zeta - a}\sum_{\alpha}\left(\frac{z-a}{\zeta-a}\right)^{\alpha}.$$

Notice that the geometric series is just a product of geometric series in one variable, and geometric series in one variable converges uniformly on compact subsets of the unit disc. Therefore the series above converges on compact subsets of Δ.

We wish to apply the Cauchy formula. We write $\Gamma = \partial\Delta_1 \times \cdots \times \partial\Delta_n$ and orient it positively.

Compute

$$\begin{aligned} f(z) &= \frac{1}{(2\pi i)^n} \int_\Gamma \frac{f(\zeta)}{\zeta - z} d\zeta \\ &= \frac{1}{(2\pi i)^n} \int_\Gamma \frac{f(\zeta)}{\zeta - a} \frac{\zeta - a}{\zeta - z} d\zeta \\ &= \frac{1}{(2\pi i)^n} \int_\Gamma \frac{f(\zeta)}{\zeta - a} \sum_\alpha \left(\frac{z-a}{\zeta - a} \right)^\alpha d\zeta \\ &= \sum_\alpha \left(\frac{1}{(2\pi i)^n} \int_\Gamma \frac{f(\zeta)}{(\zeta - a)^{\alpha+1}} d\zeta \right) (z-a)^\alpha. \end{aligned}$$

The last equality follows because the convergence of the sum is uniform in $\zeta \in \Gamma$ for a fixed z. Uniform convergence (as z moves) on compact subsets of the final series follows from the uniform convergence of the geometric series. It is also a direct consequence of the Cauchy estimates below.

We have shown that

$$f(z) = \sum_\alpha c_\alpha (z-a)^\alpha,$$

where

$$c_\alpha = \frac{1}{(2\pi i)^n} \int_\Gamma \frac{f(\zeta)}{(\zeta - z)^{\alpha+1}} d\zeta.$$

Notice how strikingly similar the computation is to one variable.

The converse follows by applying the Cauchy-Riemann equations to the series term-wise. To do this you have to show that the term-by-term derivative series also converges uniformly on compact subsets. It is left as an exercise. Then you apply the well known theorem from real analysis. Note that the proof of this is very similar to one variable series that you know.

The conclusion also follows by restricting to one variable for each variable in turn, and then using the corresponding one-variable result. □

Exercise* 1.2.1*: *Prove the claim above that if a power series converges uniformly on compact subsets of a polydisc Δ, then the term by term derivative converges. Do the proof without using the analogous result for single variable series.*

Using Leibniz rule, as long as $z \in \Delta$ and not on the boundary, we can differentiate under the integral. Let us do a single derivative to get the idea:

$$\begin{aligned} \frac{\partial f}{\partial z_1}(z) &= \frac{\partial}{\partial z_1} \left[\frac{1}{(2\pi i)^n} \int_\Gamma \frac{f(\zeta_1, \zeta_2, \ldots, \zeta_n)}{(\zeta_1 - z_1)(\zeta_2 - z_2) \cdots (\zeta_n - z_n)} d\zeta_1 \wedge d\zeta_2 \wedge \cdots \wedge d\zeta_n \right] \\ &= \frac{1}{(2\pi i)^n} \int_\Gamma \frac{f(\zeta_1, \zeta_2, \ldots, \zeta_n)}{(\zeta_1 - z_1)^2(\zeta_2 - z_2) \cdots (\zeta_n - z_n)} d\zeta_1 \wedge d\zeta_2 \wedge \cdots \wedge d\zeta_n. \end{aligned}$$

How about we do it a second time:

$$\frac{\partial^2 f}{\partial z_1^2}(z) = \frac{1}{(2\pi i)^n} \int_\Gamma \frac{2f(\zeta_1,\zeta_2,\ldots,\zeta_n)}{(\zeta_1-z_1)^3(\zeta_2-z_2)\cdots(\zeta_n-z_n)}\, d\zeta_1 \wedge d\zeta_2 \wedge \cdots \wedge d\zeta_n.$$

Notice the 2 before the f. Next time 3 is coming out, so after j derivatives in z_1 you will get the constant $j!$. It is exactly the same thing that is happening in one variable. A moment's thought will convince you that the following formula is correct for $\alpha \in \mathbb{N}_0^n$:

$$\frac{\partial^{|\alpha|} f}{\partial z^\alpha}(z) = \frac{1}{(2\pi i)^n} \int_\Gamma \frac{\alpha! f(\zeta)}{(\zeta - z)^{\alpha+1}}\, d\zeta.$$

Therefore

$$\alpha! c_\alpha = \frac{\partial^{|\alpha|} f}{\partial z^\alpha}(a).$$

And as befre, we obtain the *Cauchy estimates*:

$$\left|\frac{\partial^{|\alpha|} f}{\partial z^\alpha}(a)\right| = \left|\frac{1}{(2\pi i)^n} \int_\Gamma \frac{\alpha! f(\zeta)}{(\zeta - a)^{\alpha+1}}\, d\zeta\right| \leq \frac{1}{(2\pi)^n} \int_\Gamma \frac{\alpha! |f(\zeta)|}{\rho^{\alpha+1}} |d\zeta| \leq \frac{\alpha!}{\rho^\alpha} \|f\|_\Gamma.$$

Or

$$|c_\alpha| \leq \frac{\|f\|_\Gamma}{\rho^\alpha}.$$

As in one variable theory the Cauchy estimates prove the following proposition.

Proposition 1.2.2. *Let $U \subset \mathbb{C}^n$ be a domain. Suppose $f_j \colon U \to \mathbb{C}$ converge uniformly on compact subsets to $f \colon U \to \mathbb{C}$. If every f_j is holomorphic, then f is holomorphic and $\frac{\partial^{|\alpha|} f_j}{\partial z^\alpha}$ converge to $\frac{\partial^{|\alpha|} f}{\partial z^\alpha}$ uniformly on compact subsets.*

***Exercise* 1.2.2:** *Prove the above proposition.*

Let $W \subset \mathbb{C}^n$ be the set where a power series converges such that it diverges on the complement. The interior of W is called the *domain of convergence*. In one variable, every domain of convergence is a disc, and hence it can be described with a single number (the radius). In several variables, the domain where a series converges is not as easy to describe. We content ourselves for now with a few simple examples. For the geometric series it is easy to see that the domain of convergence is the unit polydisc, but more complicated examples exist.

Example 1.2.3: The power series

$$\sum_{k=0}^{\infty} z_1 z_2^k$$

converges absolutely exactly on the set

$$\{z : |z_2| < 1\} \cup \{z : z_1 = 0\},$$

which is not quite a polydisc. It is not even an open set.

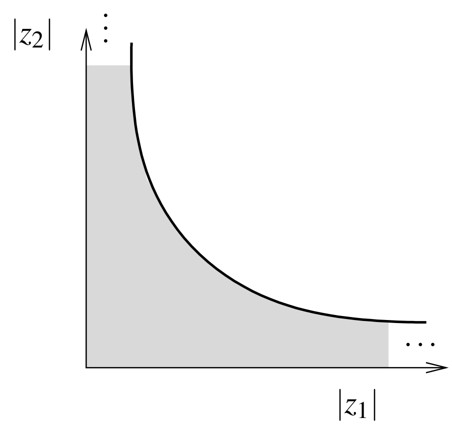
Example 1.2.4: The power series

$$\sum_{k=0}^{\infty} z_1^k z_2^k$$

converges absolutely exactly on the set

$$\{z : |z_1 z_2| < 1\}.$$

Here the picture is definitely more complicated than a polydisc:



A domain $U \subset \mathbb{C}^n$ such that if $z \in U$, then $w \in U$ whenever $|z_j| = |w_j|$ is called a *Reinhardt domain*. The domains we were drawing so far have been Reinhardt domains, they are exactly the domains that you can draw by plotting what happens for the moduli of the variables. A domain is called a *complete Reinhardt domain* if whenever $z \in U$ then for $r = (r_1, \ldots, r_n)$ where $r_j = |z_j|$ for all j, we have that the whole polydisc $\Delta_r(0) \subset U$. So a complete Reinhardt domain is a union (possibly infinite) of polydiscs centered at the origin.

Exercise* 1.2.3*: *Let $W \subset \mathbb{C}^n$ be the set where a certain power series at the origin converges. Show that the interior of W is a complete Reinhardt domain.*

Theorem 1.2.5 (Identity theorem)**.** *Let $U \subset \mathbb{C}^n$ be a domain (connected open set) and let $f : U \to \mathbb{C}$ be holomorphic. Suppose $f|_N \equiv 0$ for an open subset $N \subset U$. Then $f \equiv 0$.*

Proof. Let Z be set where all derivatives of f are zero; then $N \subset Z$. The set Z is closed in U as all derivatives are continuous. Take an arbitrary $a \in Z$. We find $\Delta_\rho(a) \subset U$. If we expand f in a power series around a. As the coefficients are given by derivatives of f, we see that the power series is identically zero and hence f is identically zero in $\Delta_\rho(a)$. Therefore Z is open in U and $Z = U$. $\square$

The theorem is often used to show that if two holomorphic functions f and g are equal on a small open set, then $f \equiv g$.

Theorem 1.2.6 (Maximum principle)**.** *Let $U \subset \mathbb{C}^n$ be a domain (connected open set). Let $f : U \to \mathbb{C}$ be holomorphic and suppose that $|f(z)|$ attains a maximum at some $a \in U$. Then $f \equiv f(a)$.*

Proof. Suppose $|f(z)|$ attains its maximum at $a \in U$. Consider a polydisc $\Delta = \Delta_1 \times \cdots \times \Delta_n \subset U$ centered at a. The function

$$z_1 \mapsto f(z_1, a_2, \ldots, a_n)$$

is holomorphic on Δ_1 and its modulus attains the maximum at the center. Therefore it is constant by maximum principle in one variable, that is, $f(z_1, a_2, \ldots, a_n) = f(a)$ for all $z_1 \in \Delta_1$. For any fixed $z_1 \in \Delta_1$ consider the function

$$z_2 \mapsto f(z_1, z_2, a_3, \ldots, a_n).$$

This function again attains its maximum modulus at the center of Δ_2 and hence is constant on Δ_2. Iterating this procedure we obtain that $f(z) = f(a)$ for all $z \in \Delta$. By the identity theorem we have that $f(z) = f(a)$ for all $z \in U$. $\square$

***Exercise* 1.2.4:** *Let V be the volume measure on $\mathbb{R}^{2n}$ and hence on $\mathbb{C}^n$. Using Cauchy formula prove that for any polydisc Δ centered at a we have:*

$$f(a) = \frac{1}{V(\Delta)} \int_\Delta f(\zeta)\, dV(\zeta).$$

That is, $f(a)$ is an average of the values on a polydisc centered at a.

***Exercise* 1.2.5:** *Prove the maximum principle by using the Cauchy formula instead. (Hint: use previous exercise)*

***Exercise* 1.2.6:** *Prove a several variables analogue of the* Schwarz's lemma*: Suppose f is holomorphic in a neighborhood of $\overline{\mathbb{D}^n}$, $f(0) = 0$, and for some $k \in \mathbb{N}$ we have $\frac{\partial^{|\alpha|} f}{\partial z^\alpha}(0) = 0$ whenever $|\alpha| < k$. Further suppose that for all $z \in \mathbb{D}^n$, $|f(z)| \leq M$ for some M. Then show that*

$$|f(z)| \leq M \|z\|^k$$

for all $z \in \overline{\mathbb{D}^n}$.

***Exercise* 1.2.7:** *Apply the one variable Liouville's theorem to prove it for several variables. That is, suppose that $f : \mathbb{C}^n \to \mathbb{C}$ is holomorphic and bounded. Prove that f is constant.*

***Exercise* 1.2.8:** *Prove the several variables version of* Montel's theorem*: Suppose $\{f_k\}$ is a sequence of holomorphic functions on $U \subset \mathbb{C}^n$ that is uniformly bounded. Show that there exists a subsequence $\{f_{k_j}\}$ that converges uniformly on compact subsets to some holomorphic function f. Hint: Mimic the one-variable proof.*

1.3 Derivatives

When you apply a conjugate to a holomorphic function you get a so-called *antiholomorphic function*. An antiholomorphic function is a function that does not depend on z, but only on $\bar{z}$. Given a holomorphic function f, we define the function $\bar{f}$ by $\overline{f(z)}$, that we write as $\bar{f}(\bar{z})$. Then for all j

$$\frac{\partial \bar{f}}{\partial z_j} = 0, \qquad \frac{\partial \bar{f}}{\partial \bar{z}_j} = \overline{\left(\frac{\partial f}{\partial z_j}\right)}.$$

Let us figure out how chain rule works for the holomorphic and antiholomorphic derivatives.

Proposition 1.3.1 (Complex chain rule)**.** *Suppose $U \subset \mathbb{C}^n$ and $V \subset \mathbb{C}^m$ are open sets and suppose that $f : U \to V$, and $g : V \to \mathbb{C}$ are differentiable functions (mappings). Write the variables as $z = (z_1, \ldots, z_n) \in U \subset \mathbb{C}^n$ and $w = (w_1, \ldots, w_m) \in V \subset \mathbb{C}^m$. Then for any $j = 1, \ldots, n$ we have*

$$\frac{\partial}{\partial z_j}[g \circ f] = \sum_{\ell=1}^{m} \left(\frac{\partial g}{\partial w_\ell}\frac{\partial f_\ell}{\partial z_j} + \frac{\partial g}{\partial \bar{w}_\ell}\frac{\partial \bar{f}_\ell}{\partial z_j} \right), \qquad \frac{\partial}{\partial \bar{z}_j}[g \circ f] = \sum_{\ell=1}^{m} \left(\frac{\partial g}{\partial w_\ell}\frac{\partial f_\ell}{\partial \bar{z}_j} + \frac{\partial g}{\partial \bar{w}_\ell}\frac{\partial \bar{f}_\ell}{\partial \bar{z}_j} \right).$$

Proof. Write $f = u + iv$, $z = x + iy$, $w = s + it$. The composition sets w to f, and so it sets s to u, and t to v. Compute

$$\begin{aligned}
\frac{\partial}{\partial z_j}[g \circ f] &= \frac{1}{2}\left(\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j}\right)[g \circ f] \\
&= \frac{1}{2}\sum_{\ell=1}^{m}\left(\frac{\partial g}{\partial s_\ell}\frac{\partial u_\ell}{\partial x_j} + \frac{\partial g}{\partial t_\ell}\frac{\partial v_\ell}{\partial x_j} - i\left(\frac{\partial g}{\partial s_\ell}\frac{\partial u_\ell}{\partial y_j} + \frac{\partial g}{\partial t_\ell}\frac{\partial v_\ell}{\partial y_j}\right)\right) \\
&= \sum_{\ell=1}^{m}\left(\frac{\partial g}{\partial s_\ell}\frac{1}{2}\left(\frac{\partial u_\ell}{\partial x_j} - i\frac{\partial u_\ell}{\partial y_j}\right) + \frac{\partial g}{\partial t_\ell}\frac{1}{2}\left(\frac{\partial v_\ell}{\partial x_j} - i\frac{\partial v_\ell}{\partial y_j}\right)\right) \\
&= \sum_{\ell=1}^{m}\left(\frac{\partial g}{\partial s_\ell}\frac{\partial u_\ell}{\partial z_j} + \frac{\partial g}{\partial t_\ell}\frac{\partial v_\ell}{\partial z_j}\right).
\end{aligned}$$

For $\ell = 1, \ldots, m$,

$$\frac{\partial}{\partial s_\ell} = \frac{\partial}{\partial w_\ell} + \frac{\partial}{\partial \bar{w}_\ell}, \qquad \frac{\partial}{\partial t_\ell} = i\left(\frac{\partial}{\partial w_\ell} - \frac{\partial}{\partial \bar{w}_\ell}\right).$$

Continuing:

$$\begin{aligned}
\frac{\partial}{\partial z_j}[g \circ f] &= \sum_{\ell=1}^{m} \left(\frac{\partial g}{\partial s_\ell}\frac{\partial u_\ell}{\partial z_j} + \frac{\partial g}{\partial t_\ell}\frac{\partial v_\ell}{\partial z_j} \right) \\
&= \sum_{\ell=1}^{m} \left(\left(\frac{\partial g}{\partial w_\ell}\frac{\partial u_\ell}{\partial z_j} + \frac{\partial g}{\partial \bar{w}_\ell}\frac{\partial u_\ell}{\partial z_j} \right) + i \left(\frac{\partial g}{\partial w_\ell}\frac{\partial v_\ell}{\partial z_j} - \frac{\partial g}{\partial \bar{w}_\ell}\frac{\partial v_\ell}{\partial z_j} \right) \right) \\
&= \sum_{\ell=1}^{m} \left(\frac{\partial g}{\partial w_\ell}\left(\frac{\partial u_\ell}{\partial z_j} + i\frac{\partial v_\ell}{\partial z_j} \right) + \frac{\partial g}{\partial \bar{w}_\ell}\left(\frac{\partial u_\ell}{\partial z_j} - i\frac{\partial v_\ell}{\partial z_j} \right) \right) \\
&= \sum_{\ell=1}^{m} \left(\frac{\partial g}{\partial w_\ell}\frac{\partial f_\ell}{\partial z_j} + \frac{\partial g}{\partial \bar{w}_\ell}\frac{\partial \bar{f}_\ell}{\partial z_j} \right).
\end{aligned}$$

The $\bar{z}$ derivative works similarly. ☐

The proposition is another reason why when we deal with arbitrary possibly nonholomorphic functions we write $f(z,\bar{z})$ and treat them as functions of z and $\bar{z}$.

Remark 1.3.2. It is good to notice the subtlety of what we just said. Formally it seems as if we are treating z and $\bar{z}$ as independent variables when taking derivatives, but in reality they are not independent if we actually wish to evaluate the function. Underneath, a smooth function that is not necessarily holomorphic is really a function of real variables x and y if $z = x + iy$.

Remark 1.3.3. Another remark to make is that we could have swapped z and $\bar{z}$, by just flipping the bars everywhere. There is no difference between the two, they are twins in effect. We just need to know which one is which. After all, it all starts with taking the two square roots of -1 and deciding which one is i. There is no "natural choice" for that, but once we make that choice we must be consistent. And once we picked which root is i, we have also picked what is holomorphic and what is antiholomorphic. This is a subtle philosophical as much as a mathematical point.

Definition 1.3.4. Let $U \subset \mathbb{C}^n$ be an open set. A mapping $f : U \to \mathbb{C}^m$ is said to be holomorphic if each component is holomorphic. That is, if $f = (f_1, \ldots, f_m)$ then each f_j is a holomorphic function.

As in one variable the composition of holomorphic functions is holomorphic.

Theorem 1.3.5. *Let $U \subset \mathbb{C}^n$ and $V \subset \mathbb{C}^m$ be open sets and suppose that $f : U \to V$, and $g : V \to \mathbb{C}^k$ are both holomorphic. Then the composition $g \circ f$ is holomorphic.*

Proof. The proof is almost trivial by chain rule. Again let g be a function of $w \in V$ and f be a function of $z \in U$. For any $j = 1, \ldots, n$ and any $p = 1, \ldots, k$ we compute

$$\frac{\partial}{\partial \bar{z}_j}[g_p \circ f] = \sum_{\ell=1}^{m} \left(\frac{\partial g_p}{\partial w_\ell}\cancelto{0}{\frac{\partial f_\ell}{\partial \bar{z}_j}} + \cancelto{0}{\frac{\partial g_p}{\partial \bar{w}_\ell}}\frac{\partial \bar{f}_\ell}{\partial \bar{z}_j} \right) = 0.$$ ☐

Let us also state the chain rule for holomorphic functions then. Again suppose that $U \subset \mathbb{C}^n$ and $V \subset \mathbb{C}^m$ are open sets and $f : U \to V$, and $g : V \to \mathbb{C}$ are holomorphic functions. Again let the variables be named $z = (z_1, \ldots, z_n) \in U \subset \mathbb{C}^n$ and $w = (w_1, \ldots, w_m) \in V \subset \mathbb{C}^m$. All the bar derivatives are zero if f and g are holomorphic. Therefore for any $j = 1, \ldots, n$,

$$\frac{\partial}{\partial z_j} [g \circ f] = \sum_{\ell=1}^{m} \frac{\partial g}{\partial w_\ell} \frac{\partial f_\ell}{\partial z_j}.$$

Definition 1.3.6. Let $U \subset \mathbb{C}^n$ be an open set. Define $\mathscr{O}(U)$ to be the *ring of holomorphic functions*. The letter $\mathscr{O}$ is used as a recognize the fundamental contribution to several complex variables by Kiyoshi Oka*.

***Exercise* 1.3.1:** *Prove that $\mathscr{O}(U)$ is actually a ring with the operations*

$$(f+g)(z) = f(z) + g(z), \qquad (fg)(z) = f(z)g(z).$$

***Exercise* 1.3.2:** *Show that $\mathscr{O}(U)$ is an integral domain (has no zero divisors) if and only if U is connected. That is, show that U being connected is equivalent to showing that if $h(z) = f(z)g(z)$ is identically zero for $f, g \in \mathscr{O}(U)$, then either $f(z)$ or $g(z)$ are identically zero.*

***Exercise* 1.3.3:** *Prove the holomorphic implicit function theorem: Let $U \subset \mathbb{C}^n \times \mathbb{C}^m$ be a domain, let $(z, w) \in \mathbb{C}^n \times \mathbb{C}^m$ be our coordinates, and let $f : U \to \mathbb{R}^m$ be a holomorphic mapping. Let $(z^0, w^0) \in U$ be a point such that $f(z^0, w^0) = 0$ and such that the $m \times m$ matrix*

$$\left[\frac{\partial f_j}{\partial w_k}(z^0, w^0) \right]_{jk}$$

is invertible. Then there exists an open set $V \subset \mathbb{C}^n$ with $z^0 \in V$, open set $W \subset \mathbb{C}^m$ with $w^0 \in W$, $V \times W \subset U$, and a holomorphic mapping $g : V \to W$, with $g(z^0) = w^0$ such that for every $z \in V$, the point $g(z)$ is the unique point in W such that

$$f\big(z, g(z)\big) = 0.$$

Hint: Check that you can use the normal implicit function theorem for C^1 functions, and then show that the g you obtain is holomorphic.

For a $U \subset \mathbb{C}^n$, a holomorphic mapping $f : U \to \mathbb{C}^m$, and a point $a \in U$, define the holomorphic derivative, sometimes called the *Jacobian matrix*:

$$Df(a) \overset{\text{def}}{=} \left[\frac{\partial f_j}{\partial z_k}(a) \right]_{jk}.$$

*See `http://en.wikipedia.org/wiki/Kiyoshi_Oka`

Sometimes the notation $f'(a) = Df(a)$ is used.

Using the holomorphic chain rule, as in the theory of real functions we get that the derivative of the composition is the composition of derivatives (multiplied as matrices).

Proposition 1.3.7 (Chain rule for holomorphic mappings)**.** *Let $U \subset \mathbb{C}^n$ and $V \subset \mathbb{C}^m$ be open sets and suppose $f: U \to V$, and $g: V \to \mathbb{C}^k$ are both holomorphic, and $a \in U$. Then*

$$D(g \circ f)(a) = Dg\big(f(a)\big)\, Df(a).$$

In short hand we often simply write $D(g \circ f) = DgDf$.

***Exercise* 1.3.4:** *Prove the proposition.*

Proposition 1.3.8. *Let $U \subset \mathbb{C}^n$ be open sets and $f: U \to \mathbb{C}^n$ be holomorphic. Then if we let $D_{\mathbb{R}} f(a)$ be the real Jacobian matrix of f (a $2n \times 2n$ real matrix), then*

$$|\det Df(a)|^2 = \det D_{\mathbb{R}} f(a).$$

The expression $\det Df(a)$ is called the *Jacobian determinant* and clearly it is important to know if we are talking about the holomorphic Jacobian determinant or the standard real Jacobian determinant $\det D_{\mathbb{R}} f(a)$. Recall from vector calculus that if the real Jacobian determinant $\det D_{\mathbb{R}} f(a)$ of a smooth function is positive, then the function preserves orientation. In particular, holomorphic maps preserve orientation.

Proof. The statement is simply about matrices. We have a complex $n \times n$ matrix A, that we rewrite as a real $2n \times 2n$ matrix B by using the identity $z = x + iy$. If we change basis from (x, y) to $(z, \bar{z})$ that is $(x + iy, x - iy)$, we are really just changing a basis via a matrix M as $M^{-1}BM$. Then we notice

$$M^{-1}BM = \begin{bmatrix} A & 0 \\ 0 & \overline{A} \end{bmatrix},$$

where $\overline{A}$ is the complex conjugate of A. Thus

$$\det(B) = \det(M^{-1}MB) = \det(M^{-1}BM) = \det(A)\det(\overline{A}) = \det(A)\overline{\det(A)} = |\det(A)|^2. \qquad \square$$

1.4 Inequivalence of ball and polydisc

Definition 1.4.1. Two domains $U \subset \mathbb{C}^n$ and $V \subset \mathbb{C}^n$ are said to be *biholomorphic* if there exists a one-to-one and onto holomorphic map $f: U \to V$ such that f^{-1} is holomorphic. The mapping if is said to be a *biholomorphic map* or a *biholomorphism*.

One of the main questions in complex analysis is to classify domains up to biholomorphic transformations. In one variable, there is the rather striking theorem due to Riemann:

Theorem 1.4.2 (Riemann mapping theorem)**.** *If $U \subset \mathbb{C}$ is a simply connected domain such that $U \neq \mathbb{C}$, then U is biholomorphic to $\mathbb{D}$.*

In one variable, a topological property on U is enough to classify a whole class of domains. It is one of the reasons why studying the disc is so important in one variable, and why many theorems are stated for the disc only. There is simply no such theorem in several variables. We will show momentarily that the unit ball

$$\mathbb{B}_n \stackrel{\text{def}}{=} \{z \in \mathbb{C}^n : \|z\| < 1\}.$$

and the polydisc $\mathbb{D}^n$ are inequivalent. Both are simply connected (have no holes), and they are the two most obvious generalizations of the disc to several dimensions. Let us stick with $n = 2$. Instead of proving that $\mathbb{B}_2$ and $\mathbb{D}^2$ are inequivalent we will prove a stronger theorem. First a definition.

Definition 1.4.3. Suppose $f: X \to Y$ is a continuous map between two topological spaces. Then f is a *proper map* if for every compact $K \subset\subset Y$, the set $f^{-1}(K)$ is compact.

Note that "$\subset\subset$" is a common notation for compact, or relatively compact subset. Often the distinction between compact and relatively compact is not important, for example in the above definition.

Vaguely, "proper" means that "boundary goes to the boundary." A continuous map, f pushes compacts to compacts; a proper map is one where the inverse does so too. If the inverse is a continuous function, then clearly f is proper, but not every proper map is invertible. For example, the map $f: \mathbb{D} \to \mathbb{D}$ given by $f(z) = z^2$ is proper, but not invertible. The codomain of f is important. If we replace f by $g: \mathbb{D} \to \mathbb{C}$, still given by $g(z) = z^2$, the map is no longer proper. We now state the main result of this section.

Theorem 1.4.4 (Rothstein 1935)**.** *There exists no proper mapping of the unit bidisc $\mathbb{D}^2 = \mathbb{D} \times \mathbb{D} \subset \mathbb{C}^2$ to the unit ball $\mathbb{B}_2 \subset \mathbb{C}^2$.*

As a biholomorphic mapping is proper, the unit bidisc is not biholomorphically equivalent to the unit ball in $\mathbb{C}^2$. This fact was first proved by Poincaré by computing the automorphism groups of $\mathbb{D}^2$ and $\mathbb{B}_2$, although his proof assumed the maps extended past the boundary. The first complete proof was by Henri Cartan in 1931, though popularly the theorem is attributed to Poincaré. It seems standard that any general audience talk about several complex variables contains a mention of Poincaré.

We need some lemmas before we get to the proof of the result. First, a certain one-dimensional object plays a very important role in the geometry of several complex variables. It allows us to apply certain one-dimensional results to several dimensions. It is especially important in understanding the boundary behavior of holomorphic functions. It also prominently appears in complex geometry.

Definition 1.4.5. A non-constant holomorphic mapping $\varphi\colon \mathbb{D} \to \mathbb{C}^n$ is called an *analytic disc*. If the mapping φ extends continuously to the closed unit disc $\overline{\mathbb{D}}$, then the mapping $\varphi\colon \overline{\mathbb{D}} \to \mathbb{C}^n$ is called a *closed analytic disc*.

Often we call the image $\Delta = \varphi(\mathbb{D})$ the analytic disc rather than the mapping. For a closed analytic disc we write $\partial\Delta = \varphi(\partial\mathbb{D})$ and call it the boundary of the analytic disc.

In some sense, analytic discs play the role of line segments in $\mathbb{C}^n$. It is important to always have in mind that there is a mapping defining the disc, even if we are more interested in the set. Obviously for a given image, the mapping φ is not unique.

Let us consider the boundaries of the unit bidisc $\mathbb{D}\times\mathbb{D} \subset \mathbb{C}^2$ and the unit ball $\mathbb{B}_2 \subset \mathbb{C}^2$. We notice that the boundary of the unit bidisc contains analytic discs $\{p\}\times\mathbb{D}$ and $\mathbb{D}\times\{p\}$ for $p \in \partial\mathbb{D}$. That is, through every point in the boundary, with the exception of the distinguished boundary $\partial\mathbb{D}\times\partial\mathbb{D}$ there exists an analytic disc lying entirely inside the boundary. On the other hand for the ball we have the following proposition.

Proposition 1.4.6. *The unit sphere $S^{2n-1} = \partial\mathbb{B}_n \subset \mathbb{C}^n$ contains no analytic discs.*

Proof. Suppose we have a holomorphic function $g\colon \mathbb{D} \to \mathbb{C}^n$ such that the image of g is inside the unit sphere. In other words

$$\|g(z)\|^2 = |g_1(z)|^2 + |g_2(z)|^2 + \cdots + |g_n(z)|^2 = 1$$

for all $z \in \mathbb{D}$. Without loss of generality (after composing with a unitary matrix) we assume that $g(0) = (1,0,0,\ldots,0)$. We look at the first component and notice that $g_1(0) = 1$. Furthermore we notice that if a sum of positive numbers is less than or equal to 1, they all are and hence $|g_1(z)| \leq 1$. By maximum principle we have that $g_1(z) = 1$ for all $z \in \mathbb{D}$. But then $g_j(z) = 0$ for all $j = 2,\ldots,n$ and all $z \in \mathbb{D}$. Therefore g is constant and thus not an analytic disc. $\square$

The fact that the sphere contains no analytic discs is the most important geometric distinction between the boundary of the polydisc and the sphere.

