Differential forms in $\mathbb{R}^3$

The classical theorems we learned this semester can be conveniently stated in a way that gives a vast generalization in one simple statement, and also allows one to more easily remember/derive the statements of the theorems, and simplify computations. We will only scratch the surface here.

In 3 dimensions there are 4 different kinds of what are called *differential forms*. There are 0-forms, 1-forms, 2-forms, 3-forms. You have seen 0-forms and 1-forms without knowing about it. Differential forms are things that are "integrated" on the geometric object of the corresponding dimension (point, path, surface, region).

0-forms

In this setup, functions are called 0-forms. 0-forms are "integrated" on points. That is, they are evaluated at points. If P is point then let

$$\int_P f = f(P).$$

For example, if $f(x, y, z) = x^2 - 1 + z$ and $P = (1, 2, 3)$, then

$$\int_P f = f(1, 2, 3) = 1^2 - 1 + 3 = 3.$$

Points can have orientation, that is positive or negative. Above we dealt with a positively oriented P. If Q is negatively oriented, then

$$\int_Q f = -f(Q).$$

For example if $Q = (2, 1, 0)$ is negatively oriented then

$$\int_Q f = -f(2, 1, 0) = -(2^2 - 1 + 0) = -3.$$

This may seem like we're making up nonsense, but it will be useful for stating the fundamental theorem of calculus as the same theorem as Green's, Stokes', divergence, etc...

1-forms

One forms are expressions of the form

$$f(x, y, z)\,dx + g(x, y, z)\,dy + h(x, y, z)\,dz.$$

For example

$$x^2 y\,dx + 3xe^z\,dy + (z + y)\,dz.$$

One forms are things that are integrated on paths. If C is a path, then we write

$$\int_C x^2 y\,dx + 3xe^z\,dy + (z + y)\,dz.$$

And you have seen this expression before. That is

$$\int_C f(x, y, z)\,dx + g(x, y, z)\,dy + h(x, y, z)\,dz$$
$$= \int_C \langle f(x, y, z), g(x, y, z), h(x, y, z)\rangle \cdot \hat{T}\,ds.$$

As you've seen in the class it is a lot of times convenient to use the following formula for 1-forms. Suppose C is parametrized by t for $a \le t \le b$. That is, x, y, z are functions of t. Then

$$\int_C f(x, y, z)\,dx + g(x, y, z)\,dy + h(x, y, z)\,dz$$
$$= \int_a^b \left(f\big(x, y, z\big)\frac{dx}{dt} + g\big(x, y, z\big)\frac{dy}{dt} + h\big(x, y, z\big)\frac{dz}{dt} \right) dt. \quad (1)$$

We often just give a name to the one-form, that we say $\omega = f(x, y, z)\,dx + g(x, y, z)\,dy + h(x, y, z)\,dz$. Then

$$\int_C \omega = \int_C f(x, y, z)\,dx + g(x, y, z)\,dy + h(x, y, z)\,dz$$

And as we've seen before, paths have orientation, so this looks similar like all the notation for one-forms.

One way that one-forms arise is as derivatives of functions. For example, let f be a function, then what you called total-derivative in Calc III, is really the d operator on 0-forms giving 1-forms. That is

$$df = \frac{\partial f}{\partial x}\,dx + \frac{\partial f}{\partial y}\,dy + \frac{\partial f}{\partial z}\,dz.$$

For example if $f(x, y, z) = x^2 e^y z$, then

$$df = 2xe^y z\,dx + x^2 e^y z\,dy + x^2 e^y\,dz.$$

Boundaries of paths and the fundamental theorem

If C is a path from point Q to point P, then we say that the boundary of C is P with positive orientation and Q with negative orientation. Sometimes this is written as $P - Q$, although you then have to be careful not to do arithmetic here despite what it looks like. We call the boundary ∂C.

