RESEARCH STATEMENT

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1. Introduction and research philosophy

My primary interests lie in complex analysis, and in CR geometry in particular. Research in CR geometry also led me to problems in many other fields of mathematics. My research philosophy is not to simply solve problems within the confines of a particular area but to look for connections and applications to other areas of mathematics and even other disciplines. In this statement, for sake of brevity, I talk only about my main interest, which is CR geometry, and I will only highlight those results which lead to some recent ongoing work or future plans. For a complete list of publications and preprints completed since my last evaluation in November 2017, see §**??**.

In CR geometry, very broadly, I study singularities and complexity. I am assuming that the reader has some basic background in several complex variables. In CR geometry, one generally studies CR submanifolds, that is, a submanifolds with a vector bundle that is the restriction of the Cauchy–Riemann equations. However, this CR structure can develop singularities when the submanifold is not a hypersurface, and we can also consider subvarieties instead of submanifolds, or perhaps even more general subsets. One part of my work deals with trying to understand such singularities, and analogues of results from the nonsingular context to the singular setting.

Another part of my work concerns the complexity of the CR structure. One way this complexity may be studied is via an analogue of the Nash embedding theorem, that is, by studying maps to spheres and hyperquadrics, which are the model CR manifolds of hypersurface type. We can ask which maps exist and how complicated such maps are.

The common technique underlying most of the work mentioned below is the study of real power series, both formal and convergent, and its application to the questions above. In particular, one way to describe the relation of a real power series to the complex structure is by the use of the Segre varieties and related techniques. One of these techniques is to "diagonalize" the hermitian matrix of coefficients of a real-analytic function to write it as $\|f(z)\|^2 - \|g(z)\|^2$ for holomorphic mappings ($\mathbb{C}^m$- or possibly ℓ^2-valued) f and g. This decomposition naturally gives rise to maps to spheres and hyperquadrics.

2. Higher codimension CR singular submanifolds

Let $M \subset \mathbb{C}^n$ be a smooth submanifold. The *CR structure* of M is the restriction of the Cauchy–Riemann equations to the complexified tangent bundle of M. If the CR structure is a vector bundle, we say that M is a *CR manifold*. When M is of higher codimension than 1, the CR structure naturally develops singularities. E. Bishop [4] first studied the nondegenerate CR singular submanifolds of real codimension 2 in $\mathbb{C}^2$. These have the form

$$w = z\bar{z} + \lambda(z^2 + \bar{z}^2) + O(3), \tag{1}$$

for $\lambda \in [0, \infty]$ ($\lambda = \infty$ means $w = z^2 + \bar{z}^2 + O(3)$).

In this direction, I am primarily interested in CR singular submanifolds of codimension 2 in $\mathbb{C}^m$ for $m > 2$. In particular, I am particularly interested in the local function theory on these manifolds. On CR manifolds, there is a natural notion of a so-called *CR function*: a function killed by vector fields in the CR structure. If both the submanifold M and the function f are real-analytic, then the classical theorem of Severi says that if f is CR then it is a restriction of a holomorphic function. On CR singular submanifolds, the notion of a CR function does not have a single obvious generalization. A current project I am working on is a discussion of several distinct and natural generalizations of the notion of CR function to a CR singular manifold.

The simplest possibility is to extend the definition above to the CR singular manifold. We will say $f\colon M \to \mathbb{C}$ is a *CR function* if near every point it is killed by every anti-holomorphic vector field tangent to M. This definition is equivalent to asking f to be a CR function on the CR points of M, a set that we denote M_{CR}.

The theorem of Severi mentioned above that a real-analytic CR function f on a real-analytic CR submanifold M is a restriction of a holomorphic function, no longer holds in the CR singular case. The simplest example is the Bishop surface $w = |z|^2$ in $\mathbb{C}^2$, where $\bar{z}$ is CR according to our definition but is not a restriction of a holomorphic function. In $\mathbb{C}^2$, we proved a Severi-type theorem in the elliptic case by introducing a certain natural moment condition.