Exercise* 1.4.1: Modify the proof to show some stronger results:*
a) If Δ is an analytic disc and $p \in \Delta\cap\partial\mathbb{B}_n$. Then Δ contains points not in $\overline{\mathbb{B}_n}$.
b) If Δ is an analytic disc such that $\mathbb{B}_n\cap\Delta = \emptyset$, but there is a point $p \in \Delta\cap\partial\mathbb{B}_n$. Then p is isolated in $\Delta\cap\overline{\mathbb{B}_n}$.

Before we prove the theorem let us prove a lemma making the statement about proper maps taking boundary to boundary precise.

Lemma 1.4.7. *Let $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^m$ be bounded domains and let $f\colon U \to V$ be a continuous. Then f is proper if and only if for every sequence $\{p_k\}$ in U such that $p_k \to p \in \partial U$, the set of limit points of $\{f(p_k)\}$ lies in ∂V.*

Proof. First suppose that f is proper. Take a sequence $\{p_k\}$ in U such that $p_k \to p \in \partial U$. Then take any convergent subsequence $\{f(p_{k_j})\}$ of $\{f(p_k)\}$ converging to some $q \in \overline{V}$. Take $E = \{f(p_{k_j})\}$ as a set. Let $\overline{E}$ be the closure of E in V (relative topology). If $q \in V$, then $\overline{E} = E \cup \{q\}$, otherwise $\overline{E} = E$. The inverse image $f^{-1}(\overline{E})$ is not compact (it contains a sequence going to $p \in \partial U$) and hence $\overline{E}$ is not compact either as f is proper. Thus $q \notin V$, and hence $q \in \partial V$. As we took an arbitrary subsequence of $\{f(p_k)\}$, q was an arbitrary limit point. Therefore, all limit points are in ∂V.

We will prove the converse by contrapositive. Suppose that for every sequence $\{p_k\}$ in U such that $p_k \to p \in \partial U$, the set of limit points of $\{f(p_k)\}$ lies in ∂V. Take a closed set $E \subset V$ (relative topology) and look at $f^{-1}(E)$. If $f^{-1}(E)$ is not compact, then there exists a sequence $\{p_k\}$ in $f^{-1}(E)$ such that $p_k \to p \in \partial U$. That is because $f^{-1}(E)$ is closed (in U) but not compact. But then the limit points of $\{f(p_k)\}$ are in ∂V and hence E has limit points in ∂V and thus is not compact. $\square$

Another useful characterization of proper maps is the following exercise:

***Exercise* 1.4.2:** *Let $f\colon X \to Y$ be a continuous function of topological spaces. Let X_∞ and Y_∞ be the one-point-compactifications of X and Y. Then f is a proper map if and only if it extends as a continuous map of X_∞ to Y_∞.*

We now have all the lemmas needed to prove the theorem of Rothstein.

Proof of Theorem 1.4.4. Suppose we have a proper holomorphic map $f\colon \mathbb{D}^2 \to \mathbb{B}_2$. Fix some $e^{i\theta}$ in the boundary of the disc $\mathbb{D}$. Take a sequence $w_k \in \mathbb{D}$ such that $w_k \to e^{i\theta}$. The functions $g_k(\zeta) = f(\zeta, w_k)$ map the unit disc into $\mathbb{B}_2$. By the standard Montel's theorem, by passing to a subsequence we assume that the sequence of functions converges (uniformly on compact subsets) to a limit $g\colon \mathbb{D} \to \overline{\mathbb{B}}_2$. As $(\zeta, w_k) \to (\zeta, e^{i\theta}) \in \partial\mathbb{D}^2$, then by Lemma 1.4.7 we have that $g(\mathbb{D}) \subset \partial\mathbb{B}_2$ and hence g must be constant.

Let g_k' denote the derivative (we differentiate each component). The functions g_k' converge (uniformly on compacts) to $g' = 0$, so for every fixed $\zeta \in \mathbb{D}$, $\frac{\partial f}{\partial z_1}(\zeta, w_k) \to 0$. Notice that this limit holds for all $e^{i\theta}$ and a subsequence of an arbitrary sequence $\{w_k\}$ where $w_k \to e^{i\theta}$. The mapping that takes w to $\frac{\partial f}{\partial z_1}(\zeta, w)$ therefore extends continuously to $\partial\mathbb{D}$ and is zero on the boundary. We apply the maximum principle or the Cauchy formula and the fact that ξ was arbitrary to find $\frac{\partial f}{\partial z_1} \equiv 0$. By symmetry $\frac{\partial f}{\partial z_2} \equiv 0$. Therefore f is constant, which is a contradiction as f was proper. $\square$

We saw that the reason why there is not even a proper mapping is the fact that the boundary of the polydisc contained analytic discs, while the sphere did not. Similar proof extends to higher dimensions as well. In fact, it is not hard to prove the following theorem.

Theorem 1.4.8. *Let $U = U' \times U'' \subset \mathbb{C}^n \times \mathbb{C}^k$ and $V \subset \mathbb{C}^m$ be bounded domains such that ∂V contains no analytic discs. Then there exist no proper holomorphic mappings $f\colon U \to V$.*

***Exercise* 1.4.3:** *Prove this theorem.*

The key take-away from this section is that in several variables, when looking at which domains are equivalent, it is the geometry of the boundaries makes a difference, not just the topology of the domains.

There is a fun exercise in one dimension about proper maps of discs:

***Exercise* 1.4.4:** *Let $f\colon \mathbb{D} \to \mathbb{D}$ be a proper map. Then*

$$f(z) = e^{i\theta} \prod_{j=1}^{k} \frac{z-a_j}{1-\bar{a}_j z},$$

for some real θ and some $a_j \in \mathbb{D}$ (that is, f is a finite Blaschke product). Hint: Consider the set $f^{-1}(0)$.

In several dimensions when $\mathbb{D}$ is replaced by a ball, this question (what are the proper maps) becomes much more involved and when the dimensions of the balls are different, it is not solved in general.

1.5 Cartan's uniqueness theorem*

The following theorem can be thought of as another analogue of Schwarz's lemma to several variables. It says that for a bounded domain, it is enough to know that a self mapping is the identity at a single point to show that it is the identity everywhere. As there are quite a few theorems named for Cartan, this one is often referred to as the *Cartan's uniqueness theorem*. It can be very useful in computing the automorphism groups of certain domains. In fact as an exercise you will use it to compute the automorphism groups of $\mathbb{B}_n$ and $\mathbb{D}^n$.

Theorem 1.5.1 (Cartan)**.** *Suppose $U \subset \mathbb{C}^n$ is a bounded domain, $a \in U$, and $f\colon U \to U$ is a holomorphic mapping such that $f(a) = a$ and $Df(a)$ is the identity. Then $f(z) = z$ for all $z \in U$.*

***Exercise* 1.5.1:** *Find a counterexample if U is not unbounded. Hint: For simplicity take $a = 0$ and $U = \mathbb{C}^n$.*

Before we get into the proof let us figure out how to write down the Taylor series of a function in a nicer way, splitting it up into parts of different degree.

A polynomial $P\colon \mathbb{C}^n \to \mathbb{C}$ is said to be *homogeneous* of degree d if

$$P(sz) = s^d P(z)$$

for all $s \in \mathbb{C}$ and $z \in \mathbb{C}^n$. A homogeneous polynomial of degree d is a polynomial whose every monomial is of total degree d. For example, $z^2w - iz^3 + 9zw^2$ is homogeneous of degree 3 in the variables $(z,w) \in \mathbb{C}^2$. A polynomial vector valued mapping is homogeneous, if each component is. If f is holomorphic near $a \in \mathbb{C}^n$, then write the power series of f at a as

$$\sum_{j=0}^{\infty} f_j(z-a),$$

where f_j is a homogeneous polynomial of degree j. The f_j is then called the *degree d homogeneous part* of f at a. The f_j would be vector valued if f is vector valued, such as in the statement of the theorem. In the proof, we will require the vector valued Cauchy estimates (exercise below)*.

***Exercise* 1.5.2:** *Prove a vector-valued version of the Cauchy estimates. Suppose that $f\colon \overline{\Delta_r(a)} \to \mathbb{C}^m$ is continuous function holomorphic on a polydisc $\Delta_r(a) \subset \mathbb{C}^n$. Let T denote the distinguished boundary of Δ. Show that for any multi-index α we get*

$$\left\lVert \frac{\partial^{|\alpha|} f}{\partial z^{\alpha}}(a) \right\rVert \leq \frac{\alpha!}{r^{\alpha}} \sup_{z \in T} \lVert f(z) \rVert .$$

Proof of Cartan's uniqueness theorem. Without loss of generality, assume $a = 0$. Write f as a power series at the origin, written in homogeneous parts:

$$f(z) = z + f_k(z) + \sum_{j=k+1}^{\infty} f_j(z),$$

where $k \geq 2$ is an integer such that $f_j(z)$ is zero for all $2 \leq j < k$. The degree 1 homogeneous part is simply the vector z as the derivative of the mapping at the origin is the identity. Compose f with itself ℓ times:

$$f^{\ell}(z) = \underbrace{f \circ f \circ \cdots \circ f}_{\ell \text{ times}}(z).$$

As $f(U) \subset U$, then f^{ℓ} is a holomorphic map of U to U. As U is bounded, there is an M such that $\lVert z \rVert \leq M$ for all U. Therefore $\lVert f(z) \rVert \leq M$ for all $z \in U$, and $\lVert f^{\ell}(z) \rVert \leq M$ for all $z \in U$.

*The normal Cauchy estimates could also be used in the proof of Cartan by applying them componentwise.

By direct computation we get

$$f^\ell(z) = z + \ell f_k(z) + \sum_{j=k+1}^{\infty} \tilde{f}_j(z),$$

for some other polynomials $\tilde{f}_j$ that are degree j homogeneous. Suppose $\Delta_r(0)$ is a polydisc whose closure is in U. Via Cauchy estimates, for any multinomial α with $|\alpha| = k$,

$$\frac{\alpha!}{r^\alpha} M \geq \left\| \frac{\partial^{|\alpha|} f^\ell}{\partial z^\alpha}(0) \right\| = \ell \left\| \frac{\partial^{|\alpha|} f}{\partial z^\alpha}(0) \right\|.$$

The inequality holds for all $\ell \in \mathbb{N}$, and so $\frac{\partial^{|\alpha|} f}{\partial z^\alpha}(0) = 0$. Therefore $f_k \equiv 0$. Hence $f(z) = z$, as there is no other nonzero homogeneous part in the expansion of f. $\square$

As an application, let us classify all biholomorphisms of all bounded circular domains that fix a point. A *circular domain* is a domain $U \subset \mathbb{C}^n$ such that if $z \in U$, then $e^{i\theta} z \in U$ for all $\theta \in \mathbb{R}$.

Corollary 1.5.2. *If $U, V \subset \mathbb{C}^n$ are bounded circular domain with $0 \in U$, $0 \in V$, and $f \colon U \to V$ is a biholomorphic map such that $f(0) = 0$. Then f is linear.*

For example $\mathbb{B}_n$ is circular and bounded, so a biholomorphism of $\mathbb{B}_n$ that fixes the origin is linear. Similarly a polydisc centered at zero is also circular and bounded.

Proof. The map $g(z) = f^{-1}\bigl(e^{-i\theta} f(e^{i\theta} z)\bigr)$ is an automorphism of U (a biholomorphic map of U to U) and via the chain-rule, $g'(0) = I$. Therefore $f^{-1}\bigl(e^{-i\theta} f(e^{i\theta} z)\bigr) = z$, or in other words

$$f(e^{i\theta} z) = e^{i\theta} f(z).$$

Write f near zero as $f(z) = \sum_{j=1}^{\infty} f_j(z)$ where f_j are homogeneous polynomials of degree j (notice that $f_0 = 0$). Then

$$e^{i\theta} \sum_{j=1}^{\infty} f_j(z) = \sum_{j=1}^{\infty} f_j(e^{i\theta} z) = \sum_{j=1}^{\infty} e^{ij\theta} f_j(z).$$

By the uniqueness of the Taylor expansion, $e^{i\theta} f_j(z) = e^{ij\theta} f_j(z)$, or $f_j(z) = e^{i(j-1)\theta} f_j(z)$, for all j, all z, and all θ. If $j \neq 1$ we obtain that $f_j \equiv 0$, which proves the claim. $\square$

***Exercise* 1.5.3:** *Show that every automorphism f of $\mathbb{D}^n$ (that is a biholomorphism $f \colon \mathbb{D}^n \to \mathbb{D}^n$) is given as*

$$f(z) = P\left(e^{i\theta_1} \frac{z_1 - a_1}{1 - \bar{a}_1 z_1}, e^{i\theta_2} \frac{z_2 - a_2}{1 - \bar{a}_2 z_2}, \dots, e^{i\theta_n} \frac{z_n - a_n}{1 - \bar{a}_n z_n} \right)$$

for $\theta \in \mathbb{R}^n$, $a \in \mathbb{D}^n$, and a permutation matrix P.

***Exercise* 1.5.4:** *Given $a \in \mathbb{B}_n$, define the linear map $P_a z = \frac{\langle z,a\rangle}{\langle a,a\rangle} a$ if $a \neq 0$ and $P_0 z = 0$. Let $s_a = \sqrt{1-\|a\|^2}$. Show that every automorphism f of $\mathbb{B}_n$ (that is a biholomorphism $f\colon \mathbb{B}_n \to \mathbb{B}_n$) can be written as*

$$f(z) = U \frac{a - P_a z - s_a (I - P_a) z}{1 - \langle z, a\rangle}$$

for a unitary matrix U and some $a \in \mathbb{B}_n$.

***Exercise* 1.5.5:** *Using the previous two exercises, show that $\mathbb{D}^n$ and $\mathbb{B}_n$, $n \geq 2$, are not biholomorphic via a method more in the spirit of what Poincaré used: Show that the groups of automorphisms of the two domains are different groups when $n \geq 2$.*

***Exercise* 1.5.6:** *Suppose $U \subset \mathbb{C}^n$ is a bounded domain, $a \in U$, and $f\colon U \to U$ is a holomorphic mapping such that $f(a) = a$. Show that every eigenvalue λ of the matrix $Df(a)$ satisfies $|\lambda| \leq 1$.*

***Exercise* 1.5.7** (Tricky)**:** *Find a domain $U \subset \mathbb{C}^n$ such that the only biholomorphism $f\colon U \to U$ is the identity $f(z) = z$. Hint: Take the polydisc (or the ball) and remove some number of points (be careful in how you choose them). Then show that f extends to a biholomorphism of the polydisc. Then see what happens to those points you took out.*

1.6 Riemann extension theorem, zero sets, and injective maps*

Let us extend a very useful theorem from one dimension to several dimensions. In one dimension if a function is holomorphic in $U \setminus \{p\}$ and locally bounded in U, in particular bounded near p, then the function extends holomorphically to U. In several variables the same theorem holds, and the analogue of a single point is the zero set of a holomorphic function.

Theorem 1.6.1 (Riemann extension theorem)**.** *Let $U \subset \mathbb{C}^n$ be a domain and let $g \in \mathscr{O}(U)$ that is not identically zero. Let $N = g^{-1}(0)$ be the zero set of g. Suppose that $f \in \mathscr{O}(U \setminus N)$ and suppose that f is locally bounded in U. Then there exists an $F \in \mathscr{O}(U)$ such that $F|_{U \setminus N} = f$.*

The proof will be to apply the Riemann extension theorem from one dimension.

Proof. Take any complex line L in $\mathbb{C}^n$, that is, an image of an affine mapping $\varphi\colon \mathbb{C} \to \mathbb{C}^n$ defined by $\varphi(\xi) = a\xi + b$, for two vectors $a, b \in \mathbb{C}^n$. If L goes through a point $p \in N$, that is say $b = p$, then $g\colon \varphi$ is holomorphic function of one variable. It either is identically zero, or the zero at $\xi = 0$. If it would be identically zero for all lines going through p, then g would be identically zero in a neighborhood of p and hence everywhere in U. So there is one line L such that $L \cap N$ has p as an isolated point.

Without loss of generality suppose that $p = 0$, and L is the line obtained by setting all $z_2, \ldots, z_n = 0$. Write $z' = (z_2, \ldots, z_n)$. There is some small $r > 0$ such that g is nonzero on the set given by $|z_1|^2 = r$ and $z' = 0$. By continuity of g, g is never zero on the set given by $|z_1|^2 = r$ and $\|z'\| < \varepsilon$.

For any fixed small $s \in \mathbb{C}^{n-1}$, with $\|s\| < \varepsilon$, the zero of g are isolated on the set where $z' = s$. For $\|z'\| < \varepsilon$, write

$$F(z_1, z') = \frac{1}{2\pi i} \int_{|\xi|=r} \frac{f(\xi, z')}{\xi - z_1} \, d\xi$$

The function $\xi \to f(\xi, z')$ extends holomorphically to the entire disc of radius r by the Riemann extension from one dimension. Therefore, F is equal to f on the points where they are both defined. By differentiating under the integral, the function F is holomorphic.

We made the extension locally so we have to show that the extension is unique. By the identity theorem, $g^{-1}(0)$ has empty interior, so as any extension is continuous, it must be unique. □

The set of zeros of a holomorphic function has nice structure at most points.

Theorem 1.6.2. *Let $U \subset \mathbb{C}^n$ be a domain and $f \in \mathscr{O}(U)$ and f is not identically zero. Let $N = f^{-1}(0)$. If N is nonempty, then an open dense set of $p \in N$ is* regular. *That is, after possibly reordering variables, N can be written locally near p as*

$$z_n = g(z_1, \ldots, z_{n-1})$$

for a holomorphic function g.

Proof. Clearly once we show that one regular point exists, then a whole neighborhood of p in N are regular points. Then we are done since we can repeat the procedure near each point of N.

Since f is not identically zero, then not all derivatives of f can vanish identically on X. So pick a derivative of order k such that all derivatives of order less than k vanish identically on X. We obtain a function $h \colon U \to \mathbb{C}$, holomorphic, and such that without loss of generality the z_n derivative does not vanish identically on N. Then there is some point $p \in N$ such that $\frac{\partial h}{\partial z_n}(p) \neq 0$. We apply the implicit function theorem at p to find g such that

$$h\big(z_1, \ldots, z_{n-1}, g(z_1, \ldots, z_{n-1})\big) = 0,$$

and the solution $z_n = g(z_1, \ldots, z_{n-1})$ is the unique one in $h = 0$ near p.

Near p we have that the zero set of h contains the zero set of f, and we need to show equality. Let $p = (p', p_n)$. Then the function

$$\xi \mapsto f(p', \xi)$$

has a zero in a small disc around p_n. By Rouche's theorem $\xi \mapsto f(z', \xi)$ must have a zero for z' sufficiently close to p'. It follows that since g was giving the unique solution near p and the zeros of f are contained in the zeros of h, we are done. □

The zero set of a holomorphic function is a so-called *analytic set*, although the general definition of an analytic set is a little more complicated, and includes more sets. The points on an analytic set that are not regular are called *singular points*. The set of regular points is what is called an $n-1$ dimensional *complex submanifold*.

Exercise* 1.6.1*: *Find all the regular points of the analytic set* $X = \{z \in \mathbb{C}^2 : z_1^2 = z_2^3\}$.

Let us now prove that a one-to-one holomorphic mapping is biholomorphic. The theorem is really about holomorphically extending the inverse of the mapping past a small set.

Theorem 1.6.3. *Suppose that $U \subset \mathbb{C}^n$ is a domain and $f \colon U \to \mathbb{C}^n$ is one-to-one. Then the Jacobian determinant is never equal to zero on U.*

In particular if $f \colon U \to V$ is one-to-one and onto for two domains $U, V \subset \mathbb{C}^n$, then f is biholomorphic.

The function f is locally biholomorphic on the set where the Jacobian determinant J_f, that is the determinant

$$J_f(z) = \det Df(z) = \det \left[\frac{\partial f_j}{\partial z_k}(z) \right]_{jk}$$

is not zero. The trick is what happens to f^{-1} at the other points.

Proof. We proceed by induction. It is standard that the theorem is true for $n = 1$, and suppose we know the theorem is true for dimension $n-1$.

Suppose for contradiction that $J_f = 0$ somewhere. The Jacobian determinant cannot be identically zero, for example, by the classical theorem of Sard the set of critical values (the image of the set where Jacobian determinant vanishes) is a null set.

Let us find a regular point q on the zero set of J_f. Write the zero set of J_f near q as

$$z_n = g(z_1, \ldots, z_{n-1})$$

for some holomorphic g. If we prove the theorem near q we are done. So without loss of generality we can assume that $q = 0$ and that U is a small neighborhood of 0. The map

$$F(z_1, \ldots, z_n) = (z_1, z_2, \ldots, z_{n-1}, z_n - g(z_1, \ldots, z_{n-1})$$

takes the zero set of J_f to the set $z_n = 0$. Let us therefore assume that the $J_f = 0$ precisely on the set $z_n = 0$.

We wish to show that all the derivatives of f in the $z_1, \ldots, z_{n-1}$ variables vanish on $z_n = 0$, this would clearly contradict f being one-to-one.

Suppose without loss of generality that $\frac{\partial f_1}{\partial z_1}$ is nonzero at the origin, and suppose that $f(0) = 0$. The map

$$G(z_1, \ldots, z_n) = \big(f_1(z), z_2, \ldots, z_n\big)$$

is biholomorphic on a small neighborhood of the origin. The function $f \circ G^{-1}$ is holomorphic and one-to-one on a small neighborhood. Furthermore by definition of G,

$$f \circ G^{-1}(w_1, \ldots, w_n) = \big(w_1, h(w)\big).$$

For any small $\xi \in \mathbb{C}$ near the origin, the mapping

$$\phi(w_2,\ldots,w_n) = h(\xi, w_2,\ldots,w_n)$$

is one-to-one holomorphic mapping of a neighborhood of the origin in $\mathbb{C}^{n-1}$. By induction hypothesis, the Jacobian determinant of φ is nowhere zero.

If we differentiate $f \circ G^{-1}$ we notice $D(f \circ G^{-1}) = Df \circ D(G^{-1})$ so

$$\det D(f \circ G^{-1}) = \big(\det Df\big)\big(\det D(G^{-1})\big) = 0.$$

We obtain a contradiction

$$\det D(f \circ G^{-1}) = \det Dh = \det D\phi \neq 0. \qquad \square$$

Exercise* 1.6.2*: *Take the analytic set $X = \{z \in \mathbb{C}^2 : z_1^2 = z_2^3\}$. Find a one-to-one holomorphic mapping $f\colon \mathbb{C} \to X$. Then note that the derivative of f vanishes at a certain point. So Theorem 1.6.3 only works in same dimension.*

Exercise* 1.6.3*: *Show that the complement of the zero set of a holomorphic function is connected.*

Exercise* 1.6.4*: *Find a continuous function $f\colon \mathbb{R} \to \mathbb{R}^2$ that is one-to-one but such that the inverse $f^{-1}\colon f(\mathbb{R}) \to \mathbb{R}$ is not continuous.*

We can now state a well-known and as yet unsolved conjecture (and most likely ridiculously hard to solve): the *Jacobian conjecture*. This conjecture asks for a converse to the above theorem in a special case. *Suppose $F\colon \mathbb{C}^n \to \mathbb{C}^n$ is a polynomial map (each component is a polynomial) and the Jacobian derivative J_F is never zero, then F is one-to-one.* Clearly F would be locally one-to-one, but proving (or disproving) the global statement is the content of the conjecture.

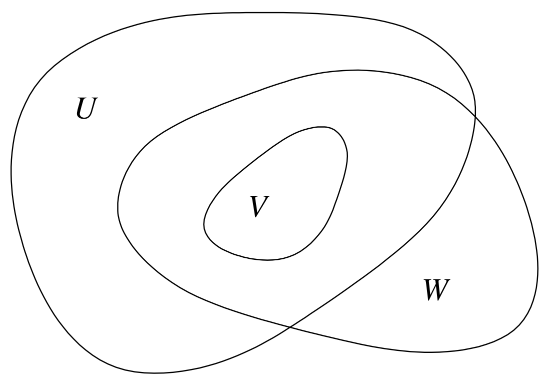
Chapter 2

Convexity and pseudoconvexity

2.1 Domains of holomorphy and holomorphic extensions

It turns out that not every domain in $\mathbb{C}^n$ is a natural domain for holomorphic functions.

Definition 2.1.1. Let $U \subset \mathbb{C}^n$ be a domain (connected open set). The set U is called a *domain of holomorphy* if there do not exist open sets V and W, with $V \subset U \cap W$, $W \not\subset U$, and W connected such that for every $f \in \mathscr{O}(U)$ there exists an $\widetilde{f} \in \mathscr{O}(W)$ such that $f(z) = \widetilde{f}(z)$ for all $z \in V$.



Example 2.1.2: The unit ball $\mathbb{B}_n \subset \mathbb{C}^n$ is a domain of holomorphy. Proof: Suppose we have V, W, and $\widetilde{f}$ as in the definition. As W is connected and open, it is path connected. There exist points in W that are not in $\mathbb{B}_n$, so there is a path γ in W that goes from a point $q \in V$ to some $p \in \partial\mathbb{B}_n \cap W$. Without loss of generality (after composing with rotations, that is unitary matrices), we assume that $p = (1,0,0,\ldots,0)$. Take the function $f(z) = \frac{1}{1-z_1}$. The function $\widetilde{f}$ must agree with f on the component of $\mathbb{B}_n \cap W$ that contains q. But that component also contains p and so $\widetilde{f}$ must blow up (in particular it cannot be holomorphic) at p. The contradiction shows that no V and W exist.

In one dimension this notion has no real meaning. Every domain is a domain of holomorphy.

Exercise* 2.1.1** (Easy):*** *In $\mathbb{C}$ every domain is a domain of holomorphy.*

Exercise* 2.1.2*: *If $U_j \subset \mathbb{C}^n$ are domains of holomorphy, then the intersection*

$$\bigcap_j U_j$$

is either empty or a domain of holomorphy.

Exercise* 2.1.3** (Easy):*** *Show that a polydisc in $\mathbb{C}^n$ is a domain of holomorphy.*

Exercise* 2.1.4*: *a) Given a $p \in \partial\mathbb{B}_n$, find a function f holomorphic on $\mathbb{B}_n$ C^∞-smooth on $\overline{\mathbb{B}_n}$ that does not extend past p. Hint: For the principal branch of $\sqrt{\cdot}$ the function $\xi \mapsto e^{-1/\sqrt{\xi}}$ is holomorphic for $\operatorname{Re}\xi > 0$ and can be extended to be continuous (even smooth) on all of $\operatorname{Re}\xi \geq 0$. b) Find a function f holomorphic on $\mathbb{B}_n$ that does not extend past any point of $\partial\mathbb{B}_n$.*

Exercise* 2.1.5*: *Show that a convex domain in $\mathbb{C}^n$ is a domain of holomorphy.*

In the following when we say $f \in \mathscr{O}(U)$ extends holomorphically to V where $U \subset V$, we will mean that there exists a function $\widetilde{f} \in \mathscr{O}(V)$ such that $f = \widetilde{f}$ on U.

Remark 2.1.3. Do note that the subtlety of the general definition is that it does not necessarily mean that the functions extend to a larger set, since we must take into account single-valuedness. For example, take f to be the principal branch of the logarithm defined on $U = \mathbb{C} \setminus \{z : \operatorname{Im} z = 0, \operatorname{Re} z \leq 0\}$. We can define locally an extension from one side through the boundary of the domain, but we cannot define an extension on a larger set that contains U. This example should be motivation why we need V to possibly be a subset of $U \cap W$, and why W need not include all of U.

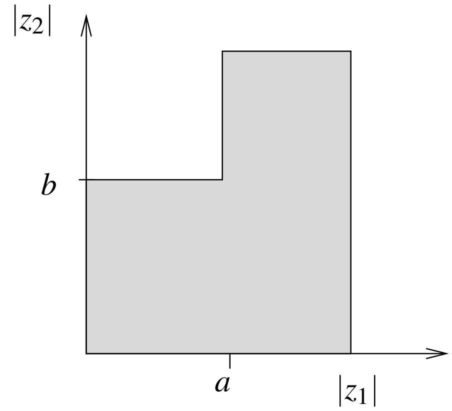
In several dimensions not every domain is a domain of holomorphy. We have the following theorem. The domain H in the theorem is called the *Hartogs figure*.

Theorem 2.1.4. *Let $(z,w) = (z_1,\ldots,z_m,w_1,\ldots,w_k) \in \mathbb{C}^m \times \mathbb{C}^k$ be the coordinates. For two numbers $0 < a,b < 1$. Let the set $H \subset \mathbb{D}^{m+k}$ be defined by*

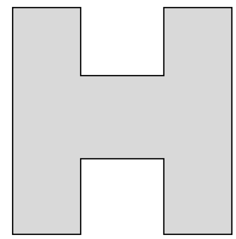
$$H = \{(z,w) \in \mathbb{D}^{m+k} : |z_j| > a \textit{ for } j = 1,\ldots,m\} \cup \{(z,w) \in \mathbb{D}^{m+k} : |w_j| < b \textit{ for } j = 1,\ldots,k\}.$$

If $f \in \mathscr{O}(H)$, then f extends holomorphically to $\mathbb{D}^{m+k}$.

In $\mathbb{C}^2$ if $m = 1$ and $k = 1$, the figure looks like:



In diagrams, often the Hartogs figure is drawn as:



Proof. Pick a $c \in (a,1)$. Let

$$\Gamma = \{ z \in \mathbb{D}^m : |z_j| = c \text{ for } j = 1, \ldots, m \}.$$

That is, Γ is the distinguished boundary of $c\mathbb{D}^m$, a polydisc centered at 0 of radius c in $\mathbb{C}^m$. We use the Cauchy formula in the first m variables and define the function F

$$F(z,w) = \frac{1}{(2\pi i)^m} \int_\Gamma \frac{f(\xi,w)}{\xi - z} \, d\xi.$$

Clearly F is well defined on all of

$$c\mathbb{D}^m \times \mathbb{D}^k$$

as ξ only ranges through Γ and so as long as $w \in \mathbb{D}^k$ then $(\xi, w) \in H$.