The advantage of all this is the easy statement of the fundamental theorem of calculus that will look like all the other statements of the fundamental theorem. We can simply write it as

$$\int_C df = \int_{\partial C} f$$

Let's interpret this. The left hand side is

$$\int_C df = \int_C \frac{\partial f}{\partial x}\, dx + \frac{\partial f}{\partial y}\, dy + \frac{\partial f}{\partial z}\, dz.$$

While the right hand side, assuming C goes from Q to P is then

$$\int_{\partial C} f = f(P) - f(Q).$$

For example, if $f(x, y, z) = x^2 e^y z$ as above, and C is the path parametrized by $\gamma(t) = \langle t, 3t, t + 1\rangle$ for $0 \le t \le 1$, so starting at $(0, 0, 1)$ and ending at $(1, 3, 2)$, then

$$\int_C df = \int_{\partial C} f = f(1, 3, 2) - f(0, 0, 1) = 1^2 e^3 2 - 0^2 e^0 1 = 2e^3.$$

2-forms

OK, so far we've only seemed to make up notation for things we already know. For 2-forms we need to be even more careful with orientation and we need to keep track of it on the form side of things. For this we introduce a new object, the so-called *wedge* or *wedge product*. It is a way to put together forms. In particular, we can write

$$dx \wedge dy, \qquad dy \wedge dz, \qquad dz \wedge dx.$$

Now we define that

$$dx \wedge dy = -dy \wedge dx, \qquad dy \wedge dz = -dz \wedge dy, \qquad dz \wedge dx = -dx \wedge dz.$$

Finally, a wedge of something with itself is just zero:

$$dx \wedge dx = 0, \qquad dy \wedge dy = 0, \qquad dz \wedge dx = 0.$$

A 2-form is an expression of the form

$$\omega = f\, dy \wedge dz + g\, dz \wedge dx + h\, dx \wedge dy.$$

If any other wedges appear we can use the above rules to convert it to this form. For example

$$x^2\, dy \wedge dz + y\, dx \wedge dz + z^2\, dx \wedge dx = x^2\, dy \wedge dz - y\, dz \wedge dx.$$

We also impose some further algebra rules on this product. It is really what we call bilinear. So if ω, η, and γ are one-forms, then

$$(\omega + \eta) \wedge \gamma = \omega \wedge \gamma + \eta \wedge \gamma,$$

and

$$\omega \wedge (\eta + \gamma) = \omega \wedge \eta + \omega \wedge \gamma.$$

Similarly we can take out functions. If f is a function, because these behave like numbers. That is

$$f\omega \wedge \eta = f(\omega \wedge \eta) = \omega \wedge (f\eta).$$

Let's see this on an example:

$$\begin{aligned}(x^2y\, dx + z^2\, dz) \wedge (e^z\, dy + 8\, dz) &= x^2y\, dx \wedge (e^z\, dy + 8\, dz) + z^2\, dz \wedge (e^z\, dy + 8\, dz)\\ &= x^2ye^z\, dx \wedge dy + 8x^2y\, dx \wedge dz + z^2e^z\, dz \wedge dy + 8z^2\, dz \wedge dz\\ &= -z^2e^z\, dy \wedge dz - 8x^2y\, dz \wedge dx + x^2ye^z\, dx \wedge dy.\end{aligned}$$

In general,

$$\begin{aligned}(f\, dx + g\, dy + h\, dz) \wedge (a\, dx + b\, dy + c\, dz) &= fa\, dx \wedge dx + fb\, dx \wedge dy + fc\, dx \wedge dz\\ &\quad + ga\, dy \wedge dx + gb\, dy \wedge dy + gc\, dy \wedge dz\\ &\quad + ha\, dz \wedge dx + hb\, dz \wedge dy + hc\, dz \wedge dz\\ &= (gc - hb)\, dy \wedge dz + (ha - fc)\, dz \wedge dx + (fb - ga)\, dx \wedge dy.\end{aligned}$$

You should recognize the formula for the cross product. That is, the result is a two form whose coefficients are $\langle f, g, h\rangle \times \langle a, b, c\rangle$. The wedge product is always the right product in the right context.

OK, now that we know what 2-forms are, what do we do with them. Well first, let's see how to differentiate 1-forms to get 2-forms, with the d operator. We want the derivative to be linear so that in particular $d(\omega + \eta) = d\omega + d\eta$. Then when we have an expression such as $f\, dx$ we define

$$d(f\, dx) = df \wedge dx.$$

Similarly for dy and dz. So let's compute the derivative of any 1-form:

