Differential Forms Crash Course

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The classical theorems we learned this semester can be conveniently stated in a way that gives a vast generalization in one simple statement, and also allows one to more easily remember/derive the statements of the theorems, and simplify computations. We will only scratch the surface (no pun intended) here. What we are aiming at is the so-called *Generalized Stokes' Theorem*:

$$\int_\Omega d\omega = \int_{\partial\Omega} \omega.$$

If rowdy mathematicians wrote graffiti on bathroom walls, this is a good candidate for what they would write. It says that the integral over an object of the derivative of something is an integral of that something over the boundary. To make all the theorems fit within this equation, we have to figure out what all the objects mean, what is a boundary, and what is a derivative. An amazing thing is that the "d" operator is the right derivative in the right context. It is the gradient when it needs to be a gradient, it is a curl when it needs to be a curl, it is a divergence when it needs to be the divergence, etc. The differential forms also include all the information needed to compute the integrals, to deal with orientation, or to change coordinates.

We will mostly consider 3 dimensions and to some extent 2 dimensions. But the ideas apply in any number of dimensions with almost no change. You will also notice that we do not write $\iint$ and $\iiint$ and such for 2 or 3 dimensional integrals. In this setup, we simply write $\int$ and note that the dimension is implied by the object that is under the integral.

In 3 dimensions, there are 4 different kinds of so-called *differential forms*. There are 0-forms, 1-forms, 2-forms, 3-forms. You have seen 0-forms and 1-forms without knowing about it. Differential forms are things that are "integrated" on the geometric object of the corresponding dimension (point, path, surface, region). In n dimensions there would be $n + 1$ different kinds of differential forms, but let us stick to 3 dimensions for simplicity.

0-forms

In the context of differential forms, functions are called 0-*forms*. These 0-forms are "integrated" on points; points are the 0-dimensional objects. That is, functions are evaluated at points: if P is a point,

$$\int_P f = f(P).$$

For example, if $f(x, y, z) = x^2 - 1 + z$ and $P = (1, 2, 3)$, then

$$\int_P f = f(1, 2, 3) = 1^2 - 1 + 3 = 3.$$

Points are "oriented" positively or negatively. The P above was positively oriented. If Q is negatively oriented,

$$\int_Q f = -f(Q).$$

For example, if $Q = (2, 1, 0)$ is negatively oriented,

$$\int_Q f = -f(2, 1, 0) = -(2^2 - 1 + 0) = -3.$$

We can add and subtract points. For example, suppose that $P = (1, 2, 3)$ and $R = (0, 0, 2)$ are both positively oriented. We write $-P$ as the negatively oriented P. Adding or subtracting two points then corresponds to adding or subtracting the functions. For example,

$$\int_{R-P} f = f(R) - f(P) = f(0, 0, 2) - f(1, 2, 3) = 1 - 3 = -2.$$

That looks a lot like the "integral" of the "boundary" of a segment of a curve that starts at P and ends at R, and this is exactly where this notation will appear. You then have to be careful not to do arithmetic on the components of $R - P$, despite what it looks like. These are points, not vectors, and when points add or subtract, it is in the sense above.

We haven't really done anything except make up new notation so far, and it may seem like we are making up nonsense, but the notation will be useful for stating the fundamental theorem of calculus as the same theorem as Green's, Stokes', divergence, etc.

1-forms

1-forms are expressions such as

$$f(x, y, z)\, dx + g(x, y, z)\, dy + h(x, y, z)\, dz.$$

For example,

$$x^2 y\, dx + 3xe^z\, dy + (z + y)\, dz.$$

So a 1-form is a combination of dx, dy, and dz. We cannot just multiply the dx, dy, dz, although more on that later. These objects keep track of how we integrate. In some sense, they are the "derivatives" of the coordinate functions x, y, and z.