The function F is holomorphic in w as we can differentiate underneath the integral and f is holomorphic in w on H. Furthermore, F is holomorphic in z as the kernel $\frac{1}{\xi - z}$ is holomorphic in z as long as $z \in c\mathbb{D}^m$.

When $|w_j| < b$ for all j, then we know that $F(z,w) = f(z,w)$ for all $z \in c\mathbb{D}^m$. Therefore, F and f are equal on an open subset of H, and hence they are equal everywhere where their domains intersect. It is now easy to see that combining F and f we obtain a holomorphic function on all of $\mathbb{D}^{m+k}$ that extends f. □

We use this theorem in many situations to extend holomorphic functions. We usually need to translating, scaling, rotating, and even take more general biholomorphic mappings of H to wherever we need it. The corresponding polydisc—or the image of the polydisc under the appropriate biholomorphic mapping if one was used—to which all holomorphic functions on H extend is denoted by $\widehat{H}$ and is called the *hull* of H.

Let us state a simple but useful case of the so-called *Hartogs phenomenon*.

Corollary 2.1.5. *Let $U \subset \mathbb{C}^n$, $n \geq 2$, be a domain and $p \in U$. Then every $f \in \mathscr{O}(U \setminus \{p\})$ extends to U. In particular, holomorphic functions in several variables have no isolated zeros.*

Proof. Without loss of generality, by translating and scaling (those operations are after all holomorphic), we can assume that $p = (\frac{3}{4}, 0, \ldots, 0)$ and the unit polydisc $\mathbb{D}^n$ is contained in U. We fit a Hartogs figure H in U by letting $m = 1$ and $k = n-1$, writing $\mathbb{C}^n = \mathbb{C}^1 \times \mathbb{C}^{n-1}$, and taking $a = b = \frac{1}{2}$. Then $H \subset U$, and $p \in \mathbb{D}^n \setminus H$. By applying the extension theorem we know that f extends to be holomorphic at p.

Next, if $f \in \mathscr{O}(U)$ had an isolated zero at p, then $\frac{1}{f}$ would be holomorphic in some neighborhood of p but not at p itself. And it would not be possible to extend f through p (not even continuously let alone holomorphically). $\square$

The extension works in an even more surprising fashion. We could take out a very large set:

***Exercise* 2.1.6:** *Suppose $U \subset \mathbb{C}^n$, $n \geq 2$, be a domain and $K \subset\subset U$ is a compact geometrically convex subset. If $f \in \mathscr{O}(U \setminus K)$ then f extends to be holomorphic in U. Hint: Find a nice point on ∂K and try extending a little bit. Then make sure your extension is single-valued.*

Convexity of K is not needed; we only need that $U \setminus K$ is connected, however, the proof is much harder. The singlevaluedness of the extension is the key point that makes the general proof harder.

Notice the surprising fact that any holomorphic function on

$$\mathbb{B}_n \setminus \overline{B_{1-\varepsilon}(0)} = \{z \in \mathbb{C}^n : 1-\varepsilon < \|z\| < 1\}$$

for any $\varepsilon > 0$ automatically extends to a holomorphic function of $\mathbb{B}_n$. We need $n > 1$, the extension result decisively does not work in one dimension; for example take $1/z$. Notice that if $n \geq 2$, then if $f \in \mathscr{O}(\mathbb{B}_n)$ the set of its zeros must "touch the boundary" or be empty. If the set of zeros were in fact compact in $\mathbb{B}_n$, then we could try to extend the function $1/f$.

***Exercise* 2.1.7** (Hartogs triangle)**:** *Let*

$$T = \{(z_1, z_2) \in \mathbb{D}^2 : |z_2| < |z_1|\}.$$

Show that T is a domain of holomorphy, then show that if

$$\widetilde{T} = T \cup B_\varepsilon(0)$$

for arbitrarily small $\varepsilon > 0$, then $\widetilde{T}$ is not a domain of holomorphy and in fact every function holomorphic on $\widetilde{T}$ extends to a holomorphic function of $\mathbb{D}^2$.

***Exercise* 2.1.8:** *Take the natural embedding of $\mathbb{R}^2 \subset \mathbb{C}^2$. Suppose $f \in \mathscr{O}(\mathbb{C}^2 \setminus \mathbb{R}^2)$. Show that f extends to be holomorphic in all of $\mathbb{C}^2$.*

Exercise* 2.1.9*: *Suppose*

$$U = \{(z,w) \in \mathbb{D}^2 : 1/2 < |z|\}.$$

Draw U. Let $\gamma = \{z : |z| = 3/4\}$ oriented positively. If $f \in \mathscr{O}(U)$, then show that the function

$$F(z,w) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(\xi,w)}{\xi - z}\, d\xi$$

is well defined in $\bigl((3/4)\mathbb{D}\bigr) \times \mathbb{D}$, holomorphic where defined, yet it is not necessarily true that $F = f$ on the intersections of their domains.

Exercise* 2.1.10*: *Suppose $U \subset \mathbb{C}^n$ is an open set such that for every $z \in \mathbb{C}^n \setminus \{0\}$, there is a $\lambda \in \mathbb{C}$ such that $\lambda z \in U$. Let $f \colon U \to \mathbb{C}$ be holomorphic with $f(\lambda z) = f(z)$ whenever $z \in U$, $\lambda \in \mathbb{C}$ and $\lambda z \in U$. a) (easy) Prove that f is constant. b) (hard) Relax the requirement on f to being meromorphic, that is $f = g/h$ for holomorphic g and h, find a nonconstant example and prove that such an f must be rational (that is g and h must be polynomials).*

2.2 Tangent vectors, the Hessian, and convexity

An exercise in the previous section showed that any convex domain is a domain of holomorphy. However, classical convexity is too strong.

Exercise* 2.2.1*: *Show that if $U \subset \mathbb{C}^m$ and $V \subset \mathbb{C}^k$ are both domains of holomorphy, then $U \times V$ is a domain of holomorphy.*

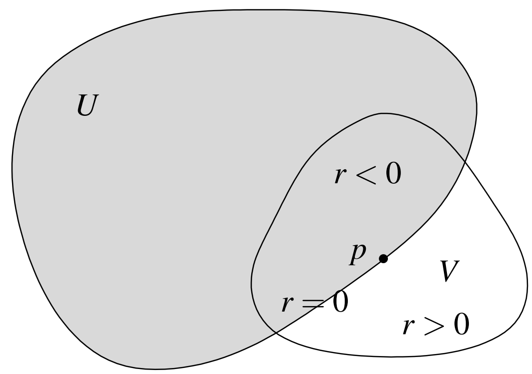
In particular, the exercise says that given any domain $U \subset \mathbb{C}$ and any domain $V \subset \mathbb{C}$, the domain $U \times V$ is a domain of holomorphy in two variables. The domains U and V, and therefore $U \times V$ can be spectacularly non-convex. But we should not discard convexity completely, there is a notion of *pseudoconvexity*, which vaguely means "convexity in the complex directions" that is the correct notion to classify which domains are domains of holomorphy.

Definition 2.2.1. A set $M \subset \mathbb{R}^n$ is a real C^k-smooth *hypersurface* if at each point $p \in M$, there exists a k-times continuously differentiable function $r \colon V \to \mathbb{R}$, defined in a neighborhood V of p with nonvanishing derivative such that $M \cap V = \{x \in V : r(x) = 0\}$. The function r is called the *defining function* (at p).

A domain U with *C^k-smooth boundary* is a domain where ∂U is a C^k-smooth hypersurface, where we further require for any defining function r of ∂U, that $r < 0$ for points in U and $r > 0$ for points not in U.

If we say simply *smooth* we mean C^∞-smooth.

In fact for simplicity in these notes we will generally deal with smooth (that is, C^∞) functions and hypersurfaces only. Dealing with C^k-smooth functions for finite k introduces technicalities that make certain arguments unnecessarily difficult.



Notice that the definition for a smooth boundary is not just that the boundary is a smooth hypersurface, that is not enough. It also says that one side of that hypersurface is in U and one side is not in U. That is because if the derivative of r never vanishes, then r must have different signs on different sides of $\{x : r(x) = 0\}$. The verification of this fact is left to the reader (Hint: look at where the gradient points).

Same definition works for $\mathbb{C}^n$ where we simply treat $\mathbb{C}^n$ as $\mathbb{R}^{2n}$. For example the ball $\mathbb{B}_n$ is a domain with smooth boundary with defining function $r(z,\bar{z}) = \|z\|^2 - 1$.

Definition 2.2.2. For any $p \in \mathbb{R}^n$, the set of tangent vectors $T_p\mathbb{R}^n$ is given by

$$T_p\mathbb{R}^n = \operatorname{span}_{\mathbb{R}} \left\{ \frac{\partial}{\partial x_1}\Big|_p, \ldots, \frac{\partial}{\partial x_n}\Big|_p \right\}.$$

That is a vector $X_p \in T_p\mathbb{R}^n$ is an object of the form

$$X_p = \sum_{j=1}^n a_j \frac{\partial}{\partial x_j}\Big|_p,$$

for real numbers a_j.

Let $M \subset \mathbb{R}^n$ be a smooth hypersurface, $p \in M$, and r is the defining function at p, then a vector $X_p \in T_p\mathbb{R}^n$ is tangent to M at p if

$$X_p r = 0, \qquad \text{or in other words} \qquad \sum_{j=1}^n a_j \frac{\partial r}{\partial x_j}\Big|_p = 0.$$

The space of tangent vectors to M is written as T_pM. Notice that the space T_pM is an $n-1$ dimensional real vector space.

***Exercise* 2.2.2:** *If r and $\tilde{r}$ are two smooth defining functions for M at p, show that there exists a nonzero smooth function g such that $\tilde{r} = gr$. Hint: First assume that $r = x_n$, so that M is simply the set $x_n = 0$, then show that there exists a g such that $\tilde{r} = x_n g$. Then think about a local change of variables that makes M into x_n. Hint for the hint: Notice the basic calculus fact that if $f(0) = 0$ and f is smooth then $s\int_0^1 f'(ts)\,dt = f(s)$ and $\int_0^1 f'(ts)\,dt$ is a smooth function of s.*

***Exercise* 2.2.3:** *Show that T_pM is independent of which defining function we take. That is prove that if $\tilde{r}$ is another defining function for M at p, then $\sum a_j \frac{\partial r}{\partial x_j}\Big|_p = 0$ if and only if $\sum a_j \frac{\partial \tilde{r}}{\partial x_j}\Big|_p = 0$.*

The disjoint union

$$T\mathbb{R}^n = \bigcup_{p\in M} T_p\mathbb{R}^n$$

is called the *tangent bundle*. Then a smooth *vector field* in $T\mathbb{R}^n$ is an object of the form

$$X = \sum_{j=1}^{n} a_j \frac{\partial}{\partial x_j}$$

where a_j are smooth functions. That is, X is a function $X\colon \mathbb{R}^n \to T\mathbb{R}^n$ such that $X(p) \in T_p\mathbb{R}^n$, and the vectors vary C^k-smoothly. Usually we write X_p rather than $X(p)$. To be more fancy we could say X is a *section* of $T\mathbb{R}^n$.

Similarly

$$TM = \bigcup_{p\in M} T_pM$$

is the tangent bundle of M. A vector field X in TM is a vector field such that $X_p \in T_pM$.

Now that we know what tangent vectors are, let us define convexity for domains with smooth boundary.

Definition 2.2.3. Suppose $U \subset \mathbb{R}^n$ is a domain with smooth boundary, and suppose that r is a defining function for ∂U at $p \in \partial U$ such that $r < 0$ on U.

If for all nonzero $X_p \in T_pM$,

$$X_p = \sum_{j=1}^{n} a_j \frac{\partial}{\partial x_j}\Big|_p,$$

we have

$$\sum_{j=1,\ell=1}^{n} a_j a_\ell \frac{\partial^2 r}{\partial x_j \partial x_\ell}\Big|_p \geq 0,$$

then U is said to be *convex* at p. If the inequality above is strict for all nonzero $X_p \in T_pM$, then U is said to be *strongly convex* at p.

The domain U is *convex* if it is convex at all $p \in \partial U$. Similarly U is *strongly convex* if it is strongly convex at all $p \in \partial U$.

The matrix

$$\left[\frac{\partial^2 r}{\partial x_j \partial x_\ell}\bigg|_p \right]_{j\ell}$$

is called the *the Hessian* of r at p. So, U is convex at $p \in \partial U$ if the Hessian of r at p as a bilinear form is positive definite (or positive semidefinite) when restricted to tangent vectors in $T_p \partial U$. This matrix is essentially the second fundamental form from Riemannian geometry in mild disguise (or perhaps it is the other way around).

Notice that we have cheated above a little bit since we have not proved that the notion is well defined. In particular for many points we have more than one defining function.

***Exercise* 2.2.4:** *Show that the definition of convexity is independent of the defining function. Hint: If $\tilde{r}$ is another defining function near p then there is a function $g > 0$ such that $\tilde{r} = gr$.*

***Exercise* 2.2.5:** *Show that if a domain is strongly convex at a point, then it is strongly convex at all nearby points. On the other hand find an example of a domain that is convex at one point p, but not convex at points arbitrarily near p.*

Example 2.2.4: Let us look at an example. Let us prove that the unit disc in $\mathbb{R}^2$ is convex (actually strongly convex). Let x, y be our coordinates and then our defining function is $r(x,y) = x^2 + y^2 - 1$.

The tangent space to the circle is one dimensional, so we simply need to find a single nonzero tangent vector at each point. Notice that $\nabla r = (2x, 2y)$, so it is easy to check that

$$y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}$$

is tangent to $r = 0$ when $x^2 + y^2 = 1$. It is also nonzero on the circle.

The Hessian matrix of r is

$$\begin{bmatrix} \frac{\partial^2 r}{\partial x^2} & \frac{\partial^2 r}{\partial x \partial y} \\ \frac{\partial^2 r}{\partial y \partial x} & \frac{\partial^2 r}{\partial x^2} \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}.$$

Applying the vector $(y, -x)$ gets us

$$\begin{bmatrix} y & -x \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} y \\ -x \end{bmatrix} = 2y^2 + 2x^2 = 2 > 0.$$

So the domain given by $r < 0$ is strongly convex at all points.

***Exercise* 2.2.6:** *Show that the domain in $\mathbb{R}^2$ defined by $x^4 + y^4 < 1$ is convex, but not strongly convex. Find all the points where the domain is not strongly convex.*

***Exercise* 2.2.7:** *Show that the domain in $\mathbb{R}^3$ defined by $(x_1^2 + x_2^2)^2 < x_3$ is strongly convex at all points except the origin, where it is just convex (but not strongly).*

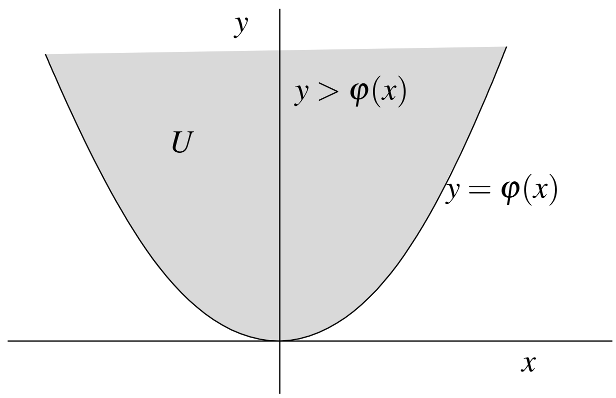
For computations it is often useful to use a more convenient defining function.

Lemma 2.2.5. *Suppose $M \subset \mathbb{R}^n$ is a smooth hypersurface, and $p \in M$. Then after a rotation and translation, p is the origin and near the origin M is defined by*

$$y = \varphi(x)$$

where $(x,y) \in \mathbb{R}^{n-1} \times \mathbb{R}$ are our coordinates and φ is a smooth function that vanishes to second order, that is $\varphi(0) = 0$ and $d\varphi(0) = 0$.

If M is the boundary of a domain U with smooth boundary and $r < 0$ on U, then the rotation can be chosen such for points in U we have $y > \varphi(x)$.



Proof. Let r be a defining equation at p. Take $v = \nabla r(p)$. By translating p to zero, and applying a rotation (an orthogonal matrix), we assume that $v = (0,0,\ldots,0,v_n)$, where $v_n < 0$. Denote our coordinates by $(x,y) \in \mathbb{R}^{n-1} \times \mathbb{R}$. As $\nabla r(0) = v$, then $\frac{\partial r}{\partial y}(0) \neq 0$. We apply the implicit function theorem to find a smooth function φ such that $r\big(x,\varphi(x)\big) = 0$ for all x in a neighborhood of the origin, and in fact that $\{(x,y) : y = \varphi(x)\}$ are all the solutions to $r = 0$ near the origin.

What is left is to show that the derivative at 0 of φ vanishes. $r\big(x,\varphi(x)\big) = 0$ for all x in a neighborhood of the origin. So let us compute for any $j = 1,\ldots,n-1$:

$$0 = \frac{\partial}{\partial x_j}\Big[r\big(x,\varphi(x)\big)\Big] = \left(\sum_{\ell=1}^{n-1} \frac{\partial r}{\partial x_\ell}\frac{\partial x_\ell}{\partial x_j}\right) + \frac{\partial r}{\partial y}\frac{\partial \varphi}{\partial x_j} = \frac{\partial r}{\partial x_j} + \frac{\partial r}{\partial y}\frac{\partial \varphi}{\partial x_j}$$

When we evaluate this at the origin, we notice that $\frac{\partial r}{\partial x_j}(0,0) = 0$ and $\frac{\partial r}{\partial y}(0,0) = v_n \neq 0$ and therefore $\frac{\partial \varphi}{\partial x_j}(0) = 0$.

To prove the final statement suppose $r < 0$ on the domain. It is enough to check that r is negative for $(0,y)$ if $y > 0$ is small, which follows as $\frac{\partial r}{\partial y}\big|_{(0,0)} < 0$. $\square$

The advantage of this representation is that the tangent plane at p can be identified with the x coordinates for the purposes of computation. Let us suppose that M is smooth for simplicity. We can write the hypersurface as

$$y = x^t H x + E(x)$$

where H is the Hessian matrix of φ at the origin, that is $H = \left[\frac{\partial^2 \varphi}{\partial x_j \partial x_k} \big|_0 \right]_{jk}$, and E vanishes to third order at the origin. That is, $E(0) = 0$, and all first and second derivatives of E vanish. If we are dealing with a domain boundary ∂U, then we pick the rotation so that $y > x^t H x + E(x)$ on U. It is an easy exercise to see that U is convex at p if H positive semidefinite and strongly convex if H is positive definite.

Exercise* 2.2.8*: *Prove the above statement about H and convexity at p.*

Exercise* 2.2.9*: *M is convex from both sides at p if and only if for a defining function r for M at p, both the set given by $r > 0$ and the set given by $r < 0$ are convex at p. Prove that if a hypersurface $M \subset \mathbb{R}^n$ is convex from both sides at all points then it is locally just a hyperplane (the zero set of a real affine function).*

There is also a geometric notion of convexity, that is, U is *geometrically convex* if for every $p, q \in U$ the line between p and q is in U, or in other words the points $tp + (1-t)q \in U$ for all $t \in [0,1]$.

Exercise* 2.2.10*: *Suppose a domain with smooth boundary is geometrically convex. Show that it is convex.*

The other direction is considerably more complicated, and we will not worry about it here. Similar difficulties will be present once we move back to several complex variables and try to relate pseudoconvexity with domains of holomorphy.

2.3 Holomorphic vectors, the Levi-form, and pseudoconvexity

As $\mathbb{C}^n$ is identified with $\mathbb{R}^{2n}$ using $z = x + iy$, we have $T_p\mathbb{C}^n = T_p\mathbb{R}^{2n}$. We write

$$\mathbb{C} \otimes T_p\mathbb{C}^n = \operatorname{span}_{\mathbb{C}} \left\{ \frac{\partial}{\partial x_1}\Big|_p, \frac{\partial}{\partial y_1}\Big|_p, \ldots, \frac{\partial}{\partial x_n}\Big|_p, \frac{\partial}{\partial y_n}\Big|_p \right\}.$$

That is, we simply replace all the real coefficients with complex ones. The space $\mathbb{C} \otimes T_p\mathbb{C}^n$ is a $2n$ dimensional complex vector space. Once we do that we notice that $\frac{\partial}{\partial z_j}\big|_p$, and $\frac{\partial}{\partial \bar{z}_j}\big|_p$ are both in

$\mathbb{C} \otimes T_p\mathbb{C}^n$, and in fact:

$$\mathbb{C} \otimes T_p\mathbb{C}^n = \operatorname{span}_{\mathbb{C}} \left\{ \frac{\partial}{\partial z_1}\Big|_p, \frac{\partial}{\partial \bar{z}_1}\Big|_p, \ldots, \frac{\partial}{\partial z_n}\Big|_p, \frac{\partial}{\partial \bar{z}_n}\Big|_p \right\}.$$

Define

$$T_p^{(1,0)}\mathbb{C}^n \overset{\text{def}}{=} \operatorname{span}_{\mathbb{C}} \left\{ \frac{\partial}{\partial z_1}\Big|_p, \ldots, \frac{\partial}{\partial z_n}\Big|_p \right\} \qquad \text{and} \qquad T_p^{(0,1)}\mathbb{C}^n \overset{\text{def}}{=} \operatorname{span}_{\mathbb{C}} \left\{ \frac{\partial}{\partial \bar{z}_1}\Big|_p, \ldots, \frac{\partial}{\partial \bar{z}_n}\Big|_p \right\}.$$

The vectors in $T_p^{(1,0)}\mathbb{C}^n$ are the *holomorphic vectors* and vectors in $T_p^{(0,1)}\mathbb{C}^n$ are the *antiholomorphic vectors*. We decompose the full tangent space as

$$\mathbb{C} \otimes T_p\mathbb{C}^n = T_p^{(1,0)}\mathbb{C}^n \oplus T_p^{(0,1)}\mathbb{C}^n.$$

A holomorphic function is one that vanishes on $T_p^{(0,1)}\mathbb{C}^n$.

Let us note what holomorphic functions do to these spaces.

Proposition 2.3.1. *Let $f: U \subset \mathbb{C}^n \to \mathbb{C}^m$ be a holomorphic function with $p \in U$. Suppose $D_{\mathbb{R}}f(p)$ is the real derivative of f at p as a mapping $D_{\mathbb{R}}f(p): T_p\mathbb{C}^n \to T_{f(p)}\mathbb{C}^m$; that is it is a real-linear mapping of $\mathbb{R}^{2n}$ to $\mathbb{R}^{2m}$. Then we naturally extend the derivative to $D_{\mathbb{C}}f(p): \mathbb{C} \otimes T_p\mathbb{C}^n \to \mathbb{C} \otimes T_{f(p)}\mathbb{C}^m$. Then*

$$D_{\mathbb{C}}f(p)\big(T_p^{(1,0)}\mathbb{C}^n\big) \subset T_{f(p)}^{(1,0)}\mathbb{C}^m \qquad \textit{and} \qquad D_{\mathbb{C}}f(p)\big(T_p^{(0,1)}\mathbb{C}^n\big) \subset T_{f(p)}^{(0,1)}\mathbb{C}^m.$$

If f is a biholomorphism, then $D_{\mathbb{C}}f(p)$ restricted to $T_p^{(1,0)}\mathbb{C}^n$ is a vector space isomorphism. Similarly for $T_p^{(0,1)}\mathbb{C}^n$.

Exercise 2.3.1: *Prove the proposition. Hint: First start with $Df(p)$ as a real $2m \times 2n$ matrix to show it extends (it is the same matrix if you think of it as a matrix). Think of $\mathbb{C}^n$ and $\mathbb{C}^m$ in terms of the zs and the $\bar{z}$s and think of f as a mapping*

$$(z, \bar{z}) \mapsto \big(f(z), \bar{f}(\bar{z})\big).$$

Write the derivative as a matrix in terms of the zs and the $\bar{z}$s and fs and $\bar{f}$s and the result will follow. That is just changing the basis.

When talking about only holomorphic functions and holomorphic vectors, when we say derivative of f, we will mean the holomorphic part of the derivative. That is at p, we mean the restriction of the real derivative of f at p to the $T_p^{(1,0)}\mathbb{C}^n$ space. In other words, if we have specific coordinates in mind, the holomorphic derivative of $f\colon \mathbb{C}^n \to \mathbb{C}^m$ can be represented as the Jacobian matrix

$$\left[\frac{\partial f_j}{\partial z_k}\right]_{jk},$$

that we have seen before.

Similarly as before we define the tangent bundles

$$\mathbb{C}\otimes T\mathbb{C}^n, \quad T^{(1,0)}\mathbb{C}^n, \quad \text{and} \quad T^{(0,1)}\mathbb{C}^n,$$

by taking the disjoint unions, and we have vector fields in these bundles.

Given a real smooth hypersurface $M \subset \mathbb{C}^n$ we can take $\mathbb{C}\otimes T_pM$. Let r be a real-valued defining function of M at p. A vector $X_p \in \mathbb{C}\otimes T_pM$ is a vector in $\mathbb{C}\otimes T_p\mathbb{C}^n$ such that $X_p r = 0$ at p. That is, write

$$X_p = \sum_{j=1}^{n} \left(a_j \frac{\partial}{\partial z_j}\Big|_p + b_j \frac{\partial}{\partial \bar{z}_j}\Big|_p \right),$$

then $X_p \in \mathbb{C}\otimes T_pM$ if

$$\sum_{j=1}^{n} \left(a_j \frac{\partial r}{\partial z_j}\Big|_p + b_j \frac{\partial r}{\partial \bar{z}_j}\Big|_p \right) = 0$$

for a defining function r of M at p. Therefore, $\mathbb{C}\otimes T_pM$ is a $2n-1$ dimensional complex vector space. We decompose $\mathbb{C}\otimes T_pM$ as

$$\mathbb{C}\otimes T_pM = T_p^{(1,0)}M \oplus T_p^{(0,1)}M \oplus B_p$$

where

$$T_p^{(1,0)}M \overset{\text{def}}{=} \left(\mathbb{C}\otimes T_pM\right) \cap \left(T_p^{(1,0)}M\right), \qquad \text{and} \qquad T_p^{(0,1)}M \overset{\text{def}}{=} \left(\mathbb{C}\otimes T_pM\right) \cap \left(T_p^{(0,1)}M\right).$$

The B_p is just the "left-over" and we really need to include it otherwise even the dimensions will not work out.

Also make sure that you understand what all the objects are. The space T_pM is a real vector space; $\mathbb{C}\otimes T_pM$, $T_p^{(1,0)}M$, $T_p^{(0,1)}M$, and B_p are complex vector spaces. Before we have that these are all vector bundles, we must have that their dimensions do not vary from point to point. The easiest way to see that is to write down a convenient local coordinates. First,

Proposition 2.3.2. *Suppose $M \subset \mathbb{C}^n$ is a smooth real hypersurface, $p \in M$, and $U \subset \mathbb{C}^n$ is a neighborhood of p. Let $f\colon U \to \mathbb{C}^n$ be a biholomorphic map. Let $D_{\mathbb{C}} f(p)$ be the complexified real derivative as before. Then*

$$D_{\mathbb{C}} f(p)\big(T_p^{(1,0)}M\big) = T_{f(p)}^{(1,0)} f(M) \qquad D_{\mathbb{C}} f(p)\big(T_p^{(0,1)}M\big) = T_{f(p)}^{(0,1)} f(M).$$

That is, the spaces are isomorphic as complex vector spaces.

Proof. Without loss of generality assume that $M \subset U$. The proof is simply the application of Proposition 2.3.1. We have

$$D_{\mathbb{C}} f(p)\big(T_p^{(1,0)}\mathbb{C}^n\big) = T_{f(p)}^{(1,0)}\mathbb{C}^n, \quad D_{\mathbb{C}} f(p)\big(T_p^{(0,1)}\mathbb{C}^n\big) = T_{f(p)}^{(0,1)}\mathbb{C}^n, \quad \text{and}$$
$$D_{\mathbb{C}} f(p)\big(\mathbb{C}\otimes T_p M\big) = \mathbb{C}\otimes T_{f(p)} f(M).$$

Then it is clear that $D_{\mathbb{C}} f(p)$ must take $T_p^{(1,0)}M$ to $T_{f(p)}^{(1,0)}f(M)$ and $T_p^{(0,1)}M$ to $T_{f(p)}^{(0,1)}f(M)$. $\square$

Proposition 2.3.3. *Suppose $M \subset \mathbb{C}^n$ is a smooth real hypersurface, $p \in M$. After a translation and rotation via a unitary matrix, $p = 0$ and near the origin M is written in variables $(z,w) \in \mathbb{C}^{n-1}\times\mathbb{C}$ as*

$$\operatorname{Im} w = \varphi(z,\bar{z},\operatorname{Re} w),$$

with the $\varphi(0)$ and $d\varphi(0) = 0$. Then

$$T_0^{(1,0)}M = \operatorname{span}_{\mathbb{C}}\left\{ \frac{\partial}{\partial z_1}\Big|_0, \ldots, \frac{\partial}{\partial z_{n-1}}\Big|_0 \right\},$$
$$T_0^{(0,1)}M = \operatorname{span}_{\mathbb{C}}\left\{ \frac{\partial}{\partial \bar{z}_1}\Big|_0, \ldots, \frac{\partial}{\partial \bar{z}_{n-1}}\Big|_0 \right\},$$
$$B_0 = \operatorname{span}_{\mathbb{C}}\left\{ \frac{\partial}{\partial(\operatorname{Re} w)}\Big|_0 \right\}.$$

In particular, $\dim_{\mathbb{C}} T_p^{(1,0)}M = \dim_{\mathbb{C}} T_p^{(0,1)}M = n-1$ *and* $\dim_{\mathbb{C}} B_p = 1$.

Proof. We apply a translation and a unitary rotation to put $p = 0$ and in the same manner as in Lemma 2.2.5, we obtain φ via a unitary. As a translation and a unitary matrix are holomorphic and in fact biholomorphic, then via Proposition 2.3.1 we obtain that the tangent spaces are all transformed correctly.

