

# RESEARCH STATEMENT

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## 1. INTRODUCTION AND RESEARCH PHILOSOPHY

My primary interests lie in complex analysis, CR geometry in particular. My research in CR geometry also led me to problems in real and complex algebraic geometry, differential equations, potential theory, discrete geometry, combinatorics, number theory, experimental mathematics using computers, and quantum computing. My research philosophy is not to simply solve problems within the confines of a particular area, but to look for connections and applications to other areas of mathematics and even other disciplines. In this statement, I talk only about my main interest, which is CR geometry.

In CR geometry, I study singularities and complexity. A real object in a complex manifold (often the complex euclidean space  $\mathbb{C}^n$ ) inherits a certain amount of the complex structure, called the CR structure, from the ambient manifold. CR geometry is the study of these objects and maps preserving the CR structure. CR complexity concerns CR maps from a real manifold to a sphere or a hyperquadric; it is the analogue of the Nash embedding in CR geometry, where the role of flat space is played by spheres and hyperquadrics. We ask which maps exist, and for those maps, we ask how complicated they are. A related question is to understand the CR geometry of singular objects or the singularities of the CR structure itself. Examples of such singularities are the Levi-flat hypersurfaces, which are pseudoconvex from both sides and foliated by complex manifolds. Singularities of these hypersurfaces are interesting from several points of view: invariant sets of holomorphic foliations, holomorphic function theory, and the intersection of real and complex algebraic (and analytic) geometry. Many fundamental questions are not yet fully answered and offer a rich array of topics for future study. Questions raised in the study of these sets have been the primary motivation for much of my research in CR geometry.

A common link between CR complexity and holomorphic maps is the equation  $\|f(z)\|^2 - \|g(z)\|^2 = 0$  for holomorphic maps  $f$  and  $g$ . When  $g$  is scalar-valued,  $f/g$  maps the solution set to the sphere, otherwise it is a hyperquadric. If  $f$  and  $g$  are scalar-valued the solution set is Levi-flat, a set with minimal CR complexity. For higher CR complexity, the target dimensions of  $f$  and  $g$  rise, until generically, the dimension is infinite. The study of CR maps between spheres and hyperquadrics has a rich combinatorial and discrete geometric aspect, lending itself naturally to computer experimentation.

## 2. LEVI-FLAT HYPERSURFACES

Pseudoconvexity is the complex variables analogue of convexity. The natural domain of holomorphic functions has a pseudoconvex boundary. A hypersurface is called *Levi-flat* if it is pseudoconvex from both sides.

Let  $H \subset \mathbb{C}^N$  be a singular codimension-one real-analytic local subvariety (a *real-hypervariety*). That is, the solution set of the equation  $r = 0$  for a real-analytic  $r$ . Denote by  $H^*$  the set of nonsingular points of top dimension and by  $H_s$  the set of points where  $H$  is not a real-analytic submanifold. We say  $H$  is a Levi-flat real-hypervariety if  $H^*$  is Levi-flat. Levi-flat hypersurfaces have a natural real-analytic foliation by complex hypersurfaces defined on the nonsingular part called the *Levi-foliation*. They arise naturally as invariant sets in foliation theory, and many questions are motivated from the point of view of singular holomorphic foliations.

A submanifold of higher codimension is *Levi-flat* if it is an intersection of Levi-flat hypersurfaces in general position. With this definition complex submanifolds are Levi-flat, and Levi-flat manifolds are a generalization of complex manifolds. A basic result about complex varieties is that the singular set is also a complex variety. For real varieties, and in particular Levi-flats, things are more complicated, but the highest stratum of the singularity must be Levi-flat.

**Theorem 2.1** (Lebl [23]). *Let  $H \subset \mathbb{C}^N$  be a local Levi-flat real-hypervariety. Then the singular set  $(\overline{H^*})_s$  is Levi-flat near points where it is a CR real-analytic submanifold. Furthermore, if  $(\overline{H^*})_s$  is a generic manifold, then  $(\overline{H^*})_s$  is a generic Levi-flat manifold of dimension  $2N - 2$ .*

An open question is: Does  $H$  have a complete Levi-flat stratification, that is, is  $H$  a union of real-analytic Levi-flat submanifolds? To this end, I began to study the CR singularities of submanifolds of higher codimension that are either Levi-flat or contained in a Levi-flat hypersurface.