In $\mathbb{C}^3$ and higher dimensions, the solution is somewhat simpler, and we found a simple sufficient condition on M. A codimension two CR singular submanifold $M \subset \mathbb{C}^{n+1}$ can be put into the form

$$w = z^* A z + \overline{z^t B z} + z^t C z + E(z, \bar{z}), \tag{2}$$

where the coordinates are $(z, w) \in \mathbb{C}^n \times \mathbb{C}$, A, B, C are complex $n \times n$ matrices, B and C are symmetric, and E is $O(\|z\|^3)$. We treat z as a column vector, z^t is the transpose, and z^* means the conjugate transpose. We have the following analogue of Severi:

Theorem 2.1 (Lebl–Noell–Ravisankar [32])**.** *Let $M \subset \mathbb{C}^{n+1}$ be a real-analytic submanifold in the form* (2) *such that* $\operatorname{rank}\left[\begin{smallmatrix} A^* \\ B \end{smallmatrix}\right] \geq 2$*. Then any real-analytic function that is CR at CR points of M is locally a restriction of a holomorphic function.*

The theorem is sharp; for A and B that do not satisfy the rank condition, there exists a counterexample M and f such that f is not the restriction of a holomorphic function.

The smooth (C^∞) case is more complicated, even in the CR case. A classical theorem of Lewy shows that a smooth CR function f on a smooth hypersurface $M \subset \mathbb{C}^m$ extends as a holomorphic function to one side of M as long as the Levi-form of M has at least one nonzero eigenvalue. If it has eigenvalues of both signs, then f is a restriction of a holomorphic function. We wanted to ask what is the analogue in higher codimension submanifolds. The first question to ask is what "extends to one side" means. The simplest case seems to be when M is holomorphically-flat, that is, if after a change of variables, $M \subset \mathbb{C}^n \times \mathbb{R}$. Conditions for being holomorphically-flat in higher dimensions have been of much interest lately; for some recent work, see [14, 20, 21] and the references within. A holomorphically-flat M is a hypersurface in the canonical Levi-flat hypersurface $\mathbb{C}^n \times \mathbb{R}$, and hence M is codimension 2 in $\mathbb{C}^{n+1}$. In this case, we put M in the form

$$w = z^* A z + \overline{z^t B z} + z^t B z + E(z, \bar{z}), \tag{3}$$

where A is hermitian, B symmetric, and E real-valued. The matrix A is the one that represents the analogue of the Levi-form in this case. Let the "positive side" of M in the $\mathbb{C}^n \times \mathbb{R}$ be

$$H_+ : \operatorname{Re} w \geq z^* A z + \overline{z^t B z} + z^t B z + E(z, \bar{z}), \quad \operatorname{Im} w = 0. \tag{4}$$

Theorem 2.2 (Lebl–Noell–Ravisankar [30])**.** *Let H_+ and M be as above, $n \geq 2$, and suppose that the real quadratic form $z^* A z + \overline{z^t B z} + z^t B z$ is nondegenerate. If A has at least two positive eigenvalues, then there exists a neighborhood $U \subset \mathbb{C}^n \times \mathbb{R}$ of the origin such that every function $f \in C^\infty(M) \cap CR(M_{CR})$ extends to a smooth CR function on $U \cap H_+$. If A also has two negative eigenvalues, then f extends to a smooth CR function on U.*

The theorem is optimal in the sense that one positive eigenvalue is not enough for extension.

In the non-holomorphically-flat case, extension may be attempted into a wedge with edge M, and such a result was proved by Tumanov [40] for CR manifolds. Recently, Noell, Ravisankar, and I [33] studied some consequences of this wedge being an open subset, including in the singular case. In particular, we proved a Cartan-uniqueness-like theorem for such sets (the classical Cartan uniqueness theorem says that if the derivative of a self-map of a bounded domain is the identity at a single fixed point, then the map is the identity).

We are currently interested in analyzing other natural possible definitions of the set of CR functions. In particular, we are interested in conditions that give an analogue of the Baouendi–Trèves theorem on polynomial approximation of CR functions, which does not hold as is for CR singular submanifolds.

3. Levi-flat hypersurfaces and Segre varieties

Levi-flat hypersurfaces, that is, hypersurfaces pseudoconvex from both sides, possess a natural foliation by complex hypersurfaces and are a much studied object in CR geometry and the theory of holomorphic foliations. E.g., a famous, and as yet not fully solved, problem in CR geometry is the claimed non-existence of a smooth Levi-flat hypersurface in the n dimensional complex projective space when $n \geq 2$. This nonexistence was proved for real-analytic hypersurfaces for $n \geq 3$ by Lins Neto [36], and has since been extended to lower regularity by many others. The case $n = 2$ stubbornly remains unsolved.