The rest of the proposition follows at once if as $\frac{\partial}{\partial(\operatorname{Im} w)}\big|_0$ is the normal vector to M at 0. $\square$

Remark 2.3.4. When M is of smaller dimension than $2n-1$ (no longer a hypersurface, but a higher codimension submanifold), then the proposition above does not hold. That is, we would still have $\dim_{\mathbb{C}} T_p^{(1,0)}M = \dim_{\mathbb{C}} T_p^{(0,1)}M$, but this number need not be constant from point to point. Fortunately, when talking about smoothly bounded domains where the boundaries are hypersurfaces, this complication does not arise.

Definition 2.3.5. Suppose $U \subset \mathbb{C}^n$ is a domain with smooth boundary, and suppose that r is a defining function for ∂U at $p \in \partial U$ such that $r < 0$ on U.

If for all nonzero $X_p \in T_p^{(1,0)}\partial U$,

$$X_p = \sum_{j=1}^{n} a_j \frac{\partial}{\partial z_j}\Big|_p,$$

we have
$$\sum_{j=1,\ell=1}^{n} \bar{a}_j a_\ell \frac{\partial^2 r}{\partial \bar{z}_j \partial z_\ell}\Big|_p \geq 0,$$
then U is said to be *pseudoconvex* at p (or *Levi pseudoconvex*). If the inequality above is strict for all nonzero $X_p \in T_p^{(1,0)} \partial U$, then U is said to be *strongly pseudoconvex*. If U is pseudoconvex, but not strongly pseudoconvex at p, then we say that U is *weakly pseudoconvex*.

The domain U is pseudoconvex if it is pseudoconvex at all $p \in \partial U$. Similarly U is strongly pseudoconvex if it is strongly pseudoconvex at all $p \in \partial U$.

For $X_p \in T_p^{(1,0)} \partial U$, the expression
$$\sum_{j=1,\ell=1}^{n} \bar{a}_j a_\ell \frac{\partial^2 r}{\partial \bar{z}_j \partial z_\ell}\Big|_p$$
is called the *Levi-form* at p. So U is pseudoconvex at $p \in \partial U$ if the Levi-form is positive (semi)definite at p.

The matrix
$$\left[\frac{\partial^2 r}{\partial \bar{z}_j \partial z_\ell}\Big|_p \right]_{j\ell}$$
is called the *the complex Hessian* of r at p. So, U is pseudoconvex at $p \in \partial U$ if the Hessian of r at p as a sesquilinear form is positive definite (or positive semidefinite) when restricted to tangent vectors in $T_p^{(1,0)} \partial U$.

Notice that the complex Hessian is not the full Hessian. Let us write down the full Hessian, using the basis of $\frac{\partial}{\partial z}$s and $\frac{\partial}{\partial \bar{z}}$s. Then the full Hessian is the symmetric matrix
$$\begin{bmatrix}
\frac{\partial^2 r}{\partial z_1 \partial z_1} & \cdots & \frac{\partial^2 r}{\partial z_n \partial z_1} & \frac{\partial^2 r}{\partial \bar{z}_1 \partial z_1} & \cdots & \frac{\partial^2 r}{\partial \bar{z}_n \partial z_1} \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
\frac{\partial^2 r}{\partial z_1 \partial z_n} & \cdots & \frac{\partial^2 r}{\partial z_n \partial z_n} & \frac{\partial^2 r}{\partial \bar{z}_1 \partial z_n} & \cdots & \frac{\partial^2 r}{\partial \bar{z}_n \partial z_n} \\
\frac{\partial^2 r}{\partial z_1 \partial \bar{z}_1} & \cdots & \frac{\partial^2 r}{\partial z_n \partial \bar{z}_1} & \frac{\partial^2 r}{\partial \bar{z}_1 \partial \bar{z}_1} & \cdots & \frac{\partial^2 r}{\partial \bar{z}_n \partial \bar{z}_1} \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
\frac{\partial^2 r}{\partial z_1 \partial \bar{z}_n} & \cdots & \frac{\partial^2 r}{\partial z_n \partial \bar{z}_n} & \frac{\partial^2 r}{\partial \bar{z}_1 \partial \bar{z}_n} & \cdots & \frac{\partial^2 r}{\partial \bar{z}_n \partial \bar{z}_n}
\end{bmatrix}.$$
So the complex Hessian is the upper right, or lower left, block. In particular it is a smaller matrix, and we also apply it only to a subspace of the complexified tangent space.

***Exercise* 2.3.2:** *If r is real valued, then the complex Hessian of r is Hermitian, that is, the matrix is equal to its conjugate transpose.*

***Exercise* 2.3.3:** *Show that pseudoconvexity is not dependent on the defining function.*

***Exercise* 2.3.4:** *Show that a convex domain is pseudoconvex, and show that strongly convex domain is strongly pseudoconvex.*

***Exercise* 2.3.5:** *Show that if a domain is strongly pseudoconvex at a point, it is strongly pseudoconvex at all nearby points.*

In particular the exercise says that the unit ball $\mathbb{B}_n$ is strongly pseudoconvex as it is strongly convex. We are generally interested what happens under a holomorphic change of variables, that is, a biholomorphic mapping. And as far as pseudoconvexity is concerned we are interested in local changes of coordinates as pseudoconvexity is a local property.

Example 2.3.6: Let us change variables to show how we write $\mathbb{B}_n$ in different local holomorphic coordinates where the Levi-form is displayed nicely. Let $\mathbb{B}_n$ be defined in the variables $Z = (Z_1,\ldots,Z_n) \in \mathbb{C}^n$ by $\|Z\| = 1$.

Let us change variables to $(z_1,\ldots,z_{n-1},w)$ where

$$z_j = \frac{Z_1}{1-Z_n} \qquad \text{for all } j = 1,\ldots,n-1, \qquad w = i\frac{1+Z_n}{1-Z_n}.$$

This is a biholomorphic mapping from the set where $Z_n \neq 1$ to the set where $w \neq -i$ (exercise). For us it is in fact sufficient to notice that the map is invertible near $(0,\ldots,0,-1)$, and that follows by simply computing the derivative. Notice that the last component is the inverse of the Cayley transform (that takes the disc to the upper half plane).

We claim that the mapping takes the unit sphere given by $\|Z\| = 1$, to the set defined by

$$\operatorname{Im} w = |z_1|^2 + \cdots + |z_{n-1}|^2,$$

and that it takes $(0,\ldots,0,-1)$ to the origin (this part is trivial). Let us check

$$\begin{aligned} |z_1|^2 + \cdots + |z_{n-1}|^2 - \operatorname{Im} w &= \left|\frac{Z_1}{1-Z_n}\right|^2 + \cdots + \left|\frac{Z_{n-1}}{1-Z_n}\right|^2 + \frac{i\frac{1+Z_n}{1-Z_n} - \overline{i\frac{1+Z_n}{1-Z_n}}}{2i} \\ &= \frac{|Z_1|^2}{|1-Z_n|^2} + \cdots + \frac{|Z_{n-1}|^2}{|1-Z_n|^2} + \frac{1+Z_n}{2(1-Z_n)} + \frac{1+\bar{Z}_n}{2(1-\bar{Z}_n)} \\ &= \frac{|Z_1|^2}{|1-Z_n|^2} + \cdots + \frac{|Z_{n-1}|^2}{|1-Z_n|^2} + \frac{1-|Z_n|^2}{|1-Z_n|^2} \end{aligned}$$

Therefore $|Z_1|^2 + \cdots + |Z_n|^2 = 1$ if and only if $\operatorname{Im} w = |z_1|^2 + \cdots + |z_n|^2$. As the map takes the point $(0,\ldots,0,-1)$ to the origin, we can think of

$$\operatorname{Im} w = |z_1|^2 + \cdots + |z_{n-1}|^2$$

as the local holomorphic coordinates at $(0,\ldots,0,-1)$ (by symmetry of the sphere we could have done this at any point by rotation). The inside of the sphere is taken to

$$\operatorname{Im} w > |z_1|^2 + \cdots + |z_{n-1}|^2.$$

In these new coordinates, the Levi-form is just the identity matrix at the origin. In particular the domain is strictly pseudoconvex.

***Exercise* 2.3.6:** *Prove the assertion in the example about the mapping being biholomorphic on the sets described above.*

Let us compute what happens to the Hessian of r under a biholomorphic change of coordinates. That is, let $f\colon U \to V$ be a biholomorphic map between two domains in $\mathbb{C}^n$, and let $r\colon V \to \mathbb{R}$ be a smooth function with nonvanishing derivative. Let us compute the Hessian of $r \circ f$. Let us compute first what happens to the non-mixed derivatives. As we have to apply chain rule twice let us write the derivatives as functions.

$$\begin{aligned}
\frac{\partial^2 (r\circ f)}{\partial z_j \partial z_k}(z) &= \frac{\partial}{\partial z_j} \sum_{\ell=1}^{n} \left(\frac{\partial r}{\partial z_\ell}\bigl(f(z)\bigr) \frac{\partial f_\ell}{\partial z_k}(z) + \frac{\partial r}{\partial \bar{z}_\ell}\bigl(f(z)\bigr) \cancelto{0}{\frac{\partial \bar{f}_\ell}{\partial z_k}(z)} \right) \\
&= \sum_{\ell,m=1}^{n} \left(\frac{\partial^2 r}{\partial z_m \partial z_\ell}\bigl(f(z)\bigr) \frac{\partial f_m}{\partial z_j}(z) \frac{\partial f_\ell}{\partial z_k}(z) + \frac{\partial^2 r}{\partial \bar{z}_m \partial z_\ell}\bigl(f(z)\bigr) \cancelto{0}{\frac{\partial \bar{f}_m}{\partial z_j}(z)} \frac{\partial f_\ell}{\partial z_k}(z) \right) \\
&\qquad + \sum_{\ell=1}^{n} \frac{\partial r}{\partial z_\ell}\bigl(f(z)\bigr) \frac{\partial^2 f_\ell}{\partial z_j \partial z_k}(z) \\
&= \sum_{\ell,m=1}^{n} \frac{\partial^2 r}{\partial z_m \partial z_\ell} \frac{\partial f_m}{\partial z_j} \frac{\partial f_\ell}{\partial z_k} + \sum_{\ell=1}^{n} \frac{\partial r}{\partial z_\ell} \frac{\partial^2 f_\ell}{\partial z_j \partial z_k}.
\end{aligned} \tag{2.1}$$

In particular, the matrix $\left[\frac{\partial^2 (r\circ f)}{\partial z_j \partial z_k}\right]$ can have different eigenvalues than the matrix $\left[\frac{\partial^2 r}{\partial z_j \partial z_k}\right]$. In fact if r has nonvanishing gradient, then using the second term, we can (locally) choose f in such a way as to make the matrix $\left[\frac{\partial^2 (r\circ f)}{\partial z_j \partial z_k}\right]$ be the zero matrix at a certain point since we can just choose the second derivatives of f arbitrarily at a point. See the exercise below. So nothing about the matrix $\left[\frac{\partial^2 r}{\partial z_j \partial z_k}\right]$ is preserved under a biholomorphic map. And that is precisely why it does not appear in the definition of pseudoconvexity. The story for $\left[\frac{\partial^2 r}{\partial \bar{z}_j \partial \bar{z}_k}\right]$ is exactly the same.

***Exercise* 2.3.7:** *Given a real function r with nonvanishing gradient at $p \in \mathbb{C}^n$. Find a local change of coordinates f at p (so f ought to be a holomorphic mapping with an invertible derivative at p) such that $\left[\frac{\partial^2 (r\circ f)}{\partial z_j \partial z_k}\right]$ and $\left[\frac{\partial^2 (r\circ f)}{\partial \bar{z}_j \partial \bar{z}_k}\right]$ are just the zero matrices.*

Let us look at the mixed derivatives:

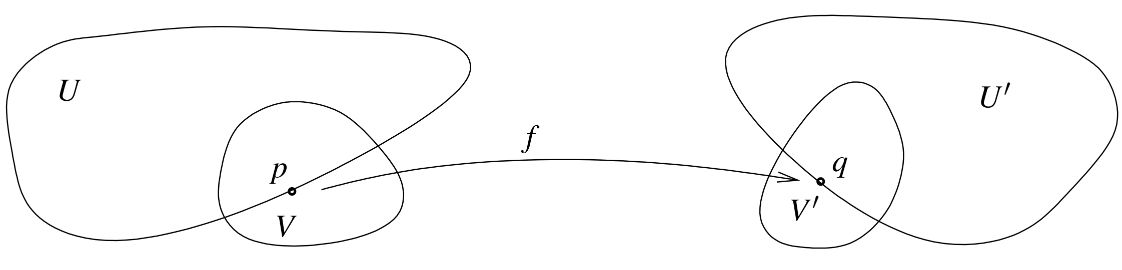
$$
\begin{aligned}
\frac{\partial^2 (r \circ f)}{\partial \bar{z}_j \partial z_k}(z) &= \frac{\partial}{\partial \bar{z}_j} \sum_{\ell=1}^{n} \left(\frac{\partial r}{\partial z_\ell}\bigl(f(z)\bigr) \frac{\partial f_\ell}{\partial z_k}(z) \right) \\
&= \sum_{\ell,m=1}^{n} \frac{\partial^2 r}{\partial \bar{z}_m \partial z_\ell}\bigl(f(z)\bigr) \frac{\partial \bar{f}_m}{\partial \bar{z}_j}(z) \frac{\partial f_\ell}{\partial z_k}(z) + \sum_{\ell=1}^{n} \frac{\partial r}{\partial z_\ell}\bigl(f(z)\bigr) \cancelto{0}{\frac{\partial^2 f_\ell}{\partial \bar{z}_j \partial z_k}(z)} \\
&= \sum_{\ell,m=1}^{n} \frac{\partial^2 r}{\partial \bar{z}_m \partial z_\ell} \frac{\partial \bar{f}_m}{\partial \bar{z}_j} \frac{\partial f_\ell}{\partial z_k}.
\end{aligned}
$$

The complex Hessian of $r \circ f$ is simply the complex Hessian H of r conjugated as D^*HD where D is the holomorphic derivative matrix of f at z and D^* is the conjugate transpose. Sylvester's Law of Inertia from linear algebra then says that the number of positive, negative, and zero eigenvalues of D^*HD is the same as that for H. The eigenvalues might have changed, but their sign did not.

In particular if H is positive definite, then D^*HD is positive definite. If a smooth hypersurface M is given by $r = 0$, then $f^{-1}(M)$ is a smooth hypersurface given by $r \circ f = 0$. The holomorphic derivative of f (given by D) takes the $T_z^{(1,0)} f^{-1}(M)$ space isomorphically to $T_{f(z)}^{(1,0)} M$. So H is positive semidefinite (resp. positive definite) on $T_{f(z)}^{(1,0)} M$ if and only if D^*HD is positive semidefinite (resp. positive definite) on $T_z^{(1,0)} f^{-1}(M)$. We have essentially proved the following theorem. That is, pseudoconvexity is a biholomorphic invariant.

Theorem 2.3.7. *Suppose $U, U' \subset \mathbb{C}^n$ are domains with smooth boundary, $p \in \partial U$, $V \subset \mathbb{C}^n$ a neighborhood of p, $q \in \partial U'$, $V' \subset \mathbb{C}^n$ a neighborhood of q, and $f : V \to V'$ a biholomorphic map with $f(p) = q$, such that $f(U \cap V) = U' \cap V'$.*

Then U is pseudoconvex at p if and only if U' is pseudoconvex at q. Similarly U is strongly pseudoconvex at p if and only if U' is strongly pseudoconvex at q.



The only thing left is to observe that if $f(U \cap V) = U' \cap V'$ then $f(\partial U \cap V) = \partial U' \cap V'$.

***Exercise* 2.3.8:** *Find an example of a bounded domain with smooth boundary that is not convex, but that is pseudoconvex.*

In fact we have proved a stronger result. We have proved that the inertia of the Levi-form is invariant under a biholomorphic change of coordinates. Let us put this together with the other observations we have made above. We find the normal form for the quadratic part of the defining equation for a smooth real hypersurface under biholomorphic transformations.

Lemma 2.3.8. *Let M be a smooth real hypersurface in $\mathbb{C}^n$ and $p \in M$. Then there exists a local holomorphic change of coordinates taking p to the origin and M to*

$$\operatorname{Im} w = \sum_{j=1}^{\alpha} |z_j|^2 - \sum_{j=\alpha+1}^{\alpha+\beta} |z_j|^2 + E(z,\bar{z},\operatorname{Re} w),$$

where E vanishes to third order, that is E and its first and second derivatives vanish at the origin. Here α is the number of positive eigenvalues of the Levi-form at p and β is the number of negative eigenvalues, and $\alpha + \beta \leq n-1$.

Below we use the *big-oh notation* and write $O(3)$ instead of E. That is, $O(3)$ means any smooth function vanishing to third order at the origin. While it is possible to do better than this proposition, it is not possible to completely get rid of the dependence on $\operatorname{Re} w$ except in the quadratic terms.

Proof. Change coordinates so that M is given by $\operatorname{Im} w = \varphi(z,\bar{z},\operatorname{Re} w)$, where φ vanishes to second order. Apply Taylor's theorem to φ up to the second order:

$$\varphi(z,\bar{z},\operatorname{Re} w) = q(z,\bar{z}) + (\operatorname{Re} w)(Lz + \overline{Lz}) + a(\operatorname{Re} w)^2 + O(3),$$

where q is quadratic, $L\colon \mathbb{C}^{n-1} \to \mathbb{C}$ is linear, and $a \in \mathbb{R}$. If $L \neq 0$, we can do a linear change of coordinates in only the z to make $Lz = z_1$. So we can assume that $Lz = \varepsilon z_1$ where $\varepsilon = 0$ or $\varepsilon = 1$.

We change variables now to let $w = w' + bw'^2 + cw'z_1$. Let us ignore $q(z,\bar{z})$ for a moment as this change of coordinates does not affect it. Also let us only look up to second order.

$$\begin{aligned}
-\operatorname{Im} w + \varepsilon(\operatorname{Re} w)(z_1+\bar{z}_1) + a(\operatorname{Re} w)^2 &= -\frac{w-\bar{w}}{2i} + \varepsilon\frac{w+\bar{w}}{2}(z_1+\bar{z}_1) + a\left(\frac{w+\bar{w}}{2}\right)^2 \\
&= -\frac{w' + bw'^2 + cw'z_1 - \bar{w}' - \bar{b}\bar{w}'^2 - \bar{c}\bar{w}'\bar{z}_1}{2i} \\
&\quad + \varepsilon\frac{w' + bw'^2 + cw'z_1 + \bar{w}' + \bar{b}\bar{w}'^2 + \bar{c}\bar{w}'\bar{z}_1}{2}(z_1+\bar{z}_1) \\
&\quad + \frac{(w' + bw'^2 + cw'z_1 + \bar{w}' + \bar{b}\bar{w}'^2 + \bar{c}\bar{w}'\bar{z}_1)^2}{4} \\
&= -\frac{w'-\bar{w}'}{2i} \\
&\quad + \frac{\big((\varepsilon i - c)w' + \varepsilon i\bar{w}'\big)z_1 + \big((\varepsilon i + \bar{c})\bar{w}' + \varepsilon i w'\big)\bar{z}_1}{2i} \\
&\quad + \frac{(ai-2b)w'^2 + (ai+2\bar{b})\bar{w}'^2 + 2iaw'\bar{w}'}{4i} + O(3).
\end{aligned}$$

We cannot quite get rid of all the quadratic terms in this equation, but we can set b and c to make the second order terms not depend on $\operatorname{Re} w'$. Setting $b = ai$ and $c = 2\varepsilon i$, and adding $q(z,\bar{z}) + O(3)$ into the mix we obtain

$$\begin{aligned}
-\operatorname{Im} w &+ q(z,\bar{z}) + \varepsilon(\operatorname{Re} w)(z_1 + \bar{z}_1) + a(\operatorname{Re} w)^2 + O(3) \\
&= -\frac{w' - \bar{w}'}{2i} + q(z,\bar{z}) - \varepsilon i \frac{w' - \bar{w}'}{2i}(z_1 - \bar{z}_1) + a\left(\frac{w' - \bar{w}'}{2i}\right)^2 + O(3) \\
&= -\operatorname{Im} w' + q(z,\bar{z}) - \varepsilon i(\operatorname{Im} w')(z_1 - \bar{z}_1) + a(\operatorname{Im} w')^2 + O(3).
\end{aligned}$$

Now the right hand side depends on $\operatorname{Im} w'$ so we have to apply the implicit function theorem to write the hypersurface as a graph again. We must solve for $\operatorname{Im} w'$. The expression for $\operatorname{Im} w'$ will vanish to second order, and therefore $-i\varepsilon(\operatorname{Im} w')(z_1 - \bar{z}_1)$ and $a(\operatorname{Im} w')^2$ vanish to third order. Therefore we can write M as a graph:

$$\operatorname{Im} w' = q(z,\bar{z}) + E(z,\bar{z},\operatorname{Re} w'),$$

where E vanishes to third order.

Next we apply the computation in (2.1). We again change variables in the w', that is, we fix the zs and we set $w' = w'' + g(z)$, where g vanishes to the second order. That is, the biholomorphic mapping is $f_j(z,w'') = z_j$ and $f_n(z,w'') = w'' + g(z)$. We let $r = -\operatorname{Im} w' + q(z,\bar{z}) + E(z,\bar{z},\operatorname{Re} w')$, so r is a function of $(z_1,\ldots,z_{n-1},w')$ and f and $(r \circ f)$ are functions of $(z_1,\ldots,z_{n-1},w'')$

The only holomorphic derivative of r that does not vanish at the origin is the w' derivative. Also the second order derivatives of r involving w' or $\bar{w}'$ all vanish at the origin. Using (2.1) at the origin for $j,k = 1,\ldots,n-1$ we get

$$\frac{\partial^2 (r \circ f)}{\partial z_j \partial z_k}\Big|_0 = \sum_{\ell,m=1}^{n-1} \frac{\partial^2 r}{\partial z_m \partial z_\ell}\Big|_0 \delta_m^j \delta_\ell^k + \frac{\partial r}{\partial w'}\Big|_0 \frac{\partial^2 g}{\partial z_j \partial z_k}\Big|_0 = \frac{\partial^2 q}{\partial z_j \partial z_k}\Big|_0 + \frac{1}{2i}\frac{\partial^2 g}{\partial z_j \partial z_k}\Big|_0.$$

Where δ_j^k is the Kronecker delta, that is, $\delta_j^j = 1$, and $\delta_j^k = 0$ if $j \neq k$. Notice the q on the right hand side. Pick the $z_j z_k$ coefficient in g such that $\frac{\partial^2 g}{\partial z_k \partial z_j}\big|_0 = \frac{-1}{2i}\frac{\partial^2 q}{\partial z_k \partial z_j}\big|_0$ making the expression vanish. The left hand side of the equation are the coefficients of the holomorphic terms in z of $r \circ f$, that is, the new q. This change of coordinates sets all the holomorphic terms of q to zero. It is left as an exercise that as q is real valued, the coefficient of $\bar{z}_j \bar{z}_k$ in q also becomes zero. Therefore, after this change of coordinates

$$q(z,\bar{z}) = \sum_{j,k=1}^{n-1} c_{jk} z_j \bar{z}_k.$$

That is, q is a sesquilinear form. Since q is real valued the matrix $C = [c_{jk}]$ must be Hermitian. In linear algebra notation, $q(z,\bar{z}) = z^* C z$, where the $*$ denotes the conjugate transpose, and we think of z as a column vector. If T is a linear transformation on the z variables we obtain $(Tz)^* C T z = z^*(T^* C T) z$. Thus, we normalize C up to $*$-congruence. A Hermitian matrix is $*$-congruent to a diagonal matrix with only 1s, -1s, and 0s on the diagonal. Writing out what that means is precisely the conclusion of the proposition. $\square$

Lemma 2.3.9 (Narasimhan)**.** *Suppose $U \subset \mathbb{C}^n$ is a domain with smooth boundary. If U is strongly pseudoconvex at $p \in \partial U$, then there exists a local holomorphic change of coordinates such that U is strongly convex in a neighborhood of p.*

> ***Exercise* 2.3.9:** *Prove the above lemma. Hint: See the proof of Lemma 2.3.8.*

The Narasimhan lemma only works at points of strong pseudoconvexity. For weakly pseudoconvex points the situation is far more complicated, and there is no simple geometric criterion.

Let us prove an easy direction of the famous *Levi-problem*. The Levi-problem was a long standing problem* in several complex variables to classify domains of holomorphy in $\mathbb{C}^n$. The answer is that a domain is a domain of holomorphy if and only if it is pseudoconvex. Just as the problem of trying to show that the classical geometric convexity is the same as convexity as we have defined it, the Levi-problem has an easier direction and a harder direction. The easier direction is to show that a domain of holomorphy is pseudoconvex, and the harder direction is to show that a pseudoconvex domain is a domain of holomorphy.

Theorem 2.3.10 (Tomato can principle)**.** *If $U \subset \mathbb{C}^n$ is a smoothly bounded domain and at some point $p \in \partial U$, the Levi-form has a negative eigenvalue, then U is not a domain of holomorphy. In particular every holomorphic function on U extends to a neighborhood of p.*

Therefore a domain of holomorphy must be pseudoconvex.

Proof. Applying what we know, we change variables so that $p = 0$, and U is given by

$$\operatorname{Im} w > -|z_1|^2 + \sum_{j=2}^{n-1} \varepsilon_j |z_j|^2 + E(z_1, z', \bar{z}_1, \bar{z}', \operatorname{Re} w),$$

where $\varepsilon_j = -1, 0, 1$, E vanishes to third order, and $z' = (z_2, \ldots, z_{n-1})$. We embed an analytic disc via $\xi \overset{\varphi}{\mapsto} (\lambda\xi, 0, 0, \ldots, 0)$ for some small $\lambda > 0$. Clearly $\varphi(0) = 0 \in \partial U$. For $\xi \neq 0$ near the origin

$$-\lambda^2|\xi|^2 + \sum_{j=2}^{n-1} \varepsilon_j |0|^2 + E(\lambda\xi, 0, \lambda\bar{\xi}, 0, 0) = -|\xi|^2 + E(\xi, 0, \bar{\xi}, 0, 0) < 0.$$

That is because by second derivative test the function has a strict minimum at $\xi = 0$. Therefore for ξ near the origin but not zero we have that $\varphi(\xi) \in U$. By picking λ small enough we can therefore assume that $\varphi(\overline{\mathbb{D}} \setminus \{0\}) \subset U$.

As $\varphi(\partial \mathbb{D})$ is compact we can "wiggle it a little" and still stay in U. In particular, for small enough $s > 0$, the disc

$$\xi \overset{\varphi_s}{\mapsto} (\lambda\xi, 0, 0, \ldots, 0, is)$$

*E. E. Levi stated the problem in 1911, but it was not completely solved until the 1950s, by Oka and others.

is entirely inside U (that is for slightly positive $\operatorname{Im} w$). Define the Hartogs figure

$$
\begin{aligned}
H = &\{(z,w) : \lambda - \varepsilon < |z_1| < \lambda + \varepsilon \text{ and } |z_j| < \varepsilon \text{ for } j = 2,\dots,n-1, \text{ and } |w - is| < s + \varepsilon\} \\
&\cup \{(z,w) : |z_1| < \lambda + \varepsilon, \text{ and } |z_j| < \varepsilon \text{ for } j = 2,\dots,n-1, \text{ and } |w - is| < \varepsilon\}.
\end{aligned}
$$

First, its because That is because the set where $|z_1| = \lambda$, $z' = 0$ and $|w| \leq s$, is inside U for all small enough s. So we can take an ε-neighborhood of that. Further for $w = is$ the whole disc where $|z_1| \leq \lambda$ is in U. So we can take an ε-neighborhood of that. We are really just taking a Hartogs figure in the z_1, w variables, and then "fattening it up" to the z' variables.

As every function holomorphic in H extends through the origin, U is not a domain of holomorphy.

□

Exercise* 2.3.10*: *Take $U \subset \mathbb{C}^2$ defined by $\operatorname{Im} w > |z|^2(\operatorname{Re} w)$. Find all the points in ∂U where U is weakly pseudoconvex and all the points where it is strongly pseudoconvex.*

Exercise* 2.3.11*: *Let $U \subset \mathbb{C}^n$ be a smoothly bounded domain that is strongly pseudoconvex at $p \in \partial U$. Show that there exists a neighborhood W of p and a smooth function $f \colon \overline{W \cap U} \to \mathbb{C}$ that is holomorphic on $W \cap U$ such that $f(p) = 1$ and $|f(z)| < 1$ for all $z \in \overline{W \cap U} \setminus \{p\}$.*

Exercise* 2.3.12*: *Suppose $U \subset \mathbb{C}^n$ is a smoothly bounded domain. Suppose there for $p \in \partial U$, there is a neighborhood W of p and a holomorphic function there is a holomorphic function $f \colon W \to \mathbb{C}$ such that the derivative of f does not vanish at p such that $f(p) = 0$ but f never zero on $W \cap U$. Show that U is pseudoconvex at p. Hint: you will need the holomorphic implicit function theorem, see Exercise 1.3.3. Note: the result does not require the derivative of f to not vanish, but is much harder to prove without that hypothesis.*

2.4 Plurisubharmonic functions and pseudoconvexity

Let us start with harmonic and subharmonic functions.

Definition 2.4.1. A C^2-smooth function $f \colon U \subset \mathbb{R}^n \to \mathbb{R}$ is called *harmonic* if*

$$\nabla^2 f = \frac{\partial^2 f}{\partial x_1^2} + \cdots + \frac{\partial^2 f}{\partial x_n^2} = 0 \quad \text{on } U.$$

A function $f \colon U \subset \mathbb{R}^n \to \mathbb{R} \cup \{-\infty\}$ is called *subharmonic* if it is upper-semicontinuous† and for every ball $B_\rho(a)$ with $\overline{B_\rho(a)} \subset U$, and every function φ harmonic on $B_\rho(a)$ and continuous on $\overline{B_\rho(a)}$ such that $f(z) \leq \varphi(z)$ for $z \in \partial B_\rho(a)$, then

$$f(z) \leq \varphi(z), \qquad \text{for all } z \in B_\rho(a).$$

*Recall that the operator ∇^2, sometimes also written Δ, is called the *Laplacian*. It is the trace of the Hessian matrix.