As for complex algebraic varieties, we study algebraic Levi-flat hypervarieties in the complex projective space  $\mathbb{P}^n$ . In high enough dimension all Levi-flat hypervarieties in  $\mathbb{P}^n$  are singular [36]. Leaves of the Levi-foliation are complex subvarieties if they are compact. The celebrated theorem of Chow says that any complex subvariety of  $\mathbb{P}^n$  is necessarily algebraic. For Levi-flats I proved an analogue of Chow's theorem.

**Theorem 2.2** (Lebl [21]). *Let  $H \subset \mathbb{P}^n$ ,  $n \geq 2$ , be an irreducible Levi-flat real-hypervariety with infinitely many compact leaves, such that locally  $H$  is defined by a meromorphic function. Then  $H$  is semialgebraic and contained in a pullback of a real-algebraic curve in  $\mathbb{C}$  by a rational function.*

Compactness of the leaves is not enough to guarantee algebraicity.

**Theorem 2.3** (Lebl [25]). *There exists a Levi-flat real-hypervariety  $H \subset \mathbb{P}^2$  with all leaves complex hyperplanes, such that  $H$  is not contained in any proper real-algebraic subvariety of  $\mathbb{P}^2$ .*

Levi-flat hypersurfaces are defined by functions satisfying a complex Monge–Ampère type equation. An interesting question is a boundary value problem for this partial differential equation. We have an analogue of the Plateau problem: when is a compact codimension-two real submanifold a boundary of a Levi-flat hypersurface. In general, the Levi-flat hypersurface may have singularities. In [29] we obtained a solution to this Plateau problem if the boundary is an image of a compact submanifold in  $\mathbb{C}^n \times \mathbb{R}$  with elliptic singularities. CR singular manifolds in dimension 2 have a long history, starting with Bishop [3] in the 1960s. In dimension 3 and higher, the problem is of a lot of recent interest, see for example [10, 17–19, 29, 30], and the references within.

### 3. HIGHER CODIMENSION CR SINGULAR SUBMANIFOLDS

In higher codimension, the CR structure naturally develops singularities. E. Bishop [3] first studied the nondegenerate CR singular submanifolds of real codimension 2 in  $\mathbb{C}^2$ . These have the form

$$w = z\bar{z} + \lambda(z^2 + \bar{z}^2) + O(3), \quad (1)$$

for  $\lambda \in [0, \infty]$  ( $\lambda = \infty$  means  $w = z^2 + \bar{z}^2 + O(3)$ ).

With Xianghong Gong [13], we studied the CR singular Levi-flat submanifolds of codimension 2 in  $\mathbb{C}^{n+1}$ ,  $n \geq 2$ . We obtained a normal form for the quadratic terms and a complete formal normal form for the nondegenerate case in  $\mathbb{C}^3$ .

In the Plateau problem mentioned above, the boundary is not completely flat, it is *holomorphically-flat*. It lies in a nonsingular Levi-flat; it is locally realizable as a hypersurface in  $\mathbb{C}^n \times \mathbb{R}$ . With Alan Noell and Sivaguru Ravisankar, we studied the extension of CR functions into the Levi-flat hypersurface. A codimension 2 CR singular submanifold  $M \subset \mathbb{C}^{n+1}$  can be given as

$$M : w = Q(z, \bar{z}) + E(z, \bar{z}) = A(z, \bar{z}) + B(z, z) + \overline{B(z, z)} + E(z, \bar{z}), \quad (2)$$

where  $Q$  is a real-valued real-quadratic form, and  $E$  is real-valued and  $O(3)$ . We split  $Q$  into a sesquilinear form  $A$  and complex symmetric form  $B$ . The form  $A$  carries the Levi-form of the manifold. The Levi-flat hypersurface  $H_+$  whose boundary is  $M$  is

$$H_+ : \operatorname{Re} w \geq Q(z, \bar{z}) + E(z, \bar{z}), \quad \operatorname{Im} w = 0 \quad (3)$$

Let us refer to the Levi-flat defined by  $\operatorname{Im} w = 0$  as  $H$ , therefore,  $H_+$  is one side of  $M$  in  $H$ .

A function is CR if it satisfies the Cauchy–Riemann equations restricted to a submanifold. A result of Lewy says that CR functions on a hypersurface in  $\mathbb{C}^n$  extend holomorphically to one side if the Levi-form of the hypersurface possesses at least one nonzero (without loss of generality positive) eigenvalue. We extended the result to CR singular submanifolds in several ways. In [29] we proved an extension to  $H_+$  in the  $C^\infty$  case when  $A$  is positive definite, an analogue of strict pseudoconvexity. We proved the following version in [31].