1-forms are integrated on (oriented) paths, as paths are one-dimensional. If C is a path, then we define

$$\int_C f(x, y, z)\, dx + g(x, y, z)\, dy + h(x, y, z)\, dz = \int_C \langle f(x, y, z), g(x, y, z), h(x, y, z) \rangle \cdot \hat{\mathbf{t}}\, ds.$$

You have seen this expression before. We use the following formula for the actual computation. Suppose the path C is parametrized by t for $a \leq t \leq b$. That is, x, y, z are functions of t. Then

$$\int_C f(x,y,z)\,dx + g(x,y,z)\,dy + h(x,y,z)\,dz = \int_a^b \left(f\big(x,y,z\big)\frac{dx}{dt} + g\big(x,y,z\big)\frac{dy}{dt} + h\big(x,y,z\big)\frac{dz}{dt} \right) dt. \quad (1)$$

For example, suppose C is the straight line from $(0,0,0)$ to $(1,2,3)$ parametrized by $x(t) = t$, $y(t) = 2t$, $z(t) = 3t$, for $0 \leq t \leq 1$. Then

$$\int_C x^2 y\,dx + 3xe^z\,dy + (z+y)\,dz = \int_0^1 \Big(\big((t)^2(2t)\big)(1) + \big(3te^{3t}\big)(2) + \big(3t+2t\big)(3) \Big)\,dt = \left[\frac{t^4}{2} + \frac{6t-2}{3}e^{3t} + \frac{15t^2}{2} \right]_{t=0}^1 = \frac{4}{3}e^3 + \frac{26}{3}.$$

We often give a name to the 1-form, as in $\omega = f(x,y,z)\,dx + g(x,y,z)\,dy + h(x,y,z)\,dz$. Then

$$\int_C \omega = \int_C f(x,y,z)\,dx + g(x,y,z)\,dy + h(x,y,z)\,dz.$$

One way 1-forms arise is as derivatives of functions. Let f be a function. Then what you called the *total derivative* in multivariable calculus, is really the "d operator" on 0-forms giving 1-forms. That is,

$$df = \frac{\partial f}{\partial x}\,dx + \frac{\partial f}{\partial y}\,dy + \frac{\partial f}{\partial z}\,dz.$$

For example, if $f(x,y,z) = x^2 e^y z$, then

$$df = 2xe^y z\,dx + x^2 e^y z\,dy + x^2 e^y\,dz.$$

Not every vector field is a gradient vector field, so similarly, not every 1-form is a derivative of a function. For example, $\omega = -y\,dx + x\,dy$ is not the total derivative of any function f. If it were, then

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial(-y)}{\partial y} = -1, \quad \text{but} \quad \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial(x)}{\partial x} = 1,$$

and that is impossible.

Notice that the dx is the derivative of x. That is, if $f(x,y,z) = x$, then

$$df = \frac{\partial f}{\partial x}\,dx + \frac{\partial f}{\partial y}\,dy + \frac{\partial f}{\partial z}\,dz = 1\,dx + 0\,dy + 0\,dz = dx.$$

Similarly for dy and dz. So the notation keeps track of changes of variables and chain rule, as we saw above. That is, dx becomes $\frac{dx}{dt}\,dt$ when we integrate with respect to t. Similarly, if we parametrize a curve with respect to x, we do not need to change the dx. Consider a curve C given by $y = x^2$, $z = x^3$ for $0 \leq x \leq 1$. Let us compute a simple integral over C:

$$\int_C x\,dx + y\,dy + z\,dz = \int_0^1 x\,dx + y\,\frac{dy}{dx}\,dx + z\,\frac{dz}{dx}\,dx = \int_0^1 \big(x + x^2(2x) + x^3(3x^2)\big)\,dx = \frac{3}{2}.$$

Boundaries of paths and the fundamental theorem

If C is a path from point Q to point P, then we say that the boundary of C, which we write as ∂C, is P with positive orientation and Q with negative orientation. In short, we write the boundary of C as