†Recall that f is *upper-semicontinuous* if $\limsup_{t \to x} f(t) \leq f(x)$ for all x.

In other words, a subharmonic function is a function that is less than any harmonic function on every ball. We will generally look at harmonic and subharmonic functions in $\mathbb{C} \cong \mathbb{R}^2$.

Let us go through some basic results on harmonic and subharmonic functions that you have seen in detail in your one-variable class. Consequently we leave some of these results as exercises.

***Exercise* 2.4.1:** *An upper-semicontinuous function achieves a maximum on compact sets.*

***Exercise* 2.4.2:** *Show that for a C^2 function $f\colon U \subset \mathbb{C} \to \mathbb{R}$,*

$$\frac{\partial^2}{\partial \bar{z} \partial z} f = \frac{1}{4} \nabla^2 f.$$

Use this fact to show that f is harmonic if and only if it is (locally) the real or imaginary part of a holomorphic function. Hint: Key is to be able to find an antiderivative of a holomorphic function.

It follows from the exercise that a harmonic function is infinitely differentiable, and by applying the Cauchy formula on a disc we obtain the following proposition.

Proposition 2.4.2 (Mean-value property and sub-mean-value property)**.** *A continuous function $f\colon U \subset \mathbb{C} \to \mathbb{R}$ is harmonic if and only if whenever $\overline{\Delta_r(a)} \subset U$ then*

$$f(a) = \frac{1}{2\pi} \int_0^{2\pi} f(a + re^{i\theta})\, d\theta.$$

An upper-semicontinuous function $f\colon U \to \mathbb{R} \cup \{-\infty\}$ is subharmonic if and only if whenever $\overline{\Delta_r(a)} \subset U$ then

$$f(a) \leq \frac{1}{2\pi} \int_0^{2\pi} f(a + re^{i\theta})\, d\theta.$$

Do note that for the sub-mean-value property we may have to use Lebesgue integral to be able to integrate a upper-semicontinuous function. Although continuous subharmonic functions are enough for this class.

***Exercise* 2.4.3:** *Fill in the details of the proof of the proposition. Feel free to prove the subharmonicity result only for continuous functions if you wish.*

***Exercise* 2.4.4:** *Show that if $f\colon U \subset \mathbb{C} \to \mathbb{R} \cup \{-\infty\}$ is subharmonic then for $z \in U$ we have*

$$\limsup_{w \to z} f(w) = f(z).$$

Proposition 2.4.3 (Maximum principle)**.** *Suppose $U \subset \mathbb{C}$ is a domain and $f\colon U \to \mathbb{R} \cup \{-\infty\}$ is subharmonic. If f attains a maximum in U then f is constant.*

Proof. If $\overline{\Delta_r(a)} \subset U$ then

$$f(a) \leq \frac{1}{2\pi} \int_0^{2\pi} f(a + re^{i\theta})\, d\theta.$$

In particular $f = f(a)$ almost everywhere on $\partial \Delta_r(a)$. By upper-semicontinuity it is true everywhere. This was true for all r with $\overline{\Delta_r(a)} \subset U$, so $f = f(a)$ on $\Delta_r(a)$ and the set where $f = f(a)$ is open. The set where an upper-semicontinuous function attains a maximum is closed, so $f = f(a)$ on U as U is connected. ☐

Proposition 2.4.4. *Suppose $U \subset \mathbb{C}$ and $f \colon U \to \mathbb{R}$ is a C^2 function. The function f is subharmonic if and only if $\nabla^2 f \geq 0$.*

Proof. We have a C^2-smooth function on a subset of $\mathbb{C} \cong \mathbb{R}^2$ with $\nabla^2 f \geq 0$ and we wish to show that it is subharmonic. Take a disc $\Delta_\rho(a)$ such that f is continuous on the closure, and take a harmonic function g on the closure $\overline{\Delta_\rho(a)}$ such that $f \leq g$ on the boundary. By noting that $\nabla^2(f-g) \geq 0$ we can assume that $g = 0$ and $f \leq 0$ on the boundary.

First suppose that $\nabla^2 f > 0$. Suppose f attains a maximum in $\Delta_\rho(a)$, call this point p. $\nabla^2 f$ is the trace of the Hessian matrix, but for f to have a maximum, the Hessian must have nonpositive eigenvalues at the critical points, which is a contradiction as the trace is the sum of the eigenvalues. So f has no maximum inside ant therefore $f \leq 0$ on all of $\overline{\Delta_\rho(a)}$

Next suppose that $\nabla^2 f \geq 0$. Let M be the maximum of $x^2 + y^2$ on $\overline{\Delta_\rho(a)}$. Take $f_n(x,y) = f(x,y) + \frac{1}{n}(x^2 + y^2) - \frac{1}{n}M$. Clearly $\nabla^2 f_n > 0$ and $f_n \leq 0$ on the boundary, so $f_n \leq 0$ on all of $\overline{\Delta_\rho(a)}$. As $f_n \to f$ we obtain that $f \leq 0$ on all of $\overline{\Delta_\rho(a)}$.

The other direction is left as an exercise. ☐

> ***Exercise* 2.4.5:** *Finish the proof of the above proposition.*

Proposition 2.4.5. *Suppose $U \subset \mathbb{C}$ is a domain and $f_\alpha \colon U \to \mathbb{R} \cup \{-\infty\}$ is a family of subharmonic functions. Let*

$$\varphi(z) = \sup_\alpha f_\alpha(z).$$

If the family is finite then φ is subharmonic. If the family is infinite and we assume that $\varphi(z) \neq \infty$ for all z and that φ is upper-semicontinuous, then φ is subharmonic.

Proof. Suppose $\overline{\Delta_r(a)} \subset U$. For any α,

$$\frac{1}{2\pi} \int_0^{2\pi} \varphi(a + re^{i\theta})\, d\theta \geq \frac{1}{2\pi} \int_0^{2\pi} f_\alpha(a + re^{i\theta})\, d\theta \geq f_\alpha(a)$$

Taking the supremum on the right over α obtains the results. ☐

***Exercise* 2.4.6:** *Prove that if $\varphi\colon \mathbb{R} \to \mathbb{R}$ is a monotonically increasing convex function and $f\colon U \subset \mathbb{C} \to \mathbb{R}$ is subharmonic, then $\varphi \circ f$ is subharmonic.*

There are too many harmonic functions in $\mathbb{C}^n$. To get the real and imaginary parts of holomorphic functions in $\mathbb{C}^n$ we require a smaller class of functions than all harmonic functions.

Definition 2.4.6. Twice differentiable function $f\colon U \subset \mathbb{C}^n \to \mathbb{R}$ is called *pluriharmonic* if for every $a, b \in \mathbb{C}^n$, the function

$$z \mapsto f(a + bz)$$

is harmonic (on the set where $a + bz \in U$). That is, if f is harmonic on every complex line.

A function $f\colon U \subset \mathbb{C}^n \to \mathbb{R} \cup \{-\infty\}$ is called *plurisubharmonic*, sometimes called *plush* or *psh* for short, if it is upper-semicontinuous and for every $a, b \in \mathbb{C}^n$, the function

$$z \mapsto f(a + bz)$$

is subharmonic (on the set where $a + bz \in U$).

***Exercise* 2.4.7:** *A C^2-smooth function $f\colon U \subset \mathbb{C}^n \to \mathbb{R}$ is pluriharmonic if and only if*

$$\frac{\partial^2 f}{\partial \bar{z}_j \partial z_k} = 0 \qquad \text{on } U$$

for all $j, k = 1, \ldots, n$.

***Exercise* 2.4.8:** *Show that a pluriharmonic function is harmonic. On the other hand, find an example of a harmonic function that is not pluriharmonic.*

***Exercise* 2.4.9:** *Show that a $f\colon U \subset \mathbb{C}^n \to \mathbb{R}$ is pluriharmonic if and only if it is locally the real or imaginary part of a holomorphic function. Hint: Using a previous exercise $\frac{\partial f}{\partial z_k}$ is holomorphic for all k. Assume that U is simply connected and $f(z^0) = 0$. Consider the line integral from $z^0 \in U$ to a nearby $z \in U$:*

$$F(z) = \int_{z_0}^{z} \sum_{k=1}^{n} \frac{\partial^2 f}{\partial z_k}(z)\, dz_k.$$

Prove that it is path independent, compute derivatives of F, and find out what is $f - F$.

Proposition 2.4.7. *A C^2-smooth function $f\colon U \subset \mathbb{C}^n \to \mathbb{R}$ is plurisubharmonic if and only if the complex Hessian matrix*

$$\left[\frac{\partial^2 f}{\partial \bar{z}_j \partial z_k} \right]_{jk}$$

is positive semidefinite at every point.

Proof. Fix a point p, and after translation assume $p = 0$. After a holomorphic linear change of variables assume that the complex Hessian $\left[\frac{\partial^2 f}{\partial \bar{z}_j \partial z_k} \Big|_0 \right]_{jk}$ is diagonal. If the complex Hessian has a negative eigenvalue, then one of the diagonal entries is negative. Without loss of generality suppose $\frac{\partial^2 f}{\partial \bar{z}_1 \partial z_1} < 0$ at the origin. The function $z_1 \mapsto f(z_1, 0, \ldots, 0)$ has a negative Laplacian and therefore is not subharmonic, and thus f itself is not plurisubharmonic.

For the other direction, suppose that the complex Hessian is positive semidefinite at all points. After a linear change of coordinates assume that the line $\xi \mapsto a + b\xi$ is simply setting all but the first variable to zero. As the complex Hessian is positive semidefinite we have $\frac{\partial^2 f}{\partial \bar{z}_1 \partial z_1} \geq 0$ for all points $(z_1, 0, \ldots, 0)$. We proved above that $\nabla^2 g \geq 0$ implies that g is subharmonic, and we are done. □

Exercise* 2.4.10*: *If $f : U \subset \mathbb{C}^n \to \mathbb{C}$ is holomorphic, show that $\log|f(z)|$ is plurisubharmonic.*

Exercise* 2.4.11*: *Show that the set of plurisubharmonic functions on a domain $U \subset \mathbb{C}^n$ is a cone in the sense that if $a, b > 0$ are constants and $f, g : U \to \mathbb{R} \cup \{-\infty\}$ are plurisubharmonic, then $af + bg$ is plurisubharmonic.*

Theorem 2.4.8. *Suppose $U \subset \mathbb{C}^n$ is a domain and $f : U \to \mathbb{R} \cup \{-\infty\}$ is plurisubharmonic. For every $\varepsilon > 0$, let $U_\varepsilon \subset U$ be the set of points at least ε away from ∂U, there exists a smooth plurisubharmonic function $f_\varepsilon : U_\varepsilon \to \mathbb{R}$ such that $f_\varepsilon(z) \geq f(z)$, and*

$$f(z) = \lim_{\varepsilon \to 0} f_\varepsilon(z).$$

That is, f is a limit of smooth plurisubharmonic functions. The idea of the proof is important and useful in many other contexts.

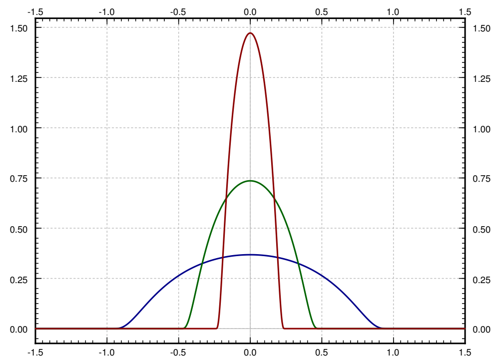
Proof. We smooth f by convolving with so-called *mollifiers*. Many different mollifiers work, but let us use a very specific one for concreteness. Define

$$g(z) = \begin{cases} Ce^{-1/(1-\|z\|^2)} & \text{if } \|z\| < 1, \\ 0 & \text{if } \|z\| \geq 1, \end{cases} \qquad \text{and} \qquad g_\varepsilon(z) = \frac{1}{\varepsilon^{2n}} g(z/\varepsilon).$$

It is left as an exercise that g, and therefore g_ε, is smooth. The function g clearly has compact support as it is only nonzero inside the unit ball. The support of g_ε is the ε-ball. Both are nonnegative. Choose C so that

$$\int_{\mathbb{C}^n} g \, dV = 1, \qquad \text{and therefore} \qquad \int_{\mathbb{C}^n} g_\varepsilon \, dV = 1.$$

The function g only depends on $\|z\|$. To get an idea of how these functions look, consider the following graphs of the functions $e^{-1/(1-x^2)}$, $\frac{1}{0.5} e^{-1/(1-(x/0.5)^2)}$, and $\frac{1}{0.25} e^{-1/(1-(x/0.25)^2)}$.



First f is bounded above on compact sets as it is upper semicontinuous. If f is not bounded below, we replace f with $\max\{f, 1/\varepsilon\}$, which is still plurisubharmonic. Therefore, without loss of generality we assume that f is locally bounded.

For $z \in U_\varepsilon$, we define f_ε as the convolution with g_ε:

$$f_\varepsilon(z) = (f * g_\varepsilon)(z) = \int_{\mathbb{C}^n} f(w) g_\varepsilon(z-w)\, dV(w) = \int_{\mathbb{C}^n} f(z-w) g_\varepsilon(w)\, dV(w).$$

The two forms of the integral follow easily via change of variables. We are perhaps abusing notation a bit since f is only defined on U, but you should think about why it is not a problem as long as $z \in U_\varepsilon$. By differentiating the first form under the integral, f_ε is smooth. Let us show that f_ε is plurisubharmonic. We need to restrict to a line $\xi \mapsto a + b\xi$. Without loss of generality, suppose that $a = 0$, $b = (1, 0, \ldots, 0)$, and that we are testing subharmonicity on a disc of radius r around $\xi = 0$.

$$\begin{aligned} \frac{1}{2\pi}\int_0^{2\pi} f_\varepsilon(re^{i\theta}, 0, \ldots, 0)\, d\theta &= \frac{1}{2\pi}\int_0^{2\pi}\int_{\mathbb{C}^n} f(re^{i\theta} - w_1, -w_2, \ldots, -w_n) g_\varepsilon(w)\, dV(w)\, d\theta \\ &= \int_{\mathbb{C}^n} \left(\frac{1}{2\pi}\int_0^{2\pi} f(re^{i\theta} - w_1, -w_2, \ldots, -w_n)\, d\theta \right) g_\varepsilon(w)\, dV(w) \\ &\geq \int_{\mathbb{C}^n} f(-w_1, -w_2, \ldots, -w_n) g_\varepsilon(w)\, dV(w) = f_\varepsilon(0). \end{aligned}$$

For the inequality we used $g_\varepsilon \geq 0$. So f_ε is plurisubharmonic.

As $g_\varepsilon(w)$ only depends on $|w_1|, \ldots, |w_n|$, we notice that $g_\varepsilon(w_1, \ldots, w_n) = g_\varepsilon(|w_1|, \ldots, |w_n|)$.

Without loss of generality we consider $z = 0$:

$$\begin{aligned}
f_\varepsilon(0) &= \int_{\mathbb{C}^n} f(-w) g_\varepsilon(|w_1|,\ldots,|w_n|)\, dV(w) \\
&= \int_0^\varepsilon \cdots \int_0^\varepsilon \left(\int_0^{2\pi} \cdots \int_0^{2\pi} f(-r_1 e^{i\theta_1},\ldots,-r_n e^{i\theta_n})\, d\theta_1 \cdots d\theta_n \right) \\
&\qquad g_\varepsilon(r_1,\ldots,r_n)\, r_1 \cdots r_n\, dr_1 \cdots dr_n \\
&\geq \int_0^\varepsilon \cdots \int_0^\varepsilon \left(\int_0^{2\pi} \cdots \int_0^{2\pi} (2\pi) f(0,-r_2 e^{i\theta_2},\ldots,-r_n e^{i\theta_n})\, d\theta_2 \cdots d\theta_n \right) \\
&\qquad g_\varepsilon(r_1,\ldots,r_n)\, r_1 \cdots r_n\, dr_1 \cdots dr_n \\
&\geq f(0) \int_0^\varepsilon \cdots \int_0^\varepsilon (2\pi)^n g_\varepsilon(r_1,\ldots,r_n)\, r_1 \cdots r_n\, dr_1 \cdots dr_n \\
&= f(0).
\end{aligned}$$

We have $\limsup_{w\to z} f(w) = f(z)$ for subharmonic, and therefore for plurisubharmonic functions. Hence for any $\delta > 0$ find an $\varepsilon > 0$ so that for $w \in B_\varepsilon(0)$ we get $f(w) - f(0) \leq \delta$.

$$\begin{aligned}
f_\varepsilon(0) - f(0) &= \int_{B_\varepsilon(0)} \bigl(f(-w) - f(0)\bigr)\, g_\varepsilon(w)\, dV(w) \\
&\leq \delta \int_{B_\varepsilon(0)} g_\varepsilon(w)\, dV(w) = \delta.
\end{aligned}$$

□

Exercise* 2.4.12*: *Show that g in the proof above is smooth on all of $\mathbb{C}^n$.*

Exercise* 2.4.13*: *If $f \colon U \subset \mathbb{C}^n \to V \subset \mathbb{C}^m$ is holomorphic and $\varphi \colon V \to \mathbb{R}$ is a C^2 plurisubharmonic function, then $\varphi \circ f$ is plurisubharmonic. Then use this to show that this holds for all plurisubharmonic functions.*

Exercise* 2.4.14*: *a) Show that for a subharmonic function $\int_0^{2\pi} f(a + re^{i\theta})\, d\theta$ is a monotone function of r (Hint: try a C^2 function first and use Green's theorem). b) Use this fact to show that $f_\varepsilon(z)$ from Theorem 2.4.8 are monotone decreasing in ε.*

Exercise* 2.4.15*: *Show that plurisubharmonicity is a local property, that is, f is plurisubharmonic if and only if f is plurisubharmonic in some neighborhood of each point.*

Exercise* 2.4.16*: *Use the computation from Theorem 2.4.8 to show that if f is pluriharmonic, then $f_\varepsilon = f$ (where that makes sense), therefore obtaining another proof that a pluriharmonic function is C^∞.*

Exercise* 2.4.17*: *Let the f in Theorem 2.4.8 be continuous and suppose $K \subset\subset U$, in particular for small enough $\varepsilon > 0$, $K \subset U_\varepsilon$. Show that f_ε converges uniformly to f on K.*

Exercise* 2.4.18*: *Let the f in Theorem 2.4.8 be C^k for some $k \geq 0$. Show that all derivatives of f_ε up to order k converge uniformly on compact sets to the corresponding derivatives of f. See also previous exercise.*

Definition 2.4.9. Let $\mathscr{F}$ be a class of (extended*) real-valued functions defined on $U \subset \mathbb{R}^n$. If $K \subset U$, we define $\widehat{K}$ be the *hull* of K with respect to $\mathscr{F}$ as the set

$$\widehat{K} \overset{\text{def}}{=} \{z \in U : f(z) \leq \sup_{w \in K} f(w) \text{ for all } f \in \mathscr{F}\}.$$

A domain U is said to be *convex with respect to* $\mathscr{F}$ if for every $K \subset\subset U$, the hull $\widehat{K} \subset\subset U$.†

Clearly $K \subset \widehat{K}$, the key thing is to show that $\widehat{K}$ is not "too large" for U. Keep in mind that the functions in $\mathscr{F}$ are defined on U so $\widehat{K}$ depends on U. A common mistake is to consider functions defined on a larger set, which obtains a smaller $\mathscr{F}$ and hence a larger $\widehat{K}$. Some authors use $\widehat{K}_{\mathscr{F}}$ to denote the dependence on $\mathscr{F}$, and to avoid having to say so in words, but we will avoid this shortcut.

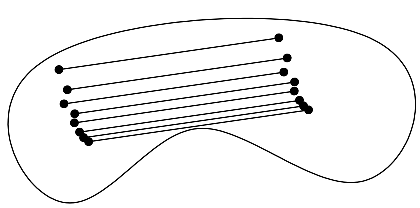
Exercise* 2.4.19*: *Show that a domain $U \subset \mathbb{R}^n$ is geometrically convex (that is, the line segment between any two points in U is contained in U) if and only if it is convex with respect to the convex functions on U.*

Exercise* 2.4.20*: *Show that any domain $U \subset \mathbb{R}^n$ is convex with respect to real polynomials.*

Theorem 2.4.10 (Kotinuitätssatz—Continuity principle)**.** *Suppose $U \subset \mathbb{C}^n$ is convex with respect to plurisubharmonic functions, then given any collection of closed analytic discs Δ_α such that $\bigcup_\alpha \partial\Delta_\alpha \subset\subset U$, we have $\bigcup_\alpha \Delta_\alpha \subset\subset U$.*

Proof. Let f be a plurisubharmonic function on U. If $\varphi_\alpha \colon \overline{\mathbb{D}} \to U$ is the holomorphic (in $\mathbb{D}$) mapping giving the closed analytic disc, then $f \circ \varphi_\alpha$ is subharmonic. By the maximum principle f on Δ_α must be less than or equal to the supremum of f on $\partial\Delta_\alpha$, so $\overline{\Delta_\alpha}$ is in the hull of $\partial\Delta_\alpha$. In other words $\bigcup_\alpha \Delta_\alpha$ is in the hull of $\bigcup_\alpha \partial\Delta_\alpha$ and therefore $\bigcup_\alpha \Delta_\alpha \subset\subset U$ by convexity. □

Let us illustrate the failure of the continuity principle. If the domain is not convex with respect to plurisubharmonic functions then you could have discs (denoted by straight line segments) that approach the boundary as in the following picture. In the diagram the boundaries of the discs are denoted by the dark dots at the end of the segments.



*By extended reals we mean $\mathbb{R} \cup \{-\infty, \infty\}$.

†Recall that $\subset\subset$ means relatively compact.

Definition 2.4.11. Let $U \subset \mathbb{C}^n$ be a domain. An $f : U \to \mathbb{R}$ is an *exhaustion function* for U if

$$\{z \in U : f(z) < r\} \subset\subset U \qquad \text{for every } r \in \mathbb{R}.$$

A domain $U \subset \mathbb{C}^n$ is *Hartogs pseudoconvex* if there exists a continuous plurisubharmonic exhaustion function. The sets $\{z \in U : f(z) < r\}$ are called the *sublevel sets* of f.

Example 2.4.12: The unit ball $\mathbb{B}_n$ is Hartogs pseudoconvex. The continuous function

$$-\log(1 - \|z\|)$$

is an exhaustion function, and it is easy to check directly that it is plurisubharmonic.

Example 2.4.13: The entire $\mathbb{C}^n$ is Hartogs pseudoconvex as $\|z\|^2$ is a continuous plurisubharmonic exhaustion function.

Theorem 2.4.14. *Suppose $U \subset \mathbb{C}^n$ is a domain. The following are equivalent:*

(i) $-\log \rho(z)$ is plurisubharmonic, where $\rho(z)$ is the distance from z to ∂U.

(ii) U has a continuous plurisubharmonic exhaustion function.

(iii) U is convex with respect to plurisubharmonic functions.

Proof. (i) $\Rightarrow$ (ii): If U is bounded, the function $-\log \rho(z)$ is clearly a continuous exhaustion function. If U is unbounded, take $z \mapsto \max\{-\log \rho(z), \|z\|^2\}$.

(ii) $\Rightarrow$ (iii): Suppose f is a continuous plurisubharmonic exhaustion function. If $K \subset\subset U$, then for some r we have $K \subset \{z \in U : f(z) < r\} \subset\subset U$. But then by definition of the hull $\widehat{K}$ we have $\widehat{K} \subset \{z \in U : f(z) < r\} \subset\subset U$.

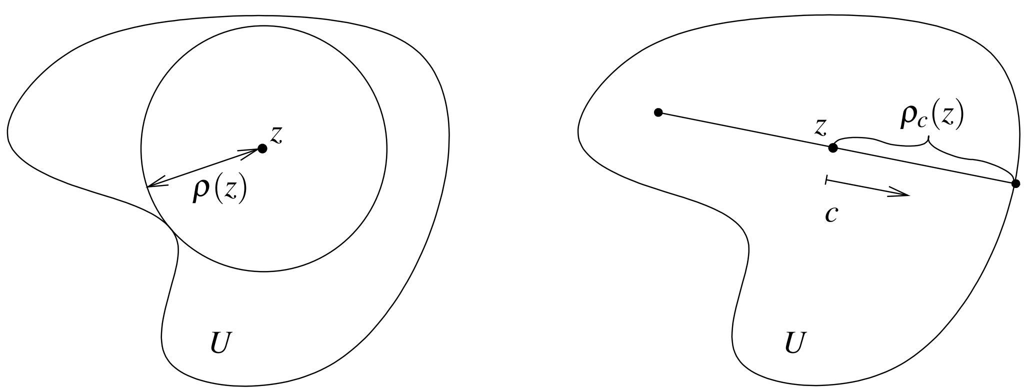
(iii) $\Rightarrow$ (i): For $c \in \mathbb{C}^n$ with $\|c\| = 1$ let us define

$$\rho_c(z) = \sup\{\lambda > 0 : z + \lambda t c \in U \text{ for all } |t| < 1\}.$$

So $\rho_c(z)$ is the radius of the largest affine disc centered at z in the direction c. As $\rho(z) = \inf_c \rho_c(z)$,

$$-\log \rho(z) = \sup_{\|c\|=1} \bigl(-\log \rho_c(z)\bigr).$$

If we prove that for any a, b, c the function $\xi \mapsto -\log \rho_c(a + b\xi)$ is subharmonic, then $\xi \mapsto -\log \rho(a + b\xi)$ is subharmonic and we are done. Here is the setup, the disc is drawn as a line:



Suppose $\Delta \subset \mathbb{C}$ is a disc such that for all $\xi \in \overline{\Delta}$, $a+b\xi \in U$. We need to show that if there is a harmonic function u on Δ continuous up to the boundary such that $-\log \rho_c(a+b\xi) \leq u(\xi)$ on $\partial\Delta$, then the inequality holds on Δ. Let $u = \operatorname{Re} f$ for a holomorphic function f. For $\xi \in \partial\Delta$ we have $-\log \rho_c(a+b\xi) \leq \operatorname{Re} f(\xi)$, or in other words

$$\rho_c(a+b\xi) \geq e^{-\operatorname{Re} f(\xi)} = \left|e^{-f(\xi)}\right|.$$

Using the definition of $\rho_c(a+b\xi)$, the statement above is equivalent to saying that whenever $|t| < 1$ then

$$(a+b\xi) + cte^{-f(\xi)} \in U.$$

This statement holds when $\xi \in \partial\Delta$. If we prove that it also holds for $\xi \in \Delta$ then we are finished.

We think of $\varphi_t(\xi) = (a+b\xi) + cte^{-f(\xi)}$ as a closed analytic disc with boundary inside U. We have a family of analytic discs, parametrized by t, whose boundaries are in U for all t with $|t| < 1$. For $t = 0$ the entire disc is inside U. Take $t_0 < 1$ such that $\varphi_t(\Delta) \subset U$ for all t with $|t| < t_0$. Then

$$\bigcup_{|t|<t_0} \varphi_t(\partial\Delta) \subset \bigcup_{|t|\leq t_0} \varphi_t(\partial\Delta) \subset\subset U,$$

because continuous functions take compact sets to compact sets. Convexity with respect to plurisubharmonic functions implies that

$$\bigcup_{|t|<t_0} \varphi_t(\Delta) \subset\subset U.$$

Again by continuity we have $\varphi_t(\Delta) \subset\subset U$ for all t with $|t| = t_0$, and consequently it is true when $|t|$ is even slightly larger than t_0. This implies that $\varphi_t(\mathbb{D}) \subset U$ for all t with $|t| < 1$. Thus $(a+b\xi) + cte^{-f(\xi)} \in U$ for all $\xi \in \Delta$ and all $|t| < 1$. And this implies that $\rho_c(a+b\xi) \geq e^{-\operatorname{Re} f(\xi)}$, which in turn implies that $-\log \rho_c(a+b\xi) \leq \operatorname{Re} f(\xi) = u(\xi)$ and therefore $-\log \rho_c(a+b\xi)$ is subharmonic. □

Exercise* 2.4.21*: *Show that if domains $U_1 \subset \mathbb{C}^n$ and $U_2 \subset \mathbb{C}^n$ are Hartogs pseudoconvex then so are all the topological components of $U_1 \cap U_2$.*

Exercise* 2.4.22*: *Show that if domains $U \subset \mathbb{C}^n$ and $V \subset \mathbb{C}^m$ are Hartogs pseudoconvex then so is $U \times V$.*

Exercise* 2.4.23*: *Show that every domain $U \subset \mathbb{C}$ is Hartogs pseudoconvex.*

Exercise* 2.4.24*: *Show that the union $\bigcup_j U_j$ of a nested sequence of Hartogs pseudoconvex domains $U_{j-1} \subset U_j \subset \mathbb{C}^n$ is Hartogs pseudoconvex.*

The statement corresponding to the last exercise on nested unions for domains of holomorphy is the Behnke-Stein theorem, which follows using this exercise and the solution of the Levi-problem. Although historically Behnke-Stein was proved independently and used to solve the Levi-problem.

It is not immediately clear from the definition, but Hartogs pseudoconvexity is a local property.

Lemma 2.4.15. *A domain $U \subset \mathbb{C}^n$ is Hartogs pseudoconvex if and only if for every point $p \in \partial U$ there exists a neighborhood W of p such that $W \cap U$ is Hartogs pseudoconvex.*

Proof. One direction is trivial, so let us consider the other direction. For $p \in \partial U$ let W be such that $U \cap W$ is Hartogs pseudoconvex. By intersecting with a ball (which is pseudoconvex) we assume that $W = B_r(p)$ (a ball centered at p). Let $B = B_{r/4}(p)$. For any $z \in B \cap U$, the distance from z to the boundary of $W \cap U$ is the same as the distance to ∂U. Let $\operatorname{dist}(x,y)$ denote the distance function. Then for $z \in B \cap U$

$$-\log \operatorname{dist}(z, \partial U) = -\log \operatorname{dist}\bigl(z, \partial(U \cap W)\bigr).$$

We know the right hand side is plurisubharmonic. We have such a ball B of positive radius around every $p \in \partial U$, so we have a plurisubharmonic exhaustion function near the boundary.