**Theorem 3.1** (Lebl–Noell–Ravisankar [31]). *Let  $H_+$ ,  $H$ , and  $M$  be as above,  $Q$  is nondegenerate, and  $n \geq 2$ . If  $A$  has at least two positive eigenvalues, then there exists a neighborhood  $U \subset H$  of the origin such that every function  $f \in C^\infty(M) \cap CR(M_{CR})$  extends to a smooth CR function on  $U \cap H_+$ . If  $A$  also has two negative eigenvalues, then  $f$  extends to a smooth CR function on  $U$ .*

We found examples showing that the hypotheses are optimal: Nondegeneracy and at least two positive eigenvalues are necessary. This contrasts with the CR nonsingular case where no nondegeneracy is needed. The difference is that in the CR case we are solving a noncharacteristic Cauchy PDE problem, and in the CR singular case, the problem is characteristic at the singularity.

When  $n = 1$  the theorem still holds in the elliptic case under a hypothesis that  $f$  extends along leaves. In the real-analytic case, the theorem holds without any condition on the eigenvalues, and the extension is holomorphic in a neighborhood of  $M$  in  $\mathbb{C}^{n+1}$ . In fact, in the real-analytic case, it is not necessary that the submanifold is holomorphically-flat. CR extension is a generic property of real-analytic CR singular submanifolds.

**Theorem 3.2** (Lebl–Noell–Ravisankar [33]). *Let  $(z, w) \in \mathbb{C}^n \times \mathbb{C}$ ,  $n \geq 2$ , be the coordinates and, let  $M \subset \mathbb{C}^{n+1}$  be a real-analytic submanifold*

$$w = z^*Az + \overline{z^tBz} + z^tCz + E(z, \bar{z}), \quad \text{such that} \quad \text{rank} \begin{bmatrix} A^* \\ B \end{bmatrix} \geq 2. \quad (4)$$

*$A, B, C$  are complex  $n \times n$  matrices,  $B$  and  $C$  are symmetric, and  $E$  is  $O(\|z\|^3)$ . Any real-analytic function that is CR at CR points of  $M$  is locally a restriction of a holomorphic function.*

The above theorem says that generically CR extension holds, but it does not hold for every CR singular submanifold. For example, if  $M$  is an image of a CR submanifold as in [28] then no extension is possible, even in the real-analytic case. Images of CR submanifolds are an interesting case, as they naturally show up when studying CR geometry of mappings.

The theorems provide existence for the Levi-flat Plateau problem in a certain case: If a compact manifold  $N \subset \mathbb{C}^{n+1}$  is an image under a CR map  $f$  of a boundary  $\partial\Omega$  with only elliptic singularities for a bounded  $\Omega \subset \mathbb{C}^n \times \mathbb{R}$ , then the map extends to a CR map  $F: \Omega \rightarrow \mathbb{C}^{n+1}$  and  $F(\Omega)$  is the possibly singular solution of the Plateau problem. In Lebl–Noell–Ravisankar [33], we studied the possible singularities of these real-analytic solutions to the Levi-flat Plateau problem and found that the only singularities that appear are self-intersections. It is tempting to conjecture that the Levi-flat Plateau problem in  $\mathbb{C}^{n+1}$  is solvable if and only if it is solvable abstractly ( $N$  being the boundary of an abstract Levi-flat hypersurface).

#### 4. CR MAPS BETWEEN SPHERES AND HYPERQUADRICS

Let  $M \subset \mathbb{C}^n$  and  $M' \subset \mathbb{C}^N$  be real submanifolds. A fundamental question is to classify CR maps between  $M$  and  $M'$ . Let  $M'$  be a hyperquadric, that is,  $M'$  is defined by  $\langle z, z \rangle = 1$ , where  $\langle \cdot, \cdot \rangle$  is a nondegenerate (not necessarily positive definite) Hermitian product. The classification of CR maps  $\varphi: M \rightarrow M'$  amounts to understanding the ideal of real functions vanishing on  $M$ . For simplicity, let  $\rho(z, \bar{z})$  be a polynomial vanishing on  $M$ . Write

$$\rho(z, \bar{z}) = \|f(z)\|^2 - \|g(z)\|^2, \quad (5)$$

where  $f$  and  $g$  are holomorphic maps to some finite-dimensional space. The target dimension of the map  $f$  (resp.  $g$ ) is the number of positive (resp. negative) eigenvalues of the matrix of coefficients of  $\rho$ . The map  $(f, g)$  induces a CR map  $\varphi: M \rightarrow M'$ , unique up to fractional linear transformations preserving  $M'$ . The problem rests in studying the signature pair (the number of positive and negative eigenvalues) of functions in the ideal generated by  $M$ .