A subset of the n-dimensional projective space naturally induces a complex cone in $\mathbb{C}^{n+1}$, that is, a singular set. So a natural type of object to consider is a real-analytic subvariety. We call a real-analytic subvariety H of codimension 1 a *real hypervariety.* Write H^* for the set of points regular points of hypersurface dimension (not necessarily the same as the set of regular points). We say H is *Levi-flat* if H^* is a Levi-flat hypersurface. Levi-flat hypervarieties have some properties of complex analytic subvarieties; however, they also possess many of the rather considerable pathologies of real-analytic subvarieties, which makes their study difficult. For example, a real subvariety may not be coherent: the defining equation of H at one point need not be the defining equation at points arbitrarily near. One of many pathologies this leads to is irreducible hypervarieties where the relative closure of H^* is a proper subset of H. Singular Levi-flat hypersurfaces were first studied by Burns–Gong [5], and many others more recently, see e.g., Fernández-Pérez [16], Pinchuk–Shafikov–Sukhov [39], and the references within. Let me list some of my own older results that are relevant to some of my current work.

(1) In [26] I proved that the singular set of the relative closure of H^* is Levi-flat where it is a real-analytic CR submanifold. In other words, the singular set of H is Levi-flat itself.
(2) In [24] I proved an analogue of the well known Chow's theorem for Levi-flat hypervarieties in $\mathbb{P}^n$, $n \geq 2$: If H is locally the pullback of a real-analytic curve via a meromorphic function, and if H has infinitely many compact leaves in its Levi-foliation, then H is contained in the pullback of a real-algebraic curve via a rational function, i.e., H is semi-algebraic.
(3) In [27] I studied Levi-flat hypersurfaces that are induced by curves in the Grassmanian, that is, those that are unions of complex hyperplanes. All singular Levi-flats are in some sense realizable as subsets of such a union of hyperplanes in much higher dimension. I also proved that a general Chow-like result cannot hold by constructing a non-algebraic Levi-flat hypervariety in $\mathbb{P}^2$.

The main technique in attacking Levi-flat hypersurfaces is the Segre variety: If H is defined near the origin by the equation $\rho(z, \bar{z}) = 0$, then the *Segre variety* is given by the equation $\rho(z, 0) = 0$. For example, for the nonsingular Levi-flat hypersurface $0 = \operatorname{Im} z_1 = \frac{z_1 - \bar{z}_1}{2i}$, the Segre variety is $z_1 = 0$. Note that this variety agrees with the leaf of the Levi-foliation. Segre varieties first gained prominence in CR geometry with the work of Webster [41] and Diederich–Fornæss [13] for the nonsingular cases. For the most part this technique still works in the singular case, but with some pathologies. First, if H is not coherent, then the germ of the equation $\rho = 0$ might not define the right ideals at points arbitrarily near the origin, and hence we cannot simply define the Segre varieties in a neighborhood with a single function ρ. The second pathology is that $\rho(z, 0)$ may in fact be identically zero. We say this point is *Segre-degenerate.* It is not difficult to show that such points are a subset of a codimension 2 complex subvariety; however, this set itself need not even be a subvariety. In [28] I proved that it has at least the structure of a semianalytic set.

Theorem 3.1. *Let X be a real hypervariety in $\mathbb{C}^n$ and $X_{[n]}$ be the set of Segre-degenerate points. Then $X_{[n]}$ is semianalytic. If X is coherent, then $X_{[n]}$ is complex analytic.*

In the same paper I give examples where $X_{[n]}$ is not a real subvariety, and also where $X_{[n]}$ does not have complex structure (not a complex subvariety). For Levi-flat hypersurfaces, the Segre-degenerate singularities correspond to dicritical singularities of the underlying foliation, hence understanding their structure is important.

A long term goal for Levi-flat hypervarieties is to prove a Levi-flat stratification. This plan requires classifying and understanding higher codimension Levi-flat submanifolds, including the CR singular ones. Together with Xianghong Gong [18], we studied the CR singular Levi-flat submanifolds of codimension 2 in $\mathbb{C}^m$, $m \geq 3$. We obtained a normal form for the quadratic terms and a complete formal normal form for the nondegenerate case in $\mathbb{C}^3$.