If U is bounded then ∂U is compact and so there is some $\varepsilon > 0$ such that $-\log \operatorname{dist}(z, \partial U)$ is plurisubharmonic if $\operatorname{dist}(z, \partial U) < 2\varepsilon$. The function

$$\varphi(z) = \max\{-\log \operatorname{dist}(z, \partial U), -\log \varepsilon\}.$$

is a continuous plurisubharmonic exhaustion function. The proof for unbounded U requires some function of $\|z\|^2$ rather than a constant and is left as an exercise. □

***Exercise* 2.4.25:** *Finish the proof of the lemma for unbounded domains.*

It may seem that we are defining a totally different concept, but it turns out that Levi and Hartogs pseudoconvexity are one and the same on domains where both concepts make sense.

Theorem 2.4.16. *Let $U \subset \mathbb{C}^n$ be a domain with smooth boundary. Then U is Hartogs pseudoconvex if and only if U is Levi pseudoconvex.*

As a consequence of this theorem we say simply "pseudoconvex" and there is no ambiguity.

Proof. Suppose $U \subset \mathbb{C}^n$ is a domain with smooth boundary that is not Levi pseudoconvex at $p \in \partial U$. As in Theorem 2.3.10, change coordinates so that $p = 0$ and U is defined by

$$\operatorname{Im} z_n > -|z_1|^2 + \sum_{j=2}^{n} \varepsilon_j |z_j|^2 + O(3).$$

For some small fixed $\lambda > 0$, the analytic discs defined by $\varphi(\xi) = (\lambda\xi, 0, \cdots, 0, is)$ are in U for all small enough $s > 0$. As the origin is in their limit set, Kontinuitätssatz is not satisfied, and U is not convex with respect to the plurisubharmonic functions. Therefore U is not Hartogs pseudoconvex.

Next suppose that U is Levi pseudoconvex. Take any $p \in \partial U$. After translation and rotation assume p is the origin and write the defining function r as

$$r(z,\bar{z}) = \varphi(z', \bar{z}', \operatorname{Re} z_n) - \operatorname{Im} z_n,$$

where $z' = (z_1, \ldots, z_{n-1})$ and φ vanishes to second order at the origin. The condition of Levi pseudoconvexity says that

$$\sum_{j=1,\ell=1}^{n} \bar{a}_j a_\ell \frac{\partial^2 r}{\partial \bar{z}_j \partial z_\ell}\Big|_q \geq 0 \quad \text{whenever} \quad \sum_{j=1}^{n} a_j \frac{\partial r}{\partial z_j}\Big|_q = 0, \tag{2.2}$$

for all $q \in \partial U$ near p. If we translate ∂U slightly in the $\operatorname{Im} z_n$ direction we still have a Levi pseudoconvex hypersurface. That is, we look at the surface $r = s$ for small real s and the condition is satisfied for $r - s$. As $\frac{\partial r}{\partial z_j} = \frac{\partial (r-s)}{\partial z_j}$ for all j and the complex Hessians of r and $r - s$ are equal, condition (2.2) holds for r for all $q \in U$ near p. We have what we need in all but one direction.

Let $\nabla_z r(q) = \left(\frac{\partial r}{\partial z_1}\Big|_q, \ldots, \frac{\partial r}{\partial z_n}\Big|_q \right)$ denote the gradient of r in the z directions only. Given $q \in U$ near p, decompose an arbitrary $c \in \mathbb{C}^n$ as $c = a + b$, where $a = (a_1, \ldots, a_n)$ satisfies

$$\sum_{j=1}^{n} a_j \frac{\partial r}{\partial z_j}\Big|_q = \left\langle a, \overline{\nabla_z r(q)} \right\rangle = 0.$$

By taking the orthogonal decomposition, we can choose b to be a scalar multiple of $\overline{\nabla_z r(q)}$. Then

$$\left| \sum_{j=1}^{n} c_j \frac{\partial r}{\partial z_j}\Big|_q \right| = \left| \sum_{j=1}^{n} b_j \frac{\partial r}{\partial z_j}\Big|_q \right| = \left| \left\langle b, \overline{\nabla_z r(q)} \right\rangle \right| = \|b\| \|\nabla_z r(q)\|.$$

As $\nabla_z r(p) = (0, \ldots, 0, -1/2i)$, then for q sufficiently near p we have that $\|\nabla_z r(q)\| \geq 1/3$. Therefore,

$$\|b\| = \frac{\left| \sum_{j=1}^{n} c_j \frac{\partial r}{\partial z_j}\Big|_q \right|}{\|\nabla_z r(q)\|} \leq 3 \left| \sum_{j=1}^{n} c_j \frac{\partial r}{\partial z_j}\Big|_q \right|.$$

As $c = a + b$ is the orthogonal decomposition we have that $\|c\| \geq \|b\|$.

Next,

$$\begin{aligned}
\sum_{j=1,\ell=1}^{n} \bar{c}_j c_\ell \frac{\partial^2 r}{\partial \bar{z}_j \partial z_\ell}\Big|_q &= \sum_{j=1,\ell=1}^{n} (\bar{a}_j + \bar{b}_j)(a_\ell + b_\ell) \frac{\partial^2 r}{\partial \bar{z}_j \partial z_\ell}\Big|_q \\
&= \sum_{j=1,\ell=1}^{n} \bar{a}_j a_\ell \frac{\partial^2 r}{\partial \bar{z}_j \partial z_\ell}\Big|_q \\
&\quad + \sum_{j=1,\ell=1}^{n} \bar{b}_j c_\ell \frac{\partial^2 r}{\partial \bar{z}_j \partial z_\ell}\Big|_q + \sum_{j=1,\ell=1}^{n} \bar{c}_j b_\ell \frac{\partial^2 r}{\partial \bar{z}_j \partial z_\ell}\Big|_q - \sum_{j=1,\ell=1}^{n} \bar{b}_j b_\ell \frac{\partial^2 r}{\partial \bar{z}_j \partial z_\ell}\Big|_q \\
&\geq \sum_{j=1,\ell=1}^{n} \bar{a}_j a_\ell \frac{\partial^2 r}{\partial \bar{z}_j \partial z_\ell}\Big|_q - M\|b\|\|c\| - M\|c\|\|b\| - M\|b\|^2 \\
&\geq -3M\|c\|\|b\|,
\end{aligned}$$

for some constant $M > 0$. Putting this together with what we know of $\|b\|$ we get:

$$\sum_{j=1,\ell=1}^{n} \bar{c}_j c_\ell \frac{\partial^2 r}{\partial \bar{z}_j \partial z_\ell}\Big|_q \geq -3M\|c\|\|b\| \geq -3^2 M\|c\| \left| \sum_{j=1}^{n} c_j \frac{\partial r}{\partial z_j}\Big|_q \right| .$$

For $z \in U$ sufficiently close to p define

$$f(z) = -\log\bigl(-r(z)\bigr) + A\|z\|^2,$$

where $A > 0$ is some constant we will choose later. The log is there to make the function blow up as we approach the boundary and the $A\|z\|^2$ is adding a constant diagonal matrix to the complex Hessian of f, which we hope is enough to make it positive semidefinite. Let us compute:

$$\frac{\partial^2 f}{\partial \bar{z}_j \partial z_\ell} = \frac{1}{r^2} \frac{\partial r}{\partial \bar{z}_j} \frac{\partial r}{\partial z_\ell} - \frac{1}{r} \frac{\partial^2 r}{\partial \bar{z}_j \partial z_\ell} + A\delta_j^\ell,$$

where δ_j^ℓ is the Kronecker delta*. Apply the complex Hessian of f to c (recall that r is negative on U and so for $z \in U$, $-r = |r|$):

$$\begin{aligned}
\sum_{j=1,\ell=1}^{n} \bar{c}_j c_\ell \frac{\partial^2 f}{\partial \bar{z}_j \partial z_\ell}\Big|_z &= \frac{1}{r^2} \left| \sum_{\ell=1}^{n} c_\ell \frac{\partial r}{\partial z_\ell}\Big|_z \right|^2 + \frac{1}{|r|} \sum_{j=1,\ell=1}^{n} \bar{c}_j c_\ell \frac{\partial^2 r}{\partial \bar{z}_j \partial z_\ell} + A\|c\|^2 \\
&\geq \frac{1}{r^2} \left| \sum_{\ell=1}^{n} c_\ell \frac{\partial r}{\partial z_\ell}\Big|_z \right|^2 - \frac{3^2 M}{|r|} \|c\| \left| \sum_{j=1}^{n} c_j \frac{\partial r}{\partial z_j}\Big|_z \right| + A\|c\|^2.
\end{aligned}$$

Now comes a somewhat funky trick. As a quadratic polynomial in $\|c\|$, the right hand side of the inequality is always nonnegative if $A > 0$ and the discriminant is negative or zero. Let us set the discriminant to zero:

$$0 = \left(\frac{3^2 M}{|r|} \left| \sum_{j=1}^{n} c_j \frac{\partial r}{\partial z_j}\Big|_z \right| \right)^2 - 4A \frac{1}{r^2} \left| \sum_{\ell=1}^{n} c_\ell \frac{\partial r}{\partial z_\ell}\Big|_z \right|^2 .$$

All the nonconstant terms go away and $A = \frac{3^4 M^2}{4}$. Thus

$$\sum_{j=1,\ell=1}^{n} \bar{c}_j c_\ell \frac{\partial^2 f}{\partial \bar{z}_j \partial z_\ell}\Big|_z \geq 0.$$

In other words, the complex Hessian of f is positive semidefinite at all points $z \in U$ near p. The function $f(z)$ goes to infinity as z approaches ∂U, so the sublevel set (where $f(z) < t$ for some $t \in \mathbb{R}$) must be a positive distance away from ∂U near p.

We have a local continuous plurisubharmonic exhaustion function for U near p. If we intersect with a small ball B centered at p we get that $U \cap B$ is Hartogs pseudoconvex. This is true at all $p \in \partial U$, so U is Hartogs pseudoconvex. □

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*Recall $\delta_j^\ell = 0$ if $j \neq \ell$ and $\delta_j^\ell = 1$ if $j = \ell$.

2.5 Holomorphic convexity

Definition 2.5.1. Let $U \subset \mathbb{C}^n$ be a domain. Define the *holomorphic hull*

$$\widehat{K}_U \overset{\text{def}}{=} \{ z \in U : |f(z)| \leq \sup_{w \in K} |f(w)| \text{ for all } f \in \mathscr{O}(U) \}.$$

The domain U is *holomorphically convex* if whenever $K \subset\subset U$, then $\widehat{K}_U \subset\subset U$. In other words, U is holomorphically convex if it is convex with respect to moduli of holomorphic functions on U.*

It is a simple exercise (see below) to show that a holomorphically convex domain is Hartogs pseudoconvex. We will show below that the converse is the Levi problem for Hartogs pseudoconvex domains and is considerably more difficult. The thing is that there are lots of plurisubharmonic functions and they are easy to construct; we can even construct them locally and then piece them together. There are far fewer holomorphic functions, and clearly we cannot just construct them locally and expect the pieces to somehow fit together.

Exercise* 2.5.1*: *Prove that a holomorphically convex domain is Hartogs pseudoconvex.*

Exercise* 2.5.2*: *Compute the hull $\widehat{K}_{\mathbb{D}^n}$ of the set $K = \{ z \in \mathbb{D}^n : |z_j| = \lambda_j \text{ for } j = 1, \ldots, n \}$ where $0 \leq \lambda_j < 1$. Prove that the unit polydisc is holomorphically convex.*

Exercise* 2.5.3*: *Prove that a geometrically convex domain $U \subset \mathbb{C}^n$ is holomorphically convex.*

Exercise* 2.5.4*: *Prove that the Hartogs figure is not holomorphically convex.*

Exercise* 2.5.5*: *Let $U \subset \mathbb{C}^n$ be a domain and $f \in \mathscr{O}(U)$. Show that if U is holomorphically convex then $\widetilde{U} = U \setminus \{ z : f(z) = 0 \}$ is holomorphically convex. Note that you must first prove that $\widetilde{U}$ is connected.*

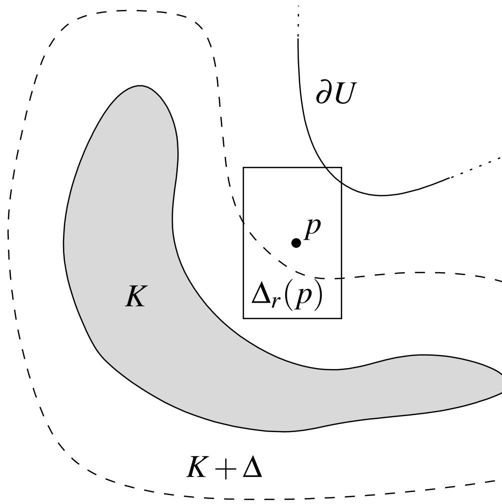
Theorem 2.5.2 (Cartan-Thullen)**.** *Let $U \subset \mathbb{C}^n$ be a domain. The following are equivalent:*

1. *U is a domain of holomorphy.*
2. *For $K \subset\subset U$,* $\operatorname{dist}(K, \partial U) = \operatorname{dist}(\widehat{K}_U, \partial U)$.
3. *U is holomorphically convex.*

Proof. Let us start with (1) $\Rightarrow$ (2). Let us suppose that there is a $K \subset\subset U$ with $\operatorname{dist}(K, \partial U) > \operatorname{dist}(\widehat{K}_U, \partial U)$. There exists a point $p \in \widehat{K}_U$ and a polydisc $\Delta = \Delta_r(0)$ with polyradius $r = (r_1, \ldots, r_n)$ such that $p + \Delta = \Delta_r(p)$ contains a point of ∂U, but

$$K + \Delta = \bigcup_{q \in K} \Delta_r(q) \subset\subset U.$$

*It is common to use $\widehat{K}_U$ rather than just $\widehat{K}$ to emphasize the dependence on U.



If $f \in \mathscr{O}(U)$, then there is an $M > 0$ such that $|f| \leq M$ on $K+\Delta$ as that is a relatively compact set. By the Cauchy estimates for each $q \in K$ we get

$$\left| \frac{\partial^\alpha f}{\partial z^\alpha}(q) \right| \leq \frac{M\alpha!}{r^\alpha}.$$

This inequality therefore holds on $\widehat{K}_U$ and hence at p. The series

$$\sum_\alpha \frac{1}{\alpha} \frac{\partial^\alpha f}{\partial z^\alpha}(p)(z-p)^\alpha$$

converges in $\Delta_r(p)$. Hence f extends to all of $\Delta_r(p)$ and $\Delta_r(p)$ contains points outside of U, in other words, U is not a domain of holomorphy.

The implication (2) $\Rightarrow$ (3) is easy.

Finally let us prove (3) $\Rightarrow$ (1). Suppose U is holomorphically convex. Let $p \in \partial U$. By convexity we choose nested compact sets $K_{j-1} \subsetneq K_j \subset\subset U$ such that $\bigcup_j K_j = U$, and $\widehat{(K_j)}_U = K_j$. We pick the sequence of K_j in such a way that there exists a sequence of points $p_j \in K_j \setminus K_{j-1}$ such that $\lim_{j\to\infty} p_j = p$.

Since p_j is not in the hull of K_{j-1}, we find a function $f_j \in \mathscr{O}(U)$ such that $|f_j| < 2^{-j}$ on K_{j-1} but such that

$$|f_j(p_j)| > j + \left| \sum_{k=1}^{j-1} f_k(p_j) \right|.$$

Finding such a function is left as an exercise below. The series $\sum_{k=1}^\infty f_k(z)$ converges uniformly on K_j as for all $k > j$, $|f_k| < 2^{-k}$ on K_j. As as the K_j exhaust U the series converges uniformly on compact subsets of U. Consequently,

$$f(z) = \sum_{k=1}^\infty f_k(z)$$

is a holomorphic function on U. We bound

$$|f(p_j)| \geq |f_j(p_j)| - \left| \sum_{k=1}^{j-1} f_k(p_j) \right| - \left| \sum_{k=j+1}^{\infty} f_k(p_j) \right| \geq j - \sum_{k=j+1}^{\infty} 2^{-k} \geq j - 1.$$

So $\lim_{j\to\infty} f(p_j) = \infty$. Clearly there cannot be any open $W \subset \mathbb{C}^n$ containing p to which f extends (see definition of domain of holomorphy). As any connected open W such that $W \setminus U \neq \emptyset$ must contain a point of ∂U, we are done. $\square$

Exercise* 2.5.6*: *Find the function $f_j \in \mathscr{O}(U)$ as indicated in the proof above.*

Exercise* 2.5.7*: *Extend the proof to show that if $U \subset \mathbb{C}^n$ is holomorphically convex then there exists a single function $f \in \mathscr{O}(U)$, that does not extend through any point $p \in U$.*

Chapter 3

CR Geometry

3.1 Real analytic functions and complexification

Definition 3.1.1. Let $U \subset \mathbb{R}^n$ be open. A function $f: U \to \mathbb{C}$ is said to be *real-analytic* (sometimes just *analytic* if clear from context) if at each point $p \in U$, the function f has a convergent power series that converges (absolutely) to f. A common notation for real-analytic is C^ω.

Before discuss the connection to holomorphic functions let us prove a simple lemma.

Lemma 3.1.2. *Let $\mathbb{R}^n \subset \mathbb{C}^n$ be the natural inclusion and suppose that $U \subset \mathbb{C}^n$ is a domain such that $U \cap \mathbb{R}^n \neq \emptyset$. Suppose $f, g: U \to \mathbb{C}$ be holomorphic functions such that $f = g$ on $U \cap \mathbb{R}^n$. Then $f = g$ on all of U.*

Proof. By taking $f - g$ we can assume that $g = 0$. Let $z = x + iy$ as usual and $\mathbb{R}^n$ is given by $y = 0$, and let us assume that $f = 0$ on $y = 0$. At every point $p \in \mathbb{R}^n \cap U$, notice that

$$0 = \frac{\partial f}{\partial x_j} = -i \frac{\partial f}{\partial y_j}.$$

Therefore, on $y = 0$,

$$\frac{\partial f}{\partial z_j} = 0.$$

Since the holomorphic function $\frac{\partial f}{\partial z_j} = 0$ on $y = 0$, then by induction all derivatives of f at p vanish, it has a zero power series. Hence f is identically zero in a neighborhood of p in $\mathbb{C}^n$ and by the identity theorem it is zero on all of U. $\square$

Let us return to $\mathbb{R}^n$ for a moment. We write a power series in $\mathbb{R}^n$ in multinomial notation as usual. Suppose that for some $a \in \mathbb{R}^n$ and some polyradius $r = (r_1, \ldots, r_n)$, the series

$$\sum_\alpha c_\alpha (x - a)^\alpha$$

converges whenever $|x_j - a_j| \leq r_j$ for all j. Here convergence is absolute convergence. That is,

$$\sum_{\alpha} |c_\alpha| |x-a|^\alpha$$

converges. If we replace $x_j \in \mathbb{R}$ with $z_j \in \mathbb{C}$ such that $|z_j - a_j| \leq |x_j - a_j|$, then the series still converges. Hence the series

$$\sum_{\alpha} c_\alpha (z-a)^\alpha$$

converges absolutely in $\Delta_r(a) \subset \mathbb{C}^n$.

Proposition 3.1.3 (Complexification part I)**.** *Suppose $U \subset \mathbb{R}^n$ is a domain and $f: U \to \mathbb{C}$ is real-analytic. Let $\mathbb{R}^n \subset \mathbb{C}^n$ be the natural inclusion. Then there exists a domain $V \subset \mathbb{C}^n$ such that $U \subset V$ and a unique holomorphic function $F: V \to \mathbb{C}$ such that $F|_U = f$.*

In particular, among many other things that follow from this proposition, we can now conclude that a real-analytic function is C^∞. Be careful and notice that U is a domain in $\mathbb{R}^n$ but is of course not an open set when considered as a subset of $\mathbb{C}^n$. Furthermore, V may be a very "thin" neighborhood around U. There is no way of finding V just from U. You need to also know f.

Proof. We have already proved the local version. But we have to prove that if we extend our f near every point, that we keep getting the same function. But that follows from the lemma above, any two such functions are equal on $\mathbb{R}^n$, and hence equal. There is a subtle topological technical point in this, so let us elaborate. Key topological fact is that we define V as a union of the polydiscs where the series converges. If there was a point where we get two distinct values, then this point must be in two distinct such polydiscs. The intersection of two polydiscs is always connected, so we can apply the lemma above. $\square$

Recall that a polynomial $P(x)$ in n real variables $(x_1, \ldots, x_n)$ is homogeneous of degree d if $P(sx) = s^d P(x)$ for all $s \in \mathbb{R}$ and $x \in \mathbb{R}^n$. That is, a homogeneous polynomial of degree d is a polynomial whose every monomial is of total degree d.If f is real-analytic near $a \in \mathbb{R}^n$, then write the power series of f at a as

$$\sum_{j=0}^{\infty} f_j(x-a),$$

where f_j is a homogeneous polynomial of degree j. The f_j is then called the *degree d homogeneous part* of f at a.

When dealing with real-analytic functions in $\mathbb{C}^n$, there is usually a better way to complexify. Suppose $U \subset \mathbb{C}^n \cong \mathbb{R}^{2n}$, and suppose $f: U \to \mathbb{C}$ is real-analytic. Let us assume that $a = 0$ for simplicity. Writing $z = x + iy$,

$$f(x,y) = \sum_{j=0}^{\infty} f_j(x,y) = \sum_{j=0}^{\infty} f_j \left(\frac{z+\bar{z}}{2}, \frac{z-\bar{z}}{2i} \right).$$

The polynomial f_j becomes a homogeneous polynomial of degree j in the variables z and $\bar{z}$. Therefore the entire series becomes a series in z and $\bar{z}$. As we mentioned before, we simply write $f(z,\bar{z})$, and when we consider the power series representation it will be in z and $\bar{z}$ rather than in x and y. In multinomial notation we write a power series at $a \in \mathbb{C}^n$ as

$$\sum_{\alpha,\beta} c_{\alpha,\beta} (z-a)^{\alpha} (\bar{z}-\bar{a})^{\beta}.$$

Notice that a holomorphic function is real-analytic, but not vice-versa. A holomorphic function is a real-analytic function that does not depend on $\bar{z}$.

Before we discuss complexification in terms of z and $\bar{z}$ we need the following lemma.

Lemma 3.1.4. *Let $U \subset \mathbb{C}^n \times \mathbb{C}^n$ be a domain, let the coordinates be $(z,\zeta) \in \mathbb{C}^n \times \mathbb{C}^n$, let*

$$D = \{ (z,\zeta) \in \mathbb{C}^n \times \mathbb{C}^n : \zeta = \bar{z} \},$$

and suppose $D \cap U \neq \emptyset$. Suppose $f,g \colon U \to \mathbb{C}$ be holomorphic functions such that $f = g$ on $D \cap U$. Then $f = g$ on all of U.

The set D is sometimes called the *diagonal*.

Proof. Again assume without loss of generality $g = 0$. Whenever $(z,\bar{z}) \in U$ we have $f(z,\bar{z}) = 0$, so using the chain rule

$$0 = \frac{\partial}{\partial \bar{z}_j} \big(f(z,\bar{z}) \big) = \frac{\partial f}{\partial \zeta_j}(z,\bar{z}).$$

Let us do this again with the z_j

$$0 = \frac{\partial}{\partial z_j} \big(f(z,\bar{z}) \big) = \frac{\partial f}{\partial z_j}(z,\bar{z}).$$

Each time we get another holomorphic function that is zero on D. By induction, for all α and β we get

$$0 = \frac{\partial^{|\alpha|+|\beta|}}{\partial z^{\alpha} \partial \bar{z}^{\beta}} \big(f(z,\bar{z}) \big) = \frac{\partial^{|\alpha|+|\beta|} f}{\partial z^{\alpha} \partial \zeta^{\beta}}(z,\bar{z}).$$

Therefore all holomorphic derivatives of f are zero on every point $(z,\bar{z})$. So f must be identically zero in a neighborhood of any point $(z,\bar{z})$. The lemma follows by the identity theorem. ☐

So let us start with a real-analytic function f. Suppose that for a polydisc $\Delta_r(a) \subset \mathbb{C}^n$ the series (in multinomial notation)

$$f(z,\bar{z}) = \sum_{\alpha,\beta} c_{\alpha,\beta} (z-a)^{\alpha} (\bar{z}-\bar{a})^{\beta}$$

converges. By convergence we mean absolute convergence as we discussed before: that is,

$$\sum_{\alpha,\beta} |c_{\alpha,\beta}| |z-a|^{\alpha} |\bar{z}-\bar{a}|^{\beta}$$

converges. Therefore the series still converges if we replace $\bar{z}_j$ with ζ_j where $|\zeta_j - \bar{a}| \leq |\bar{z}_j - \bar{a}|$. So the series

$$F(z,\zeta) = \sum_{\alpha,\beta} c_{\alpha,\beta} (z-a)^{\alpha} (\zeta - \bar{a})^{\beta}$$

converges for all $(z,\zeta) \in \Delta_r(a) \times \Delta_r(\bar{a})$.

Putting together the above with the lemma we obtain.

Proposition 3.1.5 (Complexification part II)**.** *Suppose $U \subset \mathbb{C}^n$ is a domain and $f\colon U \to \mathbb{C}$ is real-analytic. Then there exists a domain $V \subset \mathbb{C}^n \times \mathbb{C}^n$ such that*

$$\{(z,\zeta) : \zeta = \bar{z} \text{ and } z \in U\} \subset V,$$

and a unique holomorphic function $F\colon V \to \mathbb{C}$ such that $F(z,\bar{z}) = f(z,\bar{z})$ for all $z \in U$.

The function f can be thought of as the restriction of F to the set where $\zeta = \bar{z}$. We will abuse notation and write simply $f(z,\zeta)$ both for f and its extension.

Remark 3.1.6. The domain V above is not simply U times the conjugate of U. In general it is much smaller. For example a real-analytic $f\colon \mathbb{C}^n \to \mathbb{C}$ does not necessarily complexify to all of $\mathbb{C}^n \times \mathbb{C}^n$. That is because the domain of convergence for a real-analytic function on $\mathbb{C}^n$ is not necessarily all of $\mathbb{C}^n$. For example, in one dimension the function

$$f(z,\bar{z}) = \frac{1}{1+|z|^2}$$

is real-analytic on $\mathbb{C}$, but it is not a restriction to the diagonal of a holomorphic function on all of $\mathbb{C}^2$. The problem is that the complexified function

$$f(z,\zeta) = \frac{1}{1+z\zeta}$$

cannot be defined along the set where $z\zeta = 1$, which by a fluke never happens when $\zeta = \bar{z}$.

Remark 3.1.7. This form of complexification is sometimes called *polarization* due to its relation to the polarization identities*. That is, suppose A is a Hermitian matrix, we can recover A and therefore the sesquilinear form $\langle Az, w\rangle$ for $z, w \in \mathbb{C}^n$, by simply knowing the values of

$$\langle Az, z\rangle = z^* A z = \sum_{j,k=1}^{n} a_{jk} \bar{z}_j z_k$$

for all $z \in \mathbb{C}^n$. In fact Proposition 3.1.5 is really polarization in an infinite dimensional Hilbert space.

*Such as $4\langle z,w\rangle = \|z+w\|^2 - \|z-w\|^2$.

The idea of treating $\bar{z}$ as a separate variable is very powerful, and as we have just seen it is completely natural when speaking about real-analytic functions. This is one of the reasons why real-analytic functions play a special role in several complex variables.

Example 3.1.8: Not every C^∞ smooth function is real-analytic. For example, on the real line

$$f(x) = \begin{cases} e^{-1/x^2} & \text{if } x > 0, \\ 0 & \text{if } x \leq 0. \end{cases}$$

The function $f: \mathbb{R} \to \mathbb{R}$ is C^∞, $f^{(k)}(0) = 0$ for all k, and so its Taylor series at the origin does not converge to f in any neighborhood of the origin.

> ***Exercise* 3.1.1*:*** *Prove the statements of the above example.*

Definition 3.1.9. A real hypersurface $M \subset \mathbb{R}^n$ is said to be real-analytic if locally at every point it is the graph of a real-analytic function. That is near every point (that is, locally), after perhaps a rotation M can be written as

$$y = \varphi(x)$$

where φ is real-analytic.

Compare this definition to Definition 2.2.1. In fact we could define a real-analytic hypersurface as in Definition 2.2.1 and then prove an analogue of Lemma 2.2.5 to show that this would be identical to the above definition. The above definition will be sufficient and so we avoid the complication and leave it to the reader.

> ***Exercise* 3.1.2*:*** *Show that the definition above is equivalent to an analogue of Definition 2.2.1. That is, state the alternative definition of real-analytic hypersurface and then prove the analogue of Lemma 2.2.5.*

A mapping to $\mathbb{R}^m$ is real analytic if all the components are real analytic functions. Via complexification we can give a simple proof of the following result.

Proposition 3.1.10. *Let $U \subset \mathbb{R}^n$, $V \subset \mathbb{R}^k$ be domains and let $f: U \to V$ and $g: V \to \mathbb{R}^m$ be real-analytic. Then $g \circ f$ is real-analytic.*

Proof. let $x \in \mathbb{R}^n$ be our coordinates in U and $y \in \mathbb{R}^k$ be our coordinates in V. We can complexify $f(x)$ and $g(y)$ by allowing x to be a complex vector in a small neighborhood of U in $\mathbb{C}^n$ and y to be a complex vector in a small neighborhood of V in $\mathbb{C}^k$. So we treat f and g as holomorphic functions. On a certain neighborhood of U in $\mathbb{C}^n$, the composition $f \circ g$ makes sense and it is holomorphic as composition of holomorphic mappings is holomorphic. Restricting the complexified $f \circ g$ back to $\mathbb{R}^n$ we obtain a real-analytic function. $\square$

The proof demonstrates a very simple application of complexification. Many properties of holomorphic functions are easy to prove because holomorphic functions are solutions to certain PDE (the Cauchy-Riemann equations). However, there is no PDE that defines real-analytic functions, so complexification provides a useful tool to transfer certain properties of holomorphic functions to real-analytic functions. We must be careful however, hypotheses on real-analytic functions only give us hypotheses on certain points of the complexified holomorphic functions.

Exercise* 3.1.3*: *Suppose $\varphi \colon U \to \mathbb{R}$ is a pluriharmonic function for $U \subset \mathbb{C}^n$. Knowing that φ is real-analytic, let $z_0 \in U$ be fixed. Using complexification, write down a formula for a holomorphic function near z_0 whose real part is φ.*

3.2 CR functions

We first need to know what it means for a function $f \colon X \to \mathbb{C}$ to be smooth if X is not an open set, for example a hypersurface.

Definition 3.2.1. Let $X \subset \mathbb{R}^n$ be a set. The function $f \colon X \to \mathbb{C}$ is smooth (resp. real-analytic) if for each point $p \in X$ there is a neighborhood $U \subset \mathbb{R}^n$ of p and a smooth (resp. real-analytic) $F \colon U \to \mathbb{C}$ such that $F(q) = f(q)$ for $q \in X \cap U$.

For an arbitrary set X, issues surrounding this definition can be very subtle. It is very natural however if X is nice, such as a hypersurface, or if X is a closure of a domain with smooth boundary.

Proposition 3.2.2. *If $M \subset \mathbb{R}^n$ is a smooth (resp. real-analytic) real hypersurface, then $f \colon M \to \mathbb{C}$ is smooth (resp. real-analytic) if and only if whenever near some point we write M as*

$$y = \varphi(x)$$

for a smooth (resp. real-analytic) function φ, then the function $f\big(x, \varphi(x)\big)$ is a smooth (resp. real-analytic) function of x.

Exercise* 3.2.1*: *Prove the proposition above.*

Exercise* 3.2.2*: *Prove that in the definition if X is a smooth or real-analytic hypersurface, then the function F from the definition is never unique, even for a fixed neighborhood U.*

Definition 3.2.3. Let $M \subset \mathbb{C}^n$ be a smooth real hypersurface. Then a smooth function $f \colon M \to \mathbb{C}$ is a *smooth CR function* if

$$X_p f = 0$$

for all $p \in M$ and all vectors $X_p \in T_p^{(0,1)} M$.

Remark 3.2.4. Of course one only needs one-derivative in the above definition. One can also define a continuous CR function if the derivative is taken in the distribution sense, but we digress.

Remark 3.2.5. When $n = 1$, a real hypersurface $M \subset \mathbb{C}$ is a curve and $T_p^{(0,1)}M$ is trivial. Therefore, all functions are CR functions.

Proposition 3.2.6. *Let $M \subset U$ be a smooth (resp. real-analytic) real hypersurface in a domain $U \subset \mathbb{C}^n$. Suppose $F : U \to \mathbb{C}$ is a holomorphic function, then the restriction $f = F|_M$ is a smooth (resp. real-analytic) CR function.*

Proof. First let us prove that f is smooth. Given any $p \in M$ write M as $\operatorname{Im} w = \varphi(z, \bar{z}, \operatorname{Re} w)$ for a smooth φ. Then $F\bigl(z, \operatorname{Re} w + i\varphi(z, \bar{z}, \operatorname{Re} w)\bigr)$ is clearly smooth as it is a composition of smooth functions. If both M and f are real-analytic we obtain that $F\bigl(z, \operatorname{Re} w + i\varphi(z, \bar{z}, \operatorname{Re} w)\bigr)$ is real-analytic, which we could also prove directly by complexifying as before.

Let us show it is CR. We have $X_p F = 0$ for all $X_p \in T_p^{(0,1)}\mathbb{C}^n$. As $T_p^{(0,1)}M \subset T_p^{(0,1)}\mathbb{C}^n$ we have $X_p f = 0$ for all $X_p \in T_p^{(0,1)}M$. □

Not every smooth CR function is a restriction of a holomorphic function.

Example 3.2.7: Take the smooth function $f : \mathbb{R} \to \mathbb{R}$ we defined before that is not real-analytic at the origin. Take $M \subset \mathbb{C}^2$ be the set defined by $\operatorname{Im} z_2 = 0$, this is a real-analytic real hypersurface. Clearly $T_p^{(0,1)}M$ is one complex dimensional and at each point $\frac{\partial}{\partial \bar{z}_1}$ is tangent and therefore spans $T_p^{(0,1)}M$. Define $g : M \to \mathbb{C}$ by

$$g(z_1, z_2, \bar{z}_1, \bar{z}_2) = f(\operatorname{Re} z_2).$$

Then g is CR as it is independent of $\bar{z}_1$. If $G : U \subset \mathbb{C}^2 \to \mathbb{C}$ is a holomorphic function where U is some open set containing the origin, then G restricted to M must be real-analytic (a power series in $\operatorname{Re} z_1$, $\operatorname{Im} z_1$, and $\operatorname{Re} z_2$) and therefore G cannot equal to g on M.

***Exercise* 3.2.3:** *Find a smooth CR function on the sphere $S^{2n-1} \subset \mathbb{C}^n$ that is not a restriction of a holomorphic function of a neighborhood of S^{2n-1}.*

***Exercise* 3.2.4:** *Show that there is no maximum principle of CR functions. In fact, find a smooth hypersurface $M \subset \mathbb{C}^n$, $n \geq 2$, and a smooth CR function f on M such that $|f|$ attains a strict maximum at a point.*

***Exercise* 3.2.5:** *Suppose $M \subset \mathbb{C}^n$, $n \geq 2$, is the hypersurface given by $\operatorname{Im} z_n = 0$. Show that any smooth CR function on M is holomorphic in the variables $z_1, \ldots, z_{n-1}$. Use this to show that for no smooth CR function f on M can $|f|$ attain a* strict *maximum on M. But show that there do exist functions such that $|f|$ attains a (nonstrict) maximum M.*

Real-analytic CR functions on a real-analytic hypersurface M always extend to holomorphic functions of a neighborhood of M. Before we prove that fact, let us find a convenient way to write the defining equation for a real-analytic hypersurface.

Proposition 3.2.8. *Suppose $M \subset \mathbb{C}^n$ is a real-analytic hypersurface and $p \in M$. Then there are holomorphic coordinates near p taking p to 0, such that locally M is given by*

$$\bar{w} = \Phi(z, \bar{z}, w),$$

for a holomorphic function Φ defined on a neighborhood of the origin in $\mathbb{C}^{n-1} \times \mathbb{C}^{n-1} \times w$ vanishing to second order. Furthermore, the locally a basis for $T^{(0,1)}M$ vector fields is given by

$$\frac{\partial}{\partial \bar{z}_j} + \frac{\partial \Phi}{\partial \bar{z}_j} \frac{\partial}{\partial \bar{w}}, \qquad j = 1, \ldots, n-1.$$

Proof. Find coordinates such that M is given by

$$\operatorname{Im} w = \varphi(z, \bar{z}, \operatorname{Re} w),$$

Write the defining function as $r(z, w, \bar{z}, \bar{w}) = -{}^1\!/\!{}_{2i}(w - \bar{w}) + \varphi\big(z, \bar{z}, {}^1\!/\!{}_2(w + \bar{w})\big)$. Complexifying we can write $r(z, w, \zeta, \omega)$ as a holomorphic function of $2n$ variables and the derivative in ω (that is $\bar{w}$) does not vanish near the origin. We use the implicit function theorem for holomorphic functions to write $r = 0$ as

$$\omega = \Phi(z, \zeta, w).$$

Restricting to the diagonal, $\bar{w} = \omega$ and $\bar{z} = \zeta$, we get the result. The statement about the CR vector fields then follows since those vector fields annihilate the defining function $\Phi(z, \bar{z}, w) - \bar{w}$. $\square$

Proposition 3.2.9. *Suppose $M \subset \mathbb{C}^n$ is a real-analytic hypersurface and $p \in M$. For any real-analytic CR function $f \colon M \to \mathbb{C}$, there exists a holomorphic function $F \in \mathscr{O}(U)$ for a neighborhood U of p such that $F|_{M \cap U} = f$.*

Proof. Write M near p as $\bar{w} = \Phi(z, \bar{z}, w)$. Let $\mathscr{M}$ be the set in the $2n$ variables (z, w, ζ, ω) given by $\omega = \Phi(z, \zeta, w)$. Take $f(z, w, \bar{z}, \bar{w})$ and consider any real-analytic extension to a neighborhood. Complexify as before to $f(z, w, \zeta, \omega)$. On $\mathscr{M}$ we have $f(z, w, \zeta, \omega) = f\big(z, w, \zeta, \Phi(z, \zeta, w)\big)$. Let

$$F(z, w, \zeta) = f\big(z, w, \zeta, \Phi(z, \zeta, w)\big).$$

Clearly $F(z, w, \bar{z})$ equals f on M. Then as f is a CR function it is annihilated by $\frac{\partial}{\partial \bar{z}_j} + \frac{\partial \Phi}{\partial \bar{z}_j} \frac{\partial}{\partial \bar{w}}$ on M. So

$$\frac{\partial F}{\partial \zeta_j} + \frac{\partial \Phi}{\partial \zeta_j} \frac{\partial F}{\partial \omega} = \frac{\partial F}{\partial \zeta_j} = 0$$

on $\mathscr{M}$ where $\bar{z} = \zeta$ (and $\bar{w} = \omega$), and therefore it is true on $\mathscr{M}$ or in other for all z, ζ, w in a neighborhood. But that means that

$$\frac{\partial F(z, w, \bar{z})}{\partial \bar{z}_j} = 0$$

for all j and F is actually a holomorphic function in z and w only. $\square$

CR functions can often be considered as boundary values of holomorphic functions.

Proposition 3.2.10. *Suppose $U \subset \mathbb{C}^n$ is a domain with smooth boundary. Suppose $f : \overline{U} \to \mathbb{C}$ is smooth and holomorphic on U. Then $f|_{\partial U}$ is a smooth CR function.*

Proof. The function $f|_{\partial U}$ is clearly smooth.

Suppose $p \in \partial U$. If $X_p \in T_p^{(0,1)} \partial U$ is such that

$$X_p = \sum_{j=1}^{n} a_j \frac{\partial}{\partial \bar{z}_j}\Big|_p,$$

take $\{q_k\}$ in U that approaches p, then take

$$X_{q_k} = \sum_{j=1}^{n} a_j \frac{\partial}{\partial \bar{z}_j}\Big|_{q_k}.$$

If $X_{q_k} f = 0$ for all k and by continuity $X_p f = 0$. □

The boundary values of a holomorphic function define the function uniquely. In particular we have the following result you know from one variable.

Proposition 3.2.11. *Suppose $U \subset \mathbb{C}^n$ is a domain with smooth boundary and $f : \overline{U} \to \mathbb{C}$ is continuous and holomorphic on U. If $f = 0$ on an open subset of ∂U, then $f = 0$ on all of U.*

Proof. Take $p \in \partial U$ such that $f = 0$ on a neighborhood of p in ∂U. Near p write U as

$$\operatorname{Im} w > \varphi(z, \bar{z}, \operatorname{Re} w)$$

for $(z, w) \in \mathbb{C}^{n-1} \times \mathbb{C}$, and where φ vanishes to second order. Next fix any small z. Considering f as a function of w defined on $\operatorname{Im} w \geq \varphi(z, \bar{z}, \operatorname{Re} w)$ we obtain from the corresponding one dimensional result that f is identically zero inside the domain. As this held for every fixed z it holds in an open set of U and by identity it holds everywhere. □

***Exercise* 3.2.6:** *Find a domain $U \subset \mathbb{C}^n$, $n \geq 2$, with smooth boundary and a smooth CR function f on ∂U such that there is no holomorphic function on U or $\mathbb{C}^n \setminus U$ whose boundary values are f.*

***Exercise* 3.2.7:** *a) Suppose $U \subset \mathbb{C}^n$ is a bounded domain with smooth boundary. Suppose $f : \overline{U} \to \mathbb{C}$ is a continuous function holomorphic in U. Suppose $f|_{\partial U}$ is real-valued. Show that f is constant. b) Find a counterexample to the statement if you allow U to be unbounded.*

3.3 Approximation of CR functions

The following theorem (proved circa 1980) holds in much more generality, but we state its simplest version. One of the simplifications we make is that we consider only smooth CR functions here, although the theorem holds even for continuous CR functions where the CR conditions are interpreted in the sense of distributions.

Theorem 3.3.1 (Baouendi-Trèves)**.** *Suppose $M \subset \mathbb{C}^n$ is a smooth real hypersurface. Let $p \in M$ be fixed and let $z = (z_1, \ldots, z_n)$ be holomorphic coordinates near p. Then there exists a compact neighborhood $K \subset M$ of p, such that for any smooth CR function $f \colon M \to \mathbb{C}$, there exists a sequence $\{p_j\}$ of polynomials such that*

$$p_j(z) \to f(z) \qquad \textit{uniformly in } K.$$

A key point is that K cannot be chosen arbitrarily, it depends on p and M. On the other hand it does not depend on f. So given M and $p \in M$ there is a K such that *every* CR function on M is approximated uniformly on K. Also note that the theorem also applies in $n = 1$ (in one dimension), although in that case it follows from the more general Mergelyan theorem.

Example 3.3.2: Let us show that K cannot possibly be arbitrary. Let us give an example in one dimension. Let $S^1 \subset \mathbb{C}$ be the unit circle (boundary of the disc), then any smooth function on S^1 is a smooth CR function. So pick let us say a nonconstant real function such as $\operatorname{Re} z$. Let us suppose for contradiction that we could take $K = S^1$. Then $\operatorname{Re} z$ would be uniformly approximated by holomorphic polynomials on S^1. By the maximum principle, the polynomials would converge on $\mathbb{D}$ to a holomorphic function on $\mathbb{D}$ continuous on $\overline{\mathbb{D}}$ this function would have real boundary values, which is impossible. Clearly K cannot be the entire circle.

In several variables the example can easily be extended by considering $S^1 \times \mathbb{C}$, then $\operatorname{Re} z_1$ is a smooth CR function that cannot be approximated uniformly by holomorphic polynomials on $S^1 \times \{0\}$.

The technique of the above example will be used later in a more general situation, to extend CR functions using Baouendi-Trèves.

Remark 3.3.3. It is important to note the difference between Baouendi-Trèves (and similar theorems in complex analysis) and the Weierstrass approximation theorem. In Baouendi-Trèves we obtain approximation by holomorphic polynomials, while Weierstrass gives us polynomials in the real variables, or in z and $\bar{z}$. For example, via Weierstrass, any continuous function is uniformly approximable on S^1 via polynomials in $\operatorname{Re} z$ and $\operatorname{Im} w$, and therefore by polynomials in z and $\bar{z}$, but these polynomials will not in general converge anywhere but on S^1.

Exercise* 3.3.1*: *Let $z = x + iy$ as usual in $\mathbb{C}$. Find a sequence of polynomials in x and y that converge uniformly to e^{x-y} on S^1, but diverge everywhere else.*

The proof is an ingenious use of the standard technique used to prove the Weierstrass approximation theorem. Also, as we have seen mollifiers before, the technique will not be completely foreign even to the reader who does not know the Weierstrass approximation theorem. Basically what we do is use the standard convolution argument, this time against a holomorphic function. Letting $z = x + iy$ we only do the convolution in the x variables keeping $y = 0$. Then we use the fact that the function is CR to show that we get an approximation even for other y.

In the formulas below, given any vector $v = (v_1, \dots, v_n)$, it will be useful to write

$$[v]^2 \overset{\text{def}}{=} v_1^2 + \cdots + v_n^2.$$

The following lemma is a neat application of ideas from several complex variables to solve a problem that does not at first seems to involve holomorphic functions.

Lemma 3.3.4. *Let W be the set of $n \times n$ complex matrices A such that*

$$\|(\operatorname{Im} A)x\| < \|(\operatorname{Re} A)x\|$$

for all nonzero $x \in \mathbb{R}^n$ and $\operatorname{Re} A$ is positive definite. Then for all $A \in W$,

$$\int_{\mathbb{R}^n} e^{-[Ax]^2} \det A \, dx = \pi^{n/2}.$$

Proof. Suppose that A has real entries and A is positive definite (so A is also invertible). By a change of coordinates

$$\int_{\mathbb{R}^n} e^{-[Ax]^2} \det A \, dx = \int_{\mathbb{R}^n} e^{-[x]^2} \, dx = \left(\int_{\mathbb{R}} e^{-x_1^2} \, dx_1 \right) \cdots \left(\int_{\mathbb{R}} e^{-x_n^2} \, dx_n \right) = (\sqrt{\pi})^n.$$

Next suppose that A is any matrix in W. There is some $\varepsilon > 0$ such that $\|(\operatorname{Im} A)x\|^2 \leq (1 - \varepsilon^2)\|(\operatorname{Re} A)x\|^2$ for all $x \in \mathbb{R}^n$. That is because we only need to check this for x in the unit sphere, which is compact (exercise). Also note that by reality of $\operatorname{Re} A$, $\operatorname{Im} A$, and x we get $[(\operatorname{Re} A)x]^2 = \|(\operatorname{Re} A)x\|^2$ and $[(\operatorname{Im} A)x]^2 = \|(\operatorname{Im} A)x\|^2$.

$$\left| e^{-[Ax]^2} \right| \leq e^{-[(\operatorname{Re} A)x]^2 + [(\operatorname{Im} A)x]^2} \leq e^{-\varepsilon^2 [(\operatorname{Re} A)x]^2}.$$

Therefore the integral exists for all A in W.

The expression

$$\int_{\mathbb{R}^n} e^{-[Ax]^2} \det A \, dx$$

is a well-defined holomorphic function in the entries of A thinking of W as a domain (see exercises below) in $\mathbb{C}^{n^2}$. We have a holomorphic function that is constantly equal to $\pi^{n/2}$ on $W \cap \mathbb{R}^{n^2}$ and hence it is equal to $\pi^{n/2}$ everywhere on W. □

***Exercise* 3.3.2:** *Prove the existence of $\varepsilon > 0$ in the proof above.*

***Exercise* 3.3.3:** *Show that $W \subset \mathbb{C}^{n^2}$ in the proof above is a domain (open and connected).*

***Exercise* 3.3.4:** *Prove that we can really differentiate under the integral to show that the integral is holomorphic in the entries of A.*

***Exercise* 3.3.5:** *To show that some hypotheses are needed for the lemma. In particular take $n = 1$ and find the exact set of A (now just a complex number) for which the theorem is true.*

Below for an $n \times n$ matrix A, we use the standard operator norm

$$\|A\| = \sup_{\|v\|=1} \|Av\| = \sup_{v \in \mathbb{C}^n, v \neq 0} \frac{\|Av\|}{\|v\|}.$$

***Exercise* 3.3.6:** *Let W be as in Lemma 3.3.4. Let B be an $n \times n$ real matrix such that $\|B\| < 1$. Show that $I + iB \in W$.*

Proof of the theorem of Baouendi-Trèves. Suppose $M \subset \mathbb{C}^n$ is a smooth real hypersurface, and without loss of generality suppose $p = 0 \in M$. Let $z = (z_1, \ldots, z_n)$ be the holomorphic coordinates, write $z = x + iy$, $y = (y', y_n)$, and suppose that M is given by

$$y_n = \psi(x, y').$$

with ψ vanishing to second order. Note that (x, y') parametrize M near 0. In other words,

$$z_j = x_j + iy_j, \qquad \text{for } j < n, \text{ and } \qquad z_n = x_n + i\psi(x, y').$$

Write the mapping

$$\varphi(x, y') = \big(y_1, \ldots, y_{n-1}, \psi(x, y')\big).$$

We can then write $z = x + i\varphi(x, y')$ as our parametrization.

Let $r > 0$ and $d > 0$ be small numbers to be determined later. We can assume that they are small enough such that f and φ are defined on some neighborhood of the set where $\|x\| \leq r$ and $\|y'\| \leq d$.

There exists a smooth function $g \colon \mathbb{R}^n \to [0, 1]$ such that $g \equiv 1$ on $B_{r/2}(0)$ and $g \equiv 0$ outside of $B_r(0)$. Explicit formula can be given, or we can also obtain such a function by use of mollifiers on the function that is identically one on $B_{3r/4}(0)$ and zero elsewhere. Such a g is commonly called a *cutoff function*.

***Exercise* 3.3.7:** *Find an explicit formula for g without using mollifiers.*

Let
$$K' = \{(x,y') : \|x\| \leq r/4, \|y'\| \leq d\}.$$
Let $K = z(K')$, that is the image of K' under the mapping $z(x,y')$.

Let us consider the CR function f to be a function of (x,y') and write $f(x,y')$. We will also think of z as a function of (x,y'). For $\ell \in \mathbb{N}$, let α_ℓ be a differential n-form defined (thinking of $w \in \mathbb{C}^n$ as a constant parameter) by

$$\begin{aligned}
\alpha_\ell(x,y') &= \left(\frac{\ell}{\pi}\right)^{n/2} e^{-\ell[w-z]^2} g(x) f(x,y')\, dz \\
&= \left(\frac{\ell}{\pi}\right)^{n/2} e^{-\ell[w-x-i\varphi(x,y')]^2} g(x) f(x,y') \\
&\qquad (dx_1 + i dy_1) \wedge \cdots \wedge (dx_{n-1} + i dy_{n-1}) \wedge \big(dx_n + i d\psi(x,y')\big).
\end{aligned}$$

The key is the exponential which looks like the bump function mollifier, except that now we have w and z possibly complex. The exponential is holomorphic in w and that is key. And as long as we do not stray far in the y' direction it should go to zero quickly for $w \neq z$.

Fix y' with $0 < \|y'\| < d$ and let D be defined by
$$D = \{(x,s) \in \mathbb{R}^n \times \mathbb{R}^{n-1} : \|x\| < r \text{ and } s = ty' \text{ for } t \in (0,1)\}.$$

D is an $n+1$ dimensional "cylinder." That is, we take a ball in the x direction and then take a single fixed direction y'. We orient D in the standard way as if it sat in the (x,t) variables in $\mathbb{R}^n \times \mathbb{R}$. Via Stokes' theorem we get
$$\int_D d\alpha_\ell(x,s) = \int_{\partial D} \alpha_\ell(x,s).$$
Since $g(x) = 0$ if $\|x\| \geq r$ then

$$\begin{aligned}
\int_{\partial D} \alpha_\ell(x,s) = &\left(\frac{\ell}{\pi}\right)^{n/2} \int_{x\in\mathbb{R}^n} e^{-\ell[w-x-i\varphi(x,y')]^2} g(x) f(x,y')\, dx_1 \wedge \cdots \wedge dx_{n-1} \wedge \big(dx_n + i d_x\psi(x,y')\big) \\
&- \left(\frac{\ell}{\pi}\right)^{n/2} \int_{x\in\mathbb{R}^n} e^{-\ell[w-x-i\varphi(x,0)]^2} g(x) f(x,0)\, dx_1 \wedge \cdots \wedge dx_{n-1} \wedge \big(dx_n + i d_x\psi(x,0)\big),
\end{aligned} \tag{3.1}$$

where d_x means the derivative in the x directions only. That is, $d_x\psi = \frac{\partial\psi}{\partial x_1}dx_1 + \cdots + \frac{\partial\psi}{\partial x_n}dx_n$. As is usual in these types of arguments the integral extends to all of $\mathbb{R}^n$ because of g. We can ignore that f and φ are undefined where g is identically zero.

We will show that the left hand side of (3.1) goes to zero uniformly for $w \in K$ and the first term on the right hand side will go to $f(\tilde{x}, y')$ if $w = z(\tilde{x}, y')$ is in M. We define entire functions that we will show approximate f

$$f_\ell(w) = \left(\frac{\ell}{\pi}\right)^{n/2} \int_{x\in\mathbb{R}^n} e^{-\ell[w-x-i\varphi(x,0)]^2} g(x) f(x,0)\, dx_1 \wedge \cdots \wedge dx_{n-1} \wedge \big(dx_n + i d_x \psi(x,0)\big).$$

Clearly each f_ℓ is holomorphic and defined for all $w \in \mathbb{C}^n$.

In the next claim it is important that f is a CR function.

Claim 3.3.5. *We have*

$$d\alpha_\ell(x,s) = \left(\frac{\ell}{\pi}\right)^{n/2} e^{-\ell[w-z(x,s)]^2} f(x,s)\, dg(x) \wedge dz(x,s),$$

and for sufficiently small $r > 0$ and $d > 0$,

$$\lim_{\ell\to\infty} \left(\frac{\ell}{\pi}\right)^{n/2} \int_{(x,s)\in D} e^{-\ell[w-z(x,s)]^2} f(x,s)\, dg(x) \wedge dz(x,s) = 0$$

uniformly as a function of $w \in K$ and $y' \in B_d(0)$ (recall that D depends on y').

Proof. First we claim that at each point df is a linear combination of dz_1 through dz_n (recall that we are considering f and $z_1, \ldots, z_n$ as functions on M). After a complex affine change of coordinates we simply need to show this at the origin. Let the new holomorphic coordinates be $\xi_1, \ldots, \xi_n$, and suppose the $T_0^{(1,0)}M$ tangent space is spanned by $\frac{\partial}{\partial \xi_1}, \ldots, \frac{\partial}{\partial \xi_{n-1}}$, and such that $\frac{\partial}{\partial \operatorname{Re} \xi_n}$ is tangent and $\frac{\partial}{\partial \operatorname{Im} \xi_n}$ is normal. At the origin the CR conditions give us

$$df(0) = \frac{\partial f}{\partial \xi_1}(0)\, d\xi_1(0) + \cdots + \frac{\partial f}{\partial \xi_{n-1}}(0)\, d\xi_{n-1}(0) + \frac{\partial f}{\partial \operatorname{Re} \xi_n}(0)\, d(\operatorname{Re} \xi_n)(0).$$

But at the origin $d\xi_n(0) = d(\operatorname{Re} \xi_n)(0)$. As ξ is an affine function of z, then $d\xi_j$ are linear combinations of dz_1 through dz_n, an the claim follows. So for any CR function f we get that $d(f\, dz) = df \wedge dz = 0$ since $dz_j \wedge dz_j = 0$ of course.

Next note that $e^{-\ell[w-z(x,s)]^2}$ is a CR function as a function of (x,s), and so is $f(x,s)$. Therefore

$$d\alpha_\ell(x,s) = \left(\frac{\ell}{\pi}\right)^{n/2} e^{-\ell[w-z(x,s)]^2} f(x,s)\, dg(x) \wedge dz(x,s).$$

Since g is zero for $\|x\| \leq r/2$ we get that

$$\int_D d\alpha_\ell(x,s) = \left(\frac{\ell}{\pi}\right)^{n/2} \int_D e^{-\ell[w-z(x,s)]^2} f(x,s)\, dg(x) \wedge dz(x,s)$$

is only evaluated for the subset of D where $\|x\| > r/2$.

Suppose $w \in K$ and $(x,s) \in D$ with $\|x\| > r/2$. We need to estimate

$$\left|e^{-\ell[w-z(x,s)]^2}\right| = e^{-\ell \operatorname{Re}[w-z(x,s)]^2}.$$

Let $w = z(\tilde{x},\tilde{s})$. Then

$$-\operatorname{Re}[w-z]^2 = -\|\tilde{x}-x\|^2 + \|\varphi(\tilde{x},\tilde{s}) - \varphi(x,s)\|^2.$$

By the mean function theorem

$$\|\varphi(\tilde{x},\tilde{s}) - \varphi(x,s)\| \leq \|\varphi(\tilde{x},\tilde{s}) - \varphi(x,\tilde{s})\| + \|\varphi(x,\tilde{s}) - \varphi(x,s)\| \leq a\|\tilde{x}-x\| + A\|\tilde{s}-s\|,$$

where a and A are

$$a = \sup_{\|\hat{x}\|\leq r, \|\hat{y}'\|\leq d} \left\| \frac{\partial \varphi}{\partial x}(\hat{x},\hat{y}') \right\|, \qquad A = \sup_{\|\hat{x}\|\leq r, \|\hat{y}'\|\leq d} \left\| \frac{\partial \varphi}{\partial y'}(\hat{x},\hat{y}') \right\|.$$

Where $\left[\frac{\partial\varphi}{\partial x}\right]$ and $\left[\frac{\partial\varphi}{\partial y'}\right]$ are the derivatives (matrices) of φ with respect to x and y' respectively, and the norm we are taking is the operator norm. Because $\left[\frac{\partial\varphi}{\partial x}\right]$ is zero at the origin, we can pick r and d small enough (and hence K small enough) so that $a \leq 1/4$. We can furthermore pick d possibly even smaller to ensure that $d \leq \frac{r}{32A}$. We have that $r/2 \leq \|x\| \leq r$, but $\|\tilde{x}\| \leq r/4$, so

$$\frac{r}{4} \leq \|\tilde{x}-x\| \leq \frac{5r}{4}.$$

We also have $\|\tilde{s}-s\| \leq 2d$ by triangle inequality.

Therefore,

$$\begin{aligned}
-\operatorname{Re}[w-z(x,s)]^2 &\leq -\|\tilde{x}-x\|^2 + a^2\|\tilde{x}-x\|^2 + A^2\|\tilde{s}-s\|^2 + 2aA\|\tilde{x}-x\|\|\tilde{s}-s\| \\
&\leq \frac{-15}{16}\|\tilde{x}-x\|^2 + A^2\|\tilde{s}-s\|^2 + \frac{A}{2}\|\tilde{x}-x\|\|\tilde{s}-s\| \\
&\leq \frac{-r^2}{64}.
\end{aligned}$$

In other words

$$\left|e^{-\ell[w-z(x,s)]^2}\right| \leq e^{-\ell r^2/64}$$

or

$$\left| \left(\frac{\ell}{\pi}\right)^{n/2} \int_{(x,s)\in D} e^{-\ell[w-z(x,s)]^2} f(x,s)\, dg(x) \wedge dz(x,s) \right| \leq C\ell^{n/2} e^{-\ell r^2/64}.$$

For a constant C. Do notice that D depends on y'. By looking at all y' with $\|y'\| \leq d$, which is a compact set, we can make C large enough to not depend on the y' that was chosen. The claim follows. □

Claim 3.3.6. *For the given $r > 0$ and $d > 0$,*

$$\lim_{\ell\to\infty} \left(\frac{\ell}{\pi}\right)^{n/2} \int_{x\in\mathbb{R}^n} e^{-\ell[\tilde{x}+i\varphi(\tilde{x},y')-x-i\varphi(x,y')]^2} g(x) f(x,y') dx_1 \wedge \cdots \wedge dx_{n-1} \wedge \big(dx_n + i d_x\psi(x,y')\big) = f(\tilde{x},y')$$

uniformly in $(\tilde{x},y') \in K'$.