A well-studied case is when  $M = S^{2n-1} \subset \mathbb{C}^n$  and  $M' = S^{2N-1} \subset \mathbb{C}^N$  are unit spheres. When  $N < n$  no nonconstant CR maps exist. Two maps are *spherically equivalent* if they are conjugates of each other via automorphisms of the sphere. When  $n = N = 1$ , the map  $z^d$  takes the unit circle to itself and is of arbitrary degree  $d$ . A well-known theorem (Pincuk, Alexander, and others) states that if  $n = N \geq 2$ , then any CR map of spheres must be linear fractional, a rational map of degree 1. So degree-one maps are equivalent to the identity. A map is *monomial* if each component is a single monomial. For degree two maps we have:

**Theorem 4.1** (Lebl [22]). *Let  $f: S^{2n-1} \rightarrow S^{2N-1}$ ,  $n \geq 2$ , be a rational CR map of degree 2. Then  $f$  is spherically equivalent to a monomial map. Furthermore, the normal form for such a map is*

$$z \in \mathbb{C}^n \mapsto Lz \oplus (\sqrt{I - L^*Lz}) \otimes z, \quad (6)$$

where  $L$  is a diagonal matrix with nonnegative diagonal entries sorted by size, such that  $I - L^*L$  also has nonnegative entries. All maps of the form (6) are mutually spherically inequivalent.

Forstnerič [12] proved that if  $n \geq 2$  and the map is  $C^\infty$  up to the boundary, then the map is rational of degree  $d$  bounded by a function of  $n$  and  $N$ . A sharp bound on  $d$  is unknown, but D'Angelo conjectured:

$$d \leq \begin{cases} 2N - 3 & \text{if } n = 2, \\ \frac{N-1}{n-1} & \text{if } n \geq 3. \end{cases} \quad (7)$$

Monomial examples that achieve equality exist. The best currently known bound  $d \leq \frac{N(N-1)}{2(2n-3)}$  was proved by Meylan [37] for  $n = 2$  and extended to  $n \geq 3$  by D'Angelo and myself [7]. The combinatorics in the monomial seems to capture the complexity of the general problem. For monomial maps we have:

**Theorem 4.2** (D'Angelo–Kos–Riehl [4] for  $n = 2$ , and Lebl–Peters [34, 35] for  $n \geq 3$ ). *Suppose that  $f: S^{2n-1} \rightarrow S^{2N-1}$ ,  $n \geq 2$ , is a monomial CR map of degree  $d$ . Then (7) holds and is sharp.*

Spherical equivalence is not the only natural notion of equivalence for proper holomorphic maps. The obstruction in many problems in complex analysis is topological, so it is natural to study the topology of the space of proper maps. Two maps  $f: \mathbb{B}_n \rightarrow \mathbb{B}_{N_1}$  and  $g: \mathbb{B}_n \rightarrow \mathbb{B}_{N_2}$  are *homotopic in dimension  $N$*  if there exists a continuous  $H: \mathbb{B}_n \times [0, 1] \rightarrow \mathbb{B}_N$ ,  $H(z, t) = H_t(z)$ , such that  $H_t$  is a proper holomorphic map,  $\iota_1 \circ f = H_0$ , and  $\iota_2 \circ g = H_1$ , where  $\iota_j: \mathbb{B}_{N_j} \rightarrow \mathbb{B}_N$  is the linear embedding. It is also natural to restrict the regularity of the maps involved. So  $f$  and  $g$  are *homotopic through rational maps* if  $H_t$  is rational for every  $t$ . Given a large enough  $N$  (e.g.  $N = N_1 + N_2$ ), any two maps  $f$  and  $g$  are homotopic by a so-called juxtaposition:

$$H_t(z) = tf \oplus \sqrt{1 - t^2} g. \quad (8)$$

D'Angelo noticed that the juxtaposition of the identity and the so-called Whitney map gives a family of spherically inequivalent maps. By proving that given a fixed  $t_0$ , the set of parameters  $t$  such that  $H_t$  is spherically equivalent to  $H_{t_0}$  is closed, D'Angelo and I found a generalization:

**Theorem 4.3** (D'Angelo–Lebl [9]). *Let  $H_t$  be a homotopy of proper rational maps between balls such that  $H_0$  and  $H_1$  are not spherically equivalent. Then  $H_t$  contains uncountably many spherically inequivalent maps.*

A natural question is: how many equivalence classes exist for each pair  $(n, N)$ ? Any proper map of  $\mathbb{B}_1$  to  $\mathbb{B}_1$  is a Blaschke product homotopic to  $z^d$  for a unique positive integer  $d$ . Therefore when  $n = N = 1$ , there are countably many equivalence classes. If  $n = N > 1$ , then a proper map is an automorphism and thus there is exactly one homotopy equivalence class.