One connection to the CR singular submanifolds of codimension 2 is the study of Levi-flat singular hypersurfaces via their intersections with a compact CR manifold such as a sphere, generating a Milnor-link like manifold, a holomorphically-flat CR singular compact submanifold. The opposite direction, that is, starting with a compact CR singular manifold and getting a Levi-flat (possibly singular) hypersurface with this given boundary is the so-called Levi-flat Plateau problem, which we recently studied with Noell and Ravisankar in [31] for CR singular manifolds with nondegenerate elliptic singularities that are images of compact submanifolds in $\mathbb{C}^n \times \mathbb{R}$.

More recently, Bernhard Lamel and I [22] considered totally-real subvarieties, that is, subvarieties with no CR vectors at their regular points. However, their singularity, unless it is degenerate, does induce a certain "finite" CR structure via Segre varieties. Let X be such a subvariety. The Segre variety of X is a finite set. We proved that given a real-analytic function $f(z,\bar{z})$ and averaging f over the Segre variety $\{\xi_1,\ldots,\xi_k\}$, to get $\mathcal{A}f = \frac{1}{k}\sum f(z,\xi_j)$, gives an operator with the following property: f on X is the restriction of a holomorphic function if and only if

$$\mathcal{A}(f^\ell) = (\mathcal{A}f)^\ell \quad \text{for all } \ell = 1,\ldots,k. \tag{5}$$

Restrict this operator $\mathcal{A}$ to just the antiholomorphic functions and call this restricted operator $\mathcal{R}$. We proved that $\mathcal{R}$ actually contains all the information about the germ of the subvariety $(X,0)$: Finding a normal form for $\mathcal{R}$ is equivalent to finding a normal form for $(X,0)$.

We are currently working on generalizing this work to subvarieties of lower codimension, in which case the Segre variety has higher codimension, and one cannot simply average over the Segre variety.

4. CR maps between spheres and hyperquadrics

Let $M \subset \mathbb{C}^n$ and $M' \subset \mathbb{C}^N$ be real submanifolds. A fundamental question is to classify CR maps between M and M'. Let M' be a hyperquadric, that is, M' is defined by $\langle z,z\rangle = 1$, where $\langle\cdot,\cdot\rangle$ is a nondegenerate (not necessarily positive definite) Hermitian product. Hyperquadrics are the model hypersurfaces of different signatures of the Levi-form. The classification of real-analytic CR maps $\varphi\colon M \to M'$ amounts to understanding the ideal of real functions vanishing on M. For simplicity, let $\rho(z,\bar{z})$ be a polynomial vanishing on M. Write

$$\rho(z,\bar{z}) = \|f(z)\|^2 - \|g(z)\|^2, \tag{6}$$

where f and g are holomorphic maps to some k and m dimensional spaces. The numbers k and m are are the number of positive and negative eigenvalues of the matrix of coefficients of ρ. The map (f,g) induces (by dividing by one of the components) a CR map $\varphi\colon M \to M'$, unique up to fractional linear transformations preserving M', where M' is the hyperquadric of signature $(k,m-1)$ or $(k-1,m)$. The problem rests in studying the signature pair (the number of positive and negative eigenvalues) of functions in the ideal generated by M. The possible numbers can be thought of as giving the "CR complexity" of M.

A well-studied case is when $M = S^{2n-1} \subset \mathbb{C}^n$ and $M' = S^{2N-1} \subset \mathbb{C}^N$ are unit spheres, alternatively this problem studies proper holomorphic maps of $\mathbb{B}_n$ to $\mathbb{B}_N$. When $N < n$ no nonconstant

CR maps exist. Two maps are *spherically equivalent* if the they are conjugates of each other via linear fractional automorphisms of the spheres. When $n = N = 1$, the map z^d takes the unit circle to itself and is of arbitrary degree d. A well-known theorem (Pincuk [38], Alexander [1], and others) states that if $n = N \geq 2$, then any CR map of spheres must be linear fractional, a rational map of degree 1. So degree-one maps are equivalent to the identity. A map is *monomial* if each component is a single monomial. For degree-2 maps we have:

Theorem 4.1 (Lebl [25]). *Let $f \colon S^{2n-1} \to S^{2N-1}$, $n \geq 2$, be a rational CR map of degree 2. Then f is spherically equivalent to a monomial map. Furthermore, the normal form for such a map is*

$$z \in \mathbb{C}^n \mapsto Lz \oplus (\sqrt{I - L^*L}z) \otimes z, \tag{7}$$

where L is a diagonal matrix with nonnegative diagonal entries sorted by size, such that $I - L^*L$ also has nonnegative entries. All maps of the form (7) *are mutually spherically inequivalent.*

Forstnerič [17] proved that if $n \geq 2$ and the map is C^∞, then the map is rational of degree d bounded by a function of n and N. A sharp bound on d is unknown, but D'Angelo conjectured:

$$d \leq \begin{cases} 2N - 3 & \text{if } n = 2, \\ \frac{N-1}{n-1} & \text{if } n \geq 3. \end{cases} \tag{8}$$

Monomial examples that achieve equality exist. The best currently known bound $d \leq \frac{N(N-1)}{2(2n-3)}$ was proved by Meylan [37] for $n = 2$ and extended to $n \geq 3$ by D'Angelo and myself [10]. The combinatorics in the monomial case seems to capture the complexity of the general problem, and the sharp bound is known in this case:

Theorem 4.2 (D'Angelo–Kos–Riehl [8] for $n = 2$, and Lebl–Peters [34, 35] for $n \geq 3$). *Suppose that $f \colon S^{2n-1} \to S^{2N-1}$, $n \geq 2$, is a monomial CR map of degree d. Then* (8) *holds and is sharp.*

Spherical equivalence is not the only natural notion of equivalence for proper holomorphic maps. The obstruction in many problems in complex analysis is topological, so it is natural to study the topology of the space of proper maps.

Theorem 4.3 (D'Angelo–Lebl [12]). *Let S denote the set of homotopy classes of proper rational maps $f \colon \mathbb{B}_n \to \mathbb{B}_N$. If $n \geq 2$, then S is a finite set.*

On the other hand, if H_t is a homotopy of two proper maps, such that H_0 and H_1 are spherically inequivalent, then H_t contains uncountably many spherically inequivalent maps.

The result suggests that studying homotopy classes is more tractable. When only finitely many spherically inequivalent maps exist for a certain pair of dimensions, then these are also inequivalent homotopically. For example, Faran [15] proved that a proper map $f \colon \mathbb{B}_2 \to \mathbb{B}_3$, smooth up to $\partial\mathbb{B}_2$, is spherically equivalent to exactly one of four maps, $(z, w, 0)$, (z, zw, w^2), $(z^2, \sqrt{2}\, zw, w^2)$, or $(z^3, \sqrt{3}\, zw, w^3)$. Hence there are also 4 homotopy classes of maps from $\mathbb{B}_2$ to $\mathbb{B}_3$.

Unfortunately, degree is not a homotopy invariant. From known examples, however, it appears that one can construct a homotopy from a rational map to a polynomial map of the same degree by shrinking the nonconstant parts of the denominator. If this conjecture is true, then the degree bounds can be reduced to polynomial maps. Furthermore, homotopy classes would have polynomial representatives, simplifying the classification up to homotopy. Further conjecture is then that polynomial maps can be homotoped to monomial maps. It is not difficult to see that polynomial or rational maps constructed by partial tensoring can be homotoped down to monomial maps of the same or higher degree. D'Angelo's (see [7]) classification of polynomial maps by partial tensoring and untensoring then suggests that this result may be true for all maps. Furthermore, when $n = 1$ it can be proved that that all proper rational maps $f \colon \mathbb{D} \to \mathbb{B}_N$ are homotopy equivalent to a monomial map of the same degree. Thus, homotopy provides not only a new way to classify proper maps, but also a program for solving the degree bound conjecture.

Studying the denominator is related to the homotopy question asked above. Thus, I have recently found the normal form under spherical equivalence for the denominator of a proper rational map. Namely the linear terms in the denominator can be zeroed out by a unique pair of automorphisms.

Theorem 4.4 (Lebl [29])**.** *Suppose $f\colon \mathbb{B}_n \to \mathbb{B}_N$ is a rational proper map of degree d. Then there exist $0 \leq \sigma_1 \leq \cdots \leq \sigma_n \leq \frac{d-1}{2}$, and $\psi \in \operatorname{aut}(\mathbb{B}_n)$ and $\tau \in \operatorname{aut}(\mathbb{B}_N)$ such that $\tau \circ f \circ \psi = \frac{P}{G}$, where $P(0) = 0$,*

$$G(z) = 1 + \sum_{k=1}^{n} \sigma_k z_k^2 + E(z), \tag{9}$$

and $E(z)$ is of order at least 3 and degree at most $d-1$. The numbers $\sigma_1, \ldots, \sigma_n$ are spherical invariants and f is in normal form up to composition with unitary maps.

To see the power of this theorem, note that if $0 < \sigma_1 < \cdots < \sigma_n$ are all distinct, then the V above is a diagonal map with 1s and -1s on the diagonal. The U could be used to put the coefficients of P into the row echelon form with positive pivots, and then we have a normal form up to a finite group (possible flipping of signs of the variables). We can also decide if a map is equivalent to a polynomial map taking the origin to the origin by putting it into this normal form. While my previous work shows that all degree 2 maps are spherically equivalent to polynomial (monomial, in fact) maps, the generic degree 3 map is not. The theorem is a consequence of the taking the critical point of the following function to the origin.

Theorem 4.5 (Lebl [29])**.** *Suppose $f = \frac{p}{g}\colon \mathbb{B}_n \to \mathbb{B}_N$ is a rational proper map of degree d written in lowest terms. Then the function*

$$\Lambda(z, \bar{z}) = \frac{|g(z)|^2 - \|p(z)\|^2}{(1 - \|z\|^2)^d} \tag{10}$$

is a strictly plurisubharmonic exhaustion function of $\mathbb{B}_n$. Moreover, the function transforms naturally under composition with automorphisms.

By work of Catlin and D'Angelo [6], every polynomial $g(z)$ not vanishing on the closed ball is the denominator of some rational proper map (possibly of large degree). It is also possible to show that for all small enough $\sigma_1, \ldots, \sigma_n$, there is a proper map of degree 3 with the denominator being $1 + \sigma_1 z_1^2 + \cdots + \sigma_n z_n^2$. The next step in the classification up to homotopy, is to show that not only are there maps realizing the given denominator, but that these maps can be chosen to have the same target dimension and that this solution may be varied continuously.

We can generalize the manifold $\partial\mathbb{B}_n$ to a hyperquadric $Q(a, b)$ defined by $|z_1|^2 + \cdots + |z_a|^2 - |z_{a+1}|^2 - \cdots - |z_{a+b}|^2 = 1$. With D'Angelo, we proved in [11] that for any n, and all A and B with $A + B$ large enough, there exists a rational map taking $Q(n, 0) = \partial\mathbb{B}_n$ to $Q(A, B)$ in a nontrivial way, that is, the image does not lie in a complex hyperplane (the components are affine linear independent). On the other hand, with Dusty Grundmeier and Liz Vivas [19], we proved that a similar result does not hold when the source is not a sphere. Let $a \geq 2$, $b \geq 1$, and $a > b$. We proved that if f is a real-analytic CR map of an open piece of $Q(a, b)$ to $Q(A, B)$ whose image does not lie in a complex hyperplane, then $A \leq C(a, b, B)$, where $C(a, b, B)$ is a constant depending only on a, b, and B. We constructed maps for all (A, B) in a sector with $A + B$ large enough. When the codimension is small, even more can be proved. Baouendi–Huang [3] proved that if $f\colon Q(a, b) \to Q(A, b)$ is CR with $a > b \geq 1$ and $A \geq b > 1$, then f is equivalent to a linear embedding. Baouendi–Ebenfelt–Huang [2] proved that if $f\colon Q(a, b) \to Q(A, B)$ is CR with $a > b \geq 1$ and $A \geq B > 1$ and $B < 2b - 1$, then f maps to a complex hyperplane.

Embedding a CR manifold into $Q(A, B)$ in a nontrivial way is the CR analogue of the Nash embedding theorem, which fails for general submanifolds. Those manifolds that can be embedded into some $Q(A, B)$, can be considered as having a "finite CR complexity", and it is a natural question to ask, which (A, B) allow such a map. In the results above, we have partial answers for the hyperquadric itself, and we can already see that the answer is nontrivial. Despite much work by various authors, this question is not completely answered even for the sphere-to-sphere case.

5. Future plans

All the results mentioned suggest fertile ground for further work, and I am actively pursuing some of these avenues as noted above. Let me very briefly summarize my current work and future plans. Alan Noell, Sivaguru Ravisankar, and I are working on further understanding the set of CR functions (and the correct definition). Together with Bernhard Lamel, we are working on generalizing the technique of the averaging operator to subvarieties of lower codimension. More generally and along these two directions, I am working on understanding singular sets in CR geometry, namely the Levi-flat singular subvarieties. I am also working on refining the normal form for proper maps and extending the work to more general context. I hope this work will lead to better classification of these and more general hyperquadric maps, and hopefully also a complete solution to the degree bounds conjecture.

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