That is, we look at (3.1) and we plug in $w = z(\tilde{x},y') \in K$. Notice that the g (as usual) makes sure we never evaluate f, ψ, or ϕ at points where they are not defined.

Proof. The change of variables formula implies

$$dx_1 \wedge \cdots \wedge dx_{n-1} \wedge \big(dx_n + i d_x\psi(x,y')\big) = d_x z(x,y') = \det\left[\frac{\partial z}{\partial x}(x,y')\right] dx. \tag{3.2}$$

where $\left[\frac{\partial z}{\partial x}(x,y)\right]$ is the matrix corresponding to the derivative of the mapping z with respect to the x variables evaluated at (x,y').

Let us change variables of integration via $\xi = \sqrt{\ell}(x-\tilde{x})$:

$$\begin{aligned}\left(\frac{\ell}{\pi}\right)^{n/2} &\int_{x\in\mathbb{R}^n} e^{-\ell[\tilde{x}+i\varphi(\tilde{x},y')-x-i\varphi(x,y')]^2} g(x) f(x,y') \det\left[\frac{\partial z}{\partial x}(x,y')\right] dx \\ &= \left(\frac{1}{\pi}\right)^{n/2} \int_{\xi\in\mathbb{R}^n} e^{-\left[\xi+i\sqrt{\ell}\left(\varphi\left(\tilde{x}+\frac{\xi}{\sqrt{\ell}},y'\right)-\varphi(\tilde{x},y')\right)\right]^2} \\ &\qquad g\left(\tilde{x}+\frac{\xi}{\sqrt{\ell}}\right) f\left(\tilde{x}+\frac{\xi}{\sqrt{\ell}},y'\right) \det\left[\frac{\partial z}{\partial x}\left(\tilde{x}+\frac{\xi}{\sqrt{\ell}},y'\right)\right] d\xi.\end{aligned}$$

We now wish to take a limit as $\ell \to \infty$ and for this we need to apply the dominated convergence theorem. So we need to dominate the integrand.

As a function of ξ,

$$g\left(\tilde{x}+\frac{\xi}{\sqrt{\ell}}\right) f\left(\tilde{x}+\frac{\xi}{\sqrt{\ell}},y'\right) \det\left[\frac{\partial z}{\partial x}\left(\tilde{x}+\frac{\xi}{\sqrt{\ell}},y'\right)\right]$$

is globally bounded independent of ℓ, because it has compact support and it is continuous.

Hence it is enough to worry about the exponential term. We also only need to consider those ξ where the integrand is not zero. Recall that r and d are small enough that

$$\sup_{\|\hat{x}\|\le r, \|\hat{y}'\|\le d} \left\|\frac{\partial\varphi}{\partial x}(\hat{x},\hat{y}')\right\| \le \frac{1}{4},$$

and as $\|\tilde{x}\| \leq r/4$ (as $(\tilde{x},y') \in K$) and $\left\|\tilde{x}+\frac{\xi}{\sqrt{\ell}}\right\| \leq r$ (because g is zero otherwise) then

$$\left\|\varphi\left(\tilde{x}+\frac{\xi}{\sqrt{\ell}},y'\right)-\varphi(\tilde{x},y')\right\| \leq \frac{1}{4}\left\|\tilde{x}+\frac{\xi}{\sqrt{\ell}}-\tilde{x}\right\| = \frac{\|\xi\|}{4\sqrt{\ell}}.$$

So under the same conditions we have

$$\begin{aligned}\left|e^{-\left[\xi+i\sqrt{\ell}\left(\varphi\left(\tilde{x}+\frac{\xi}{\sqrt{\ell}},y'\right)-\varphi(\tilde{x},y')\right)\right]^2}\right| &= e^{-\operatorname{Re}\left[\xi+i\sqrt{\ell}\left(\varphi\left(\tilde{x}+\frac{\xi}{\sqrt{\ell}},y'\right)-\varphi(\tilde{x},y')\right)\right]^2} \\ &= e^{-\|\xi\|^2+\ell\left\|\varphi\left(\tilde{x}+\frac{\xi}{\sqrt{\ell}},y'\right)-\varphi(\tilde{x},y')\right\|^2} \\ &\leq e^{-(15/16)\|\xi\|^2}.\end{aligned}$$

Therefore we can take the pointwise limit under the integral to obtain

$$\left(\frac{1}{\pi}\right)^{n/2}\int_{\xi\in\mathbb{R}^n} e^{-\left[\xi+i\left[\frac{\partial\varphi}{\partial x}(\tilde{x},y')\right]\xi\right]^2} g(\tilde{x})f(\tilde{x},y')\det\left[\frac{\partial z}{\partial x}(\tilde{x},y')\right]d\xi.$$

Notice how in the exponent we actually had an expression for the derivative in the ξ direction with y' fixed. If $(\tilde{x},y') \in K'$, then $g(\tilde{x}) = 1$ and so we can ignore it.

Letting $A = I + i\left[\frac{\partial\varphi}{\partial x}(\tilde{x},y')\right]$. Then using Lemma 3.3.4 we obtain

$$\left(\frac{1}{\pi}\right)^{n/2}\int_{\xi\in\mathbb{R}^n} e^{-\left[\xi+i\left[\frac{\partial\varphi}{\partial x}(\tilde{x},y')\right]\xi\right]^2} f(\tilde{x},y')\det\left[\frac{\partial z}{\partial x}(\tilde{x},y')\right]d\xi = f(\tilde{x},y').$$

The convergence of the integrand is pointwise in ξ but uniform in $(\tilde{x},y') \in K'$. That is left as an exercise. Hence the limit of the integrals converges uniformly in $(\tilde{x},y') \in K'$ and we are done. □

***Exercise* 3.3.8:** *In the proof of the above claim, show that for a fixed ξ, the integrand converges uniformly in $(\tilde{x},y') \in K'$.*

We are essentially done with the proof of the theorem. The two claims together with (3.1) show that f_ℓ are entire holomorphic functions that approximate f uniformly on K. Entire holomorphic functions can be approximated by polynomials uniformly on compact subsets; simply take the partial sums of Taylor series at the origin. □

***Exercise* 3.3.9:** *Explain why being approximable by (holomorphic) polynomials does not mean that f is real-analytic. That is, real-analytic means that f is on some compact neighborhood uniformly approximated by the Taylor polynomials.*

***Exercise* 3.3.10:** *Suppose $M \subset \mathbb{C}^n$ is given by $\operatorname{Im} z_n = 0$. Use the standard Weierstrass approximation theorem to show that for any $K \subset\subset M$ an arbitrary CR function $f\colon M \to \mathbb{C}$ can be uniformly approximated by holomorphic polynomials on K.*

3.4 Extension of CR functions

We will now apply the so-called “technique of analytic discs” together with Baouendi-Trèves to prove the Lewy extension theorem. Lewy's original proof was different and predates Baouendi-Trèves.

Theorem 3.4.1 (Lewy)**.** *Suppose $M \subset \mathbb{C}^n$ is a smooth real hypersurface and $p \in M$. There exists some small neighborhood U of p with the following property. Suppose $r\colon U \to \mathbb{R}$ a defining function for $M \cap U$, denote by $U_- \subset U$ the set where r is negative and $U_+ \subset U$ the set where r is positive. Let $f\colon M \to \mathbb{R}$ be a smooth CR function. Then:*

(i) If the Levi-form with respect to r has a positive eigenvalue at p, then f extends to a holomorphic function on U_- continuous up to M.

(ii) If the Levi-form with respect to r has a negative eigenvalue at p, then f extends to a holomorphic function on U_+ continuous up to M.

(iii) If the Levi-form with respect to r has eigenvalues of both signs at p, then any smooth CR f extends to a function holomorphic on U.

In particular, note that if the Levi-form has eigenvalues of both signs, then near p the CR function is in fact a restriction of a holomorphic function on all of U. The function r can really be any defining function for M, either one can extend it to all of U or we could take a smaller U such that r is defined on U. As we have noticed before, once we pick sides (where r is positive and where it is negative), then the number of positive eigenvalues and the number of negative eigenvalues of the Levi-form is fixed. Taking a different r at can at most flip U_- and U_+, but the conclusion of the theorem is exactly the same.

Proof. Without loss of generality, it is enough to suppose that M has one positive eigenvalue to prove the first two items, otherwise just take $-r$. So suppose that $p = 0$ and M is given in some neighborhood Ω of the origin as

$$\operatorname{Im} w = |z_1|^2 + \sum_{j=2}^{n-1} \varepsilon_j |z_j|^2 + E(z_1, z', \bar{z}_1, \bar{z}', \operatorname{Re} w),$$

where $\varepsilon_j = -1, 0, 1$, E vanishes to third order, and $z' = (z_2, \dots, z_{n-1})$. Let Ω_- be given by

$$0 > r = |z_1|^2 + \sum_{j=2}^{n-1} \varepsilon_j |z_j|^2 + E(z_1, z', \bar{z}_1, \bar{z}', \operatorname{Re} w) - \operatorname{Im} w.$$

The (real) Hessian of the function

$$|z_1|^2 + E(z_1, 0, \bar{z}_1, 0, 0)$$

is positive definite in an entire neighborhood of the origin and the function has a strict minimum at 0. There is some small disc $D \subset \mathbb{C}$ such that this function is strictly positive on ∂D. We can also assume that the Hessian is positive definite on the closed disc $\overline{D}$.

Therefore, for $(z',w) \in W$ in some small neighborhood W of the origin in $\mathbb{C}^{n-1}$, the function

$$z_1 \mapsto |z_1|^2 + \sum_{j=2}^{n} \varepsilon_j |z_j|^2 + E(z_1, z', \bar{z}_1, \bar{z}', \operatorname{Re} w) - \operatorname{Im} w$$

has a positive definite Hessian (as a function of z_1 only) on D and it is still strictly positive on ∂D.

We wish to apply Baouendi-Trèves and so let K be the compact neighborhood of the origin from the theorem. Take D and W small enough such that $(D \times W) \cap M \subset K$. Find the polynomials p_j that approximate f uniformly on K. Take $z_1 \in D$ and fix $(z',w) \in W$ such that $(z_1, z', w) \in \Omega_-$. Let $D_- = D \times \{(z',w)\} \cap \Omega_-$. Then each connected component* Δ of D_- is a closed analytic disc with $\partial \Delta \subset M$. We know this because r is positive on $(\partial D) \times \{(z',w)\}$ and hence $r = 0$ on the boundary of Δ. As $D \times W \cap M \subset K$, we have that $\partial \Delta \subset K$.

As $p_j \to f$ uniformly on K then $p_j \to f$ uniformly on $\partial \Delta$. As p_j are holomorphic, then by maximum principle p_j converge uniformly on all of Δ. In fact, as (z_1, z', w) was an arbitrary point in $(D \times W) \cap \Omega_-$, the polynomials p_j converge uniformly on $(D \times W) \cap \{\overline{\Omega_-}\}$. Let $U = D \times W$, then $U_- = (D \times W) \cap \Omega_-$. Notice U depends on K, but not on f. So p_j converge to a continuous function F on $\overline{U_-}$ and F is holomorphic on U_-. Clearly F equals f on $M \cap U$.

To prove the last item, pick a side, and then use one of the first two items to extend the function to that side. Via the tomato can principle (Theorem 2.3.10) the function also extends across M and therefore to a whole neighborhood of p. $\square$

We state the next corollary for a strongly convex domain, even though it holds with far more generality. In fact, a bounded domain with smooth boundary and connected complement will work without any assumptions on the Levi-form, but a different approach would have to be taken.

Corollary 3.4.2. *Suppose $U \subset \mathbb{C}^n$, $n \geq 2$, is a bounded domain with smooth boundary that is strongly convex and $f \colon U \to \mathbb{C}$ is a smooth CR function, then there exists a continuous function $F \colon \overline{U} \to \mathbb{C}$ holomorphic in U such that $F|_{\partial U} = f$.*

Proof. A strongly convex domain is strongly pseudoconvex, so f must extend to the inside locally near any point. The extension is locally unique as any two extensions have the same boundary values. Therefore, there exists a set $K \subset\subset U$ such that f extends to $U \setminus K$. Via an exercise below we can assume that K is strongly convex and therefore we can apply the special case of Hartogs phenomenon that you proved in Exercise 2.1.6 to find an extension holomorphic in U. $\square$

*If we choose W small enough there is only one component, but it is not necessary for our argument.

Exercise* 3.4.1*: *Prove the existence of the strongly convex K above.*

Exercise* 3.4.2*: *Show by example that the corollary is not true when $n = 1$. Explain where in the proof have we used that $n \geq 2$.*

Exercise* 3.4.3*: *Suppose $f : \partial \mathbb{B}_2 \to \mathbb{C}$ is a smooth CR function. Write down an explicit formula for the extension F.*

Exercise* 3.4.4*: *If $M \subset \mathbb{C}^3$ is defined by $\operatorname{Im} w = |z_1|^2 - |z_2|^2 + O(3)$ defines a smooth hypersurface and f is a real-valued smooth CR function on M. Show that $|f|$ does not attain a maximum at the origin.*

Exercise* 3.4.5*: *Suppose $M \subset \mathbb{C}^n$, $n \geq 3$, is a real-analytic hypersurface such that the Levi-form at $p \in M$ has eigenvalues of both signs. Show that every smooth CR function f on M is in fact real-analytic in a neighborhood of p.*

Exercise* 3.4.6*: *Let $M \subset \mathbb{C}^3$ be defined by $\operatorname{Im} w = |z_1|^2 - |z_2|^2$. a) Show that an arbitrary compact subset $K \subset\subset M$ will work for the conclusion Baouendi-Trèves. b) Use this to show that every smooth CR function $f : M \to \mathbb{C}$ is a restriction of an entire holomorphic function $F : \mathbb{C}^3 \to \mathbb{C}$.*

Exercise* 3.4.7*: *Find an $M \subset \mathbb{C}^n$, $n \geq 2$, such that near some $p \in M$, for every neighborhood W of p in M. There is a CR function $f : W \to \mathbb{C}$ that does not extend to either side of M at p.*

Chapter 4

The $\bar{\partial}$-problem

4.1 The generalized Cauchy integral formula

Before we get into the $\bar{\partial}$-problem, let us prove a more general version of Cauchy's formula using Stokes' theorem. Sometimes this is called the *Cauchy-Pompeiu integral formula*.

Theorem 4.1.1. *Let $U \subset \mathbb{C}$ be a bounded domain with smooth boundary and let $\varphi \colon \overline{U} \to \mathbb{C}$ be a smooth function, then for $z \in U$:*

$$\varphi(z) = \frac{1}{2\pi i} \int_{\partial U} \frac{\varphi(\zeta)}{\zeta - z} \, d\zeta + \frac{1}{2\pi i} \int_{U} \frac{\frac{\partial \varphi}{\partial \bar{z}}(\zeta)}{\zeta - z} \, d\zeta \wedge d\bar{\zeta}.$$

If φ is holomorphic the second term is zero and we obtain the standard Cauchy formula.

***Exercise* 4.1.1:** *Note the singularity in the second term, and prove that the integral still makes sense (the function is integrable). Hint: polar coordinates.*

***Exercise* 4.1.2:** *Why can we not differentiate in $\bar{z}$ under the integral in the second term? Notice that would lead to an impossible result.*

Proof. Fix $z \in U$. Let $\Delta_r(z)$ be a small disc such that $\Delta_r(z) \subset\subset U$. Via Stokes we get

$$\int_{\partial U} \frac{\varphi(\zeta)}{\zeta - z} \, d\zeta - \int_{\partial \Delta_r(z)} \frac{\varphi(\zeta)}{\zeta - z} \, d\zeta = \int_{U \setminus \Delta_r(z)} d\left(\frac{\varphi(\zeta)}{\zeta - z} \, d\zeta \right) = \int_{U \setminus \Delta_r(z)} \frac{\frac{\partial \varphi}{\partial \bar{\zeta}}(\zeta)}{\zeta - z} \, d\bar{\zeta} \wedge d\zeta.$$

The second equality follows because holomorphic derivatives in ζ will have a $d\zeta$ and when we wedge with $d\zeta$ we just get zero. We now wish to let the radius r go to zero. Via the exercise above we have that $\frac{\frac{\partial \varphi}{\partial \bar{\zeta}}(\zeta)}{\zeta - z} \, d\bar{\zeta} \wedge d\zeta$ is integrable over all of U and therefore

$$\lim_{r \to 0} \int_{U \setminus \Delta_r(z)} \frac{\frac{\partial \varphi}{\partial \bar{\zeta}}(\zeta)}{\zeta - z} \, d\bar{\zeta} \wedge d\zeta = \int_U \frac{\frac{\partial \varphi}{\partial \bar{\zeta}}(\zeta)}{\zeta - z} \, d\bar{\zeta} \wedge d\zeta = -\int_U \frac{\frac{\partial \varphi}{\partial \bar{\zeta}}(\zeta)}{\zeta - z} \, d\zeta \wedge d\bar{\zeta}.$$

The second equality is just swapping the order of the $d\zeta$ and $d\bar{\zeta}$. Next by continuity of φ we get

$$\lim_{r \to 0} \frac{1}{2\pi i} \int_{\partial \Delta_r(z)} \frac{\varphi(\zeta)}{\zeta - z} \, d\zeta = \lim_{r \to 0} \frac{1}{2\pi} \int_0^{2\pi} \varphi(z + re^{i\theta}) \, d\theta = \varphi(z).$$

The theorem follows. □

***Exercise* 4.1.3:** *Let $U \subset \mathbb{C}$ be a domain with smooth boundary and suppose that $\varphi \colon \overline{U} \to \mathbb{C}$ is a smooth function such that $\frac{\partial \varphi}{\partial \bar{z}}$ goes to zero as z goes to ∂U. Prove that $\varphi|_{\partial U}$ are the boundary values of a holomorphic function on U.*

4.2 Simple case of the $\bar{\partial}$-problem

For a smooth function ψ we have the exterior derivative

$$d\psi = \frac{\partial \psi}{\partial z_1} dz_1 + \cdots + \frac{\partial \psi}{\partial z_n} dz_n + \frac{\partial \psi}{\partial \bar{z}_1} d\bar{z}_1 + \cdots + \frac{\partial \psi}{\partial \bar{z}_n} d\bar{z}_n.$$

So let us give a name to the two parts of the derivative:

$$\partial \psi \overset{\text{def}}{=} \frac{\partial \psi}{\partial z_1} dz_1 + \cdots + \frac{\partial \psi}{\partial z_n} dz_n, \qquad \bar{\partial} \psi \overset{\text{def}}{=} \frac{\partial \psi}{\partial \bar{z}_1} d\bar{z}_1 + \cdots + \frac{\partial \psi}{\partial \bar{z}_n} d\bar{z}_n.$$

Then $d\psi = \partial \psi + \bar{\partial} \psi$. Notice that ψ is holomorphic if and only if $\bar{\partial} \psi = 0$.

The so-called *inhomogeneous $\bar{\partial}$-problem* (pronounced D-bar) is to solve the equation

$$\bar{\partial} \psi = g,$$

for ψ given a one-form g:

$$g = g_1 d\bar{z}_1 + \cdots + g_n d\bar{z}_n.$$

Such a g is called a *$(0,1)$-form*. The fact that the partial derivatives of ψ will commute forces certain compatibility conditions on g to have any hope of getting a solution (see below).

***Exercise* 4.2.1:** *Find an explicit example of a g in $\mathbb{C}^2$ such that no corresponding ψ can exist.*

On any open set where $g = 0$, ψ is holomorphic. So for a general g, what we are doing is finding a function that is not holomorphic in a very specific way.

Theorem 4.2.1. *Suppose g is a $(0,1)$-form on $\mathbb{C}^n$, $n \geq 2$, given by*

$$g = g_1 d\bar{z}_1 + \cdots + g_n d\bar{z}_n,$$

where $g_j \colon \mathbb{C}^n \to \mathbb{C}$ are compactly supported smooth functions satisfying the compatibility conditions

$$\frac{\partial g_k}{\partial \bar{z}_\ell} = \frac{\partial g_\ell}{\partial \bar{z}_k}.$$

Then there exists a compactly supported smooth function ψ such that

$$\bar{\partial} \psi = g.$$

The compatibility conditions are necessary, but the compactness is not. However in that case the boundary of the domain where the equation lives would come into play. Let us not worry about this, and prove this simple compactly supported version always has a solution. Without the compact support the solution is clearly not unique. Given any holomorphic f, $\bar{\partial}(\psi + f) = g$. But since the difference of any two solutions ψ_1 and ψ_2 is holomorphic, and the only holomorphic compactly supported function is 0, then the compactly supported solution ψ is unique.

Proof. We really have n different smooth functions, $g_1, \ldots, g_n$. The equation $\bar{\partial}\psi = g$ is then the n equations

$$\frac{\partial \psi}{\partial \bar{z}_k} = g_k,$$

where the functions g_k satisfy the compatibility conditions.

We claim that the following is an explicit solution:

$$\psi(z) = \frac{1}{2\pi i} \int_{\mathbb{C}} \frac{g_1(\zeta, z_2, \ldots, z_n)}{\zeta - z_1} d\zeta \wedge d\bar{\zeta} = \frac{1}{2\pi i} \int_{\mathbb{C}} \frac{g_1(\zeta + z_1, z_2, \ldots, z_n)}{\zeta} d\zeta \wedge d\bar{\zeta}.$$

To show that the singularity does not matter for integrability is the same idea as for the generalized Cauchy formula.

Let us check we have the solution. We use the generalized Cauchy formula on the z_1 variable. Take R large enough so that $g_j(\zeta, z_2, \ldots, z_n)$ is zero when $|\zeta| \geq R$ for all j. For any j we get

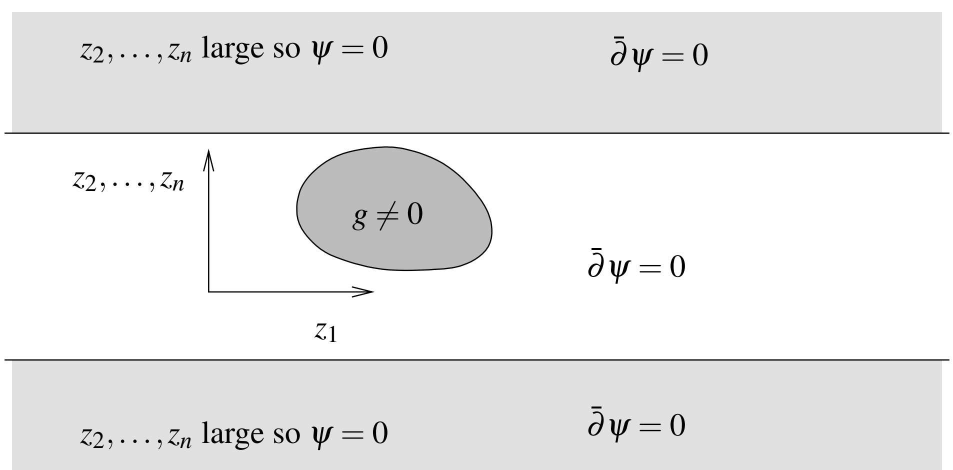
$$\begin{aligned} g_j(z_1, \ldots, z_n) &= \frac{1}{2\pi i} \int_{|\zeta|=R} \frac{g_j(\zeta, z_2, \ldots, z_n)}{\zeta - z_1} d\zeta + \frac{1}{2\pi i} \int_{|\zeta| \leq R} \frac{\frac{\partial g_j}{\partial \bar{z}_1}(\zeta, z_2, \ldots, z_n)}{\zeta - z_1} d\zeta \wedge d\bar{\zeta} \\ &= \frac{1}{2\pi i} \int_{\mathbb{C}} \frac{\frac{\partial g_j}{\partial \bar{z}_1}(\zeta, z_2, \ldots, z_n)}{\zeta - z_1} d\zeta \wedge d\bar{\zeta}. \end{aligned}$$

Using the second form of the definition of ψ, the compatibility condition, and the above computation we get

$$\begin{aligned}
\frac{\partial \psi}{\partial \bar{z}_j}(z) &= \frac{1}{2\pi i}\int_{\mathbb{C}} \frac{\frac{\partial g_1}{\bar{z}_j}(\zeta + z_1, z_2, \dots, z_n)}{\zeta} d\zeta \wedge d\bar{\zeta} \\
&= \frac{1}{2\pi i}\int_{\mathbb{C}} \frac{\frac{\partial g_j}{\bar{z}_1}(\zeta + z_1, z_2, \dots, z_n)}{\zeta} d\zeta \wedge d\bar{\zeta} \\
&= \frac{1}{2\pi i}\int_{\mathbb{C}} \frac{\frac{\partial g_j}{\bar{z}_1}(z_1, z_2, \dots, z_n)}{\zeta - z_1} d\zeta \wedge d\bar{\zeta} = g_j(z).
\end{aligned}$$

***Exercise* 4.2.2:** *Show that we were allowed to differentiate under the integral in the computation above.*

That ψ has compact support follows because g_1 has compact support and analytic continuation. In particular, ψ is holomorphic for very large z since $\bar{\partial}\psi = g = 0$ when z is large. When $z_2, \dots, z_n$ are large, then ψ is identically zero simply from its definition. By analytic continuation then ψ is identically zero for all large z. See the following diagram, where we use analytic continuation to show that as ψ is holomorphic and zero on the light gray area and holomorphic on the light gray and white area, it is also zero on the white area:



□

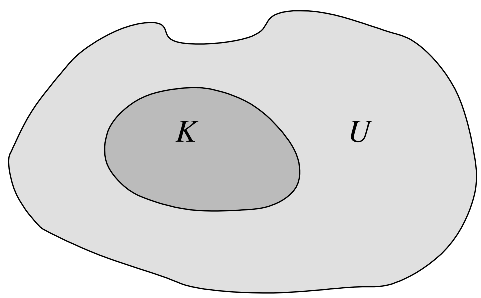
The first part of the proof still works when $n = 1$, so we do get a solution ψ. However in this case the last bit of the proof does not work, so ψ will not have compact support.

***Exercise* 4.2.3:** *a) Show that if g is supported in $K \subset\subset \mathbb{C}^n$, $n \geq 2$, then ψ is supported in the complement of the unbounded component of $\mathbb{C}^n \setminus K$. In particular, show that if K is the support of g and $\mathbb{C}^n \setminus K$ is connected, then the support of ψ is K. b) Find an explicit example where the support of ψ is strictly larger than the support of g.*

4.3 The general Hartogs phenomenon

We can now prove the general Hartogs phenomenon as an application of the solution of the compactly supported inhomogeneous $\bar{\partial}$-problem. We proved special versions of this phenomenon using Hartogs figures before.

Theorem 4.3.1 (Hartogs phenomenon)**.** *Let $U \subset \mathbb{C}^n$ be a domain, $n \geq 2$, and let $K \subset\subset U$ be a compact set such that $U \setminus K$ is connected. Every holomorphic $f : U \setminus K \to \mathbb{C}$ extends uniquely to a holomorphic function on U.*



The idea of the proof is extending in some way and then using the solution to the $\bar{\partial}$-problem to correct the result to make it holomorphic.

Proof. First find a smooth function φ that is 1 in a neighborhood of K and is compactly supported in U (exercise below). Let $f_0 = (1-\varphi)f$ on $U \setminus K$ and $f_0 = 0$ on K. The function f_0 is smooth on U and it is holomorphic and equal to f near the boundary of U, where φ is 0. We let $g = \bar{\partial} f_0$, that is $g_k = \frac{\partial f_0}{\partial \bar{z}_k}$. Let us see why g_k is compactly supported. The only place to check is on $U \setminus K$ as elsewhere we have 0 automatically. Note that f is holomorphic and compute

$$\frac{\partial f_0}{\partial \bar{z}_k} = \frac{\partial}{\partial \bar{z}_k}\big((1-\varphi)f\big) = \frac{\partial f}{\partial \bar{z}_k} - \varphi \frac{\partial f}{\partial \bar{z}_k} - \frac{\partial \varphi}{\partial \bar{z}_k} f = -\frac{\partial \varphi}{\partial \bar{z}_k} f.$$

And $\frac{\partial \varphi}{\partial \bar{z}_k}$ must be compactly supported in U. Now apply the solution of the compactly supported $\bar{\partial}$-problem to find a compactly supported function ψ such that $\bar{\partial}\psi = g$. Set $F = f_0 - \psi$. Let us check that F is the desired extension. It is holomorphic:

$$\frac{\partial F}{\partial \bar{z}_k} = \frac{\partial f_0}{\partial \bar{z}_k} - \frac{\partial \psi}{\partial \bar{z}_k} = g_k - g_k = 0.$$

Next, a bit of thought and the fact that $U \setminus K$ is connected reveals that ψ must be compactly supported in U. This means that F agrees with f near the boundary (in particular on an open set) and thus everywhere in $U \setminus K$ since $U \setminus K$ is connected. $\square$

Exercise* 4.3.1*: *Show that φ exists. Hint: Use mollifiers.*

Exercise* 4.3.2*: *Suppose $U \subset \mathbb{C}^n$, $n \geq 2$, is a bounded domain with smooth boundary that is strongly pseudoconvex and $f\colon U \to \mathbb{C}$ is a smooth CR function, then prove there exists a continuous function $F\colon \overline{U} \to \mathbb{C}$ holomorphic in U such that $F|_{\partial U} = f$.*

Exercise* 4.3.3*: *Suppose $U \subset \mathbb{C}^n$, $n \geq 2$, is a domain and the sphere $S^{2n-1} \subset U$. Suppose that $f\colon U \to \mathbb{C}^n$ is a holomorphic mapping such that locally near every point of S^{2n-1}, f is a local biholomorphism (that is f is locally invertible, i.e. the derivative is invertible at every point of S^{2n-1}). Then show that f takes the ball $\mathbb{B}_n$ biholomorphically to some domain $f(\mathbb{B}_n)$ with smooth boundary.*

Exercise* 4.3.4*: *Find an example of a smooth function $g\colon \mathbb{C} \to \mathbb{C}$ with compact support, such that no solution $\psi\colon \mathbb{C} \to \mathbb{C}$ to $\frac{\partial \psi}{\partial \bar{z}} = g$ (at least one of which always exists) is of compact support.*

…and that is how using sheep's bladders can prevent earthquakes!

Further Reading

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