Faran [11] proved that a proper map  $f: \mathbb{B}_2 \rightarrow \mathbb{B}_3$ , smooth up to  $\partial\mathbb{B}_2$ , is spherically equivalent to exactly one of four maps,  $(z, w, 0)$ ,  $(z, zw, w^2)$ ,  $(z^2, \sqrt{2}zw, w^2)$ , or  $(z^3, \sqrt{3}zw, w^3)$ . We proved these maps are in distinct homotopy classes (through rational maps) in dimension 3. The number of equivalence classes for the pair  $(n, N) = (2, 3)$  is therefore exactly 4. In general we have:

**Theorem 4.4** (D’Angelo–Lebl [9]). *Let  $S$  denote the set of homotopy classes in target dimension  $N$  of proper rational maps  $f: \mathbb{B}_n \rightarrow \mathbb{B}_N$ . If  $n \geq 2$ , then  $S$  is a finite set.*

Define the hyperquadric

$$Q(a, b) = \{(z, w) \in \mathbb{C}^a \times \mathbb{C}^b : \|z\|^2 - \|w\|^2 = 1\}. \quad (9)$$

A linear fractional transformation normalizes  $Q(a, b)$  so that  $a > b$ . If  $f: M \rightarrow Q(A, B)$  is a CR map whose image lies in a hyperplane (affine space)  $W$ , that is,  $f(M) \subset W \cap Q(A, B)$ , then for some  $k, \tilde{A}, \tilde{B}$  and an affine change of coordinates,  $W \cap Q(A, B) \cong Q(\tilde{A}, \tilde{B}) \times \mathbb{C}^k$ . Therefore  $f$  is a direct sum of  $f_1: M \rightarrow Q(\tilde{A}, \tilde{B})$  and an arbitrary  $f_2: M \rightarrow \mathbb{C}^k$ . Hence we consider only those maps whose image does not lie in a hyperplane.

Given  $M$ , the natural question is to find the set of  $(A, B)$  such that a map from  $M$  to  $Q(A, B)$  exists. Hyperquadrics can be thought of as flat models, and this question is the analogue of Nash embedding. In the CR case, most manifolds do not admit an embedding into any  $Q(A, B)$ . For those that do, such as the algebraic manifolds, the set of  $(A, B)$  is the so-called CR complexity.

**Theorem 4.5** (D’Angelo–Lebl [8]). *Let  $n \geq 2, A > B \geq 0$ . There exists a  $K$  such that whenever  $A + B \geq K$ , there exists a CR map  $f: S^{2n-1} = Q(n, 0) \rightarrow Q(A, B)$ , with the image not contained in a complex hyperplane. If  $Q(A, B)$  is not a sphere, a rational  $f$  of arbitrary degree exists.*

The sphere case differs considerably from the hyperquadric case. A result such as the above theorem does not hold if the source is not equivalent to a sphere, i.e.  $b = 0$ . Baouendi–Huang [2] proved that if  $f: Q(a, b) \rightarrow Q(A, b)$  is CR with  $a > b \geq 1$  and  $A \geq b > 1$ , then  $f$  is equivalent to a linear embedding. Baouendi–Ebenfelt–Huang [1] proved that if  $f: Q(a, b) \rightarrow Q(A, B)$  is CR with  $a > b \geq 1$  and  $A \geq B > 1$  and  $B < 2b - 1$ , then  $f$  maps to a complex hyperplane.

**Theorem 4.6** (Grundmeier–Lebl–Vivas [15]). *Let  $a \geq 2, b \geq 1$ , and  $a > b$ . Let  $U \subset Q(a, b)$  be a connected open set and  $f: U \rightarrow Q(A, B)$  be a real-analytic CR map such that  $f(U)$  does not lie in a complex hyperplane then*

$$A \leq C(a, b, B), \quad (10)$$

where  $C = C(a, b, B)$  is a constant depending only  $a, b$ , and  $B$ .