$$
\begin{aligned}
d(f\,dx + g\,dy + h\,dz) &= df \wedge dx + dg \wedge dy + dh \wedge dz \\
&= \frac{\partial f}{\partial x}\,dx \wedge dx + \frac{\partial f}{\partial y}\,dy \wedge dx + \frac{\partial f}{\partial z}\,dz \wedge dx \\
&\quad + \frac{\partial g}{\partial x}\,dx \wedge dy + \frac{\partial g}{\partial y}\,dy \wedge dy + \frac{\partial g}{\partial z}\,dz \wedge dy \\
&\quad + \frac{\partial h}{\partial x}\,dx \wedge dz + \frac{\partial h}{\partial y}\,dy \wedge dz + \frac{\partial h}{\partial z}\,dz \wedge dz \\
&= \left(\frac{\partial h}{\partial y} - \frac{\partial g}{\partial z}\right) dy \wedge dz + \left(\frac{\partial f}{\partial z} - \frac{\partial h}{\partial x}\right) dz \wedge dx + \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dy \wedge dx.
\end{aligned}
$$

You should recognise the formula for the curl. That is, if the functions f, g, h are coefficients of a vector field, then the coefficients of the derivative of the one form are the coefficients of the curl of the vector field.

For example,

$$d(x\,dx + y^2\,dz) = 2y\,dy \wedge dz.$$

The formula $\nabla \times \nabla f = 0$ shows itself in the fact that

$$d(df) = 0.$$

This will in fact be a feature of the d operator and it is sometimes written as $d^2 = 0$.

OK, now that we have the derivative, we also want to integrate 2-forms. 2-forms are integrated over surfaces. So let S be an oriented surface where $\hat{n}$ is the unit normal that gives the orientation. We define

$$\int_S f\,dy \wedge dz + g\,dz \wedge dx + h\,dx \wedge dy = \iint_S \langle f, g, h\rangle \cdot \hat{n}\,dS.$$

We use only one integral sign for integrals of forms by convention.

There is another way to see what this integral is using the change of variables formula from Calculus III. Denote

$$\frac{\partial(x,y)}{\partial(u,v)} = \det\left(\begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix}\right) = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}.$$

This expression is the determinant of the derivative from the change of variables formula for 2 dimensional integrals from Calc III. Sometimes this formula is called the Jacobian determinant. Let S be parametrized by (u, v) ranging over a domain D, where the ordering u and then v gives the orientation of S. Then

$$\int_S f\,dy\wedge dz+g\,dz\wedge dx+h\,dx\wedge dy = \iint_D \left(f\frac{\partial(y,z)}{\partial(u,v)} + g\frac{\partial(z,x)}{\partial(u,v)} + h\frac{\partial(x,y)}{\partial(u,v)}\right) du\,dv.$$

Compare this to how we computed 1-form integrals above in equation (1), and it will feel very familiar.

Stokes' theorem

The classical Stokes' theorem can now be stated. Let S be an oriented surface and ∂S be the boundary curve of S oriented according to the right hand rule as we have for the classical Stokes theorem. Let ω be a 1-form. Then

$$\int_S d\omega = \int_{\partial S} \omega.$$

If $\omega = f\,dx + g\,dy + h\,dz$, then $d\omega$ as we saw above is really the 2-form whose coefficients are the components of $\nabla \times \langle f, g, h\rangle$. So the left

$$\int_S d\omega = \iint_S \nabla \times \langle f, g, h\rangle \cdot \hat{n}\,dS.$$

The right hand side is simply the integral

$$\int_{\partial S} \omega = \int_{\partial S} \langle f, g, h\rangle \cdot \hat{T}\,ds.$$

And we have the classical Stokes'. Notice how the expression

$$\int_S d\omega = \int_{\partial S} \omega$$

is now the same for both the Stokes' theorem and the fundamental theorem of calculus. All that changes is if S is a surface or a curve, and if ω is a 0-form (function) or a 1-form.

3-forms

If we take one more wedge we find that the only forms that survive our rules, namely that $dx \wedge dx = dy \wedge dy = dz \wedge dz = 0$, are the ones that look like

$$f\,dx \wedge dy \wedge dz.$$

Notice that

$$dx \wedge dy \wedge dz = dz \wedge dx \wedge dy = dy \wedge dz \wedge dx$$
$$= -dy \wedge dx \wedge dz = -dx \wedge dz \wedge dy = -dz \wedge dy \wedge dx.$$

Integrating 3-forms is easy. Write the 3-form as $f\, dx \wedge dy \wedge dz$ and then, given a region R in 3-space, we have

$$\int_R f\, dx \wedge dy \wedge dz = \iiint_R f\, dV,$$

where dV is the volume measure. We also put orientation on R, and the above is for positive orientation. If orientation is not mentioned, we always mean the positive orientation. We would get the negative of the integral for negative orientation. Let us not worry about it, and just do positively oriented regions in 3 space.