The proof for rational maps translates the question into a problem in commutative algebra and applies Green’s hyperplane restriction theorem [14]. For analytic maps, we proved a more general version of Green’s theorem. The main point of the theorem is to translate the problem into a combinatorial question about initial monomial ideals. As long as  $A$  and  $B$  are comparable

$$\frac{B - b + 3}{A} \geq \frac{b}{a} \quad \text{and} \quad \frac{A - b + 2}{B + 1} \geq \frac{b}{a}, \quad (11)$$

and  $A + B$  is sufficiently large, we obtain nontrivial CR maps. The sharp value  $C(a, b, B)$  in (10) is not known, although it must go to infinity as  $B$  grows. The explicit bound we obtained is

$$A \leq C(a, b, B) \leq \left( \prod_{\ell=0}^{a-1} \left( \frac{a + b - 1 - \ell}{a + b - \ell} \right)^{\frac{a+b}{a+b-\ell}} \right) B^{\frac{a+b}{b}} + (\text{lower-order terms in } B). \quad (12)$$

We conjecture that the power of  $B$  in the estimate should be 1 rather than  $\frac{a+b}{b}$ .

Grundmeier and I [16] extended the techniques of [15] to show that the generic initial monomial ideal of the map and its quotient are invariants of rational maps between spheres and hyperquadrics. If we look at the span of the generators rather than the entire ideal, the technique extends beyond the rational case. The generic initial monomial space of a vector subspace of germs of holomorphic functions at a point is an invariant under biholomorphisms. The advantage of initial monomial ideal techniques is that they are readily computable via computer algebra systems.

A related problem to the degree bound problem is to study the  $L^1$  norm of the coefficients rather than the rank. This idea ties the degree bound problem to the compressed sensing techniques in applied mathematics. This is ongoing work, and some initial results were published in D’Angelo–Grundmeier–Lebl [5].

There is a computational, and thus experimental, aspect of these problems. Many questions, can be answered computationally. In [27] Daniel Lichtblau (Wolfram Research) and I proved new results about sharp monomial CR maps from  $S^3$  to  $S^{2N-1}$  and we have used these results together with independent computer code to classify all such maps up to degree  $d = 17$ . For this we have used *Mathematica*, my own mathematics software package Genius [26], and hand-tuned C code. I extended the computations up to degree  $d = 21$  in [24].

## 5. FUTURE PLANS

All the results mentioned suggest fertile ground for further work, and I am actively pursuing this plan. I am currently working on new results on CR singular manifolds whose CR structure extends through the singularities. These manifolds appear naturally as images of CR manifolds, and interestingly, they are in some sense a degenerate case of CR singularities. For example, the CR extension phenomenon does not hold without extra conditions.

For singularities of Levi-flat hypersurfaces, my goal is to prove a stratification theorem similar to complex analytic varieties. A more short term goal is to understand the CR singular sets of higher codimension Levi-flat submanifolds. I am working to understand the exact dimensions of such singularities. For example, I believe that in  $\mathbb{C}^n$  for a Levi-flat  $H$ , a real dimension  $2n - 3$  singularity only occurs at a generic point as a union of a nonsingular Levi-flat and a higher codimension component. Therefore, a coherent variety will not have such a singularity.

With Bernhard Lamel, we have discovered an averaging operator that can be thought of as a projection onto the set of holomorphic functions from the set of real-analytic functions on a totally-real subvariety. We are currently working on trying to use this operator to find normal forms of certain CR singular submanifolds and in general of totally-real subvarieties. The hope is to use this averaging technique to provide a better understanding of the CR structure at a singular point of a real subvariety of higher dimension, such as a Levi-flat hypervariety.

For CR maps of spheres and hyperquadrics, the conjectured degree bound is still open for rational maps and developing new techniques is needed to attack this problem. It remains my goal to prove this conjecture. The current thinking on this problem is that a single integer is not the right measure of complexity, and perhaps a more refined invariant is needed. In particular, the rational map degree is not invariant under homotopy, while it appears that at least in examples the number of tensor operations needed in the construction of polynomial maps appears to be invariant, at least in restricted situations. The hope is to find and prove the exact invariant at play and read off the degree bound. Recently, we started studying the  $L^1$  norm of the coefficients, which ties the degree bound problem to the compressed sensing techniques in applied mathematics. I am working on this angle with John D’Angelo and Dusty Grundmeier.

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