Example: Let R be the region defined by$-1 < x < 2$, $2 < y < 3$, $0 < z < 1$. Then

$$\int_R x^2 y e^z dx \wedge dy \wedge dz = \int_{-1}^{2} \int_{2}^{3} \int_{0}^{y} x^2 y e^z\, dz\, dy\, dx = \int_{-1}^{2} \int_{2}^{3} x^2 y (e-1)\, dy\, dx$$
$$= \int_{-1}^{2} x^2 \left(\frac{3^2}{2} - \frac{2^2}{2} \right) (e-1)\, dx = \left(\frac{2^3}{3} - \frac{(-1)^3}{3} \right) \left(\frac{3^2}{2} - \frac{2^2}{2} \right) (e-1).$$

Next, how do we differentiate 2-forms to get 3-forms? We apply essentially the same formula as before:

$$d(f\, dy \wedge dz + g\, dz \wedge dx + h\, dx \wedge dy) = df \wedge dy \wedge dz + dg \wedge dz \wedge dx + dh \wedge dx \wedge dy.$$

Let us carry this through. For example, let's start with the first term:

$$df \wedge dy \wedge dz = \left(\frac{\partial f}{\partial x}\, dx + \frac{\partial f}{\partial y}\, dy + \frac{\partial f}{\partial z}\, dz \right) \wedge dy \wedge dz$$
$$= \frac{\partial f}{\partial x}\, dx \wedge dy \wedge dz + \frac{\partial f}{\partial y}\, dy \wedge dy \wedge dz + \frac{\partial f}{\partial z}\, dz \wedge dy \wedge dz = \frac{\partial f}{\partial x}\, dx \wedge dy \wedge dz.$$

In the second term, it is only the $\frac{\partial g}{\partial y}$ term to survive, and in the third term it is only the $\frac{\partial h}{\partial z}$ term.

All in all we find that for $\omega = f\, dy \wedge dz + g\, dz \wedge dx + h\, dx \wedge dy$, that

$$d\omega = d(f\, dy \wedge dz + g\, dz \wedge dx + h\, dx \wedge dy) = \left(\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z} \right) dx \wedge dy \wedge dz.$$

And again, notice the expression for the divergence pops up. We are then not surprised that the Divergence theorem

$$\iiint_R \nabla \cdot \langle f, g, h \rangle\, dV = \iint_{\partial R} \langle f, g, h \rangle \cdot \hat{n}\, dS,$$

where ∂R is the boundary of R oriented with the outward unit normal $\hat{n}$, takes the form

$$\int_R d\omega = \int_{\partial R} \omega.$$

The formula

$$\int_\Omega d\omega = \int_{\partial\Omega} \omega.$$

is called the *generalized Stokes' theorem*. Here ω is a $(k-1)$-form and Ω is a k-dimensional geometric object over which to integrate. In 3-space this is either a path (1-dimensional), a surface (2-dimensional), or a region (3-dimensional).

Applying in the plane

In the plane you can think of everything as if it were in three space but with no z dependence, so no dz. So there are only 0-forms, 1-forms and 2-forms. And in fact the only 2-form that appears is the $dx \wedge dy$ since the other possiblity gets you $dy \wedge dx = -dx \wedge dy$.

$$\begin{aligned} d(f\,dx + g\,dy) &= df \wedge dx + dg \wedge dy \\ &= \left(\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy\right) \wedge dx + + \left(\frac{\partial g}{\partial x}dx + \frac{\partial g}{\partial y}dy\right) \wedge dy \\ &= \frac{\partial f}{\partial y}dy \wedge dx + + \frac{\partial g}{\partial x}dx \wedge dy \\ &= \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right)\, dx \wedge dy. \end{aligned}$$

If R is a region in the plane and ∂R is its boundary, then Stokes' theorem says:

$$\int_{\partial R} f\,dx + g\,dy = \int_R d(f\,dx + g\,dy) = \int_R \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right)\, dx \wedge dy.$$

And you will recognize Green's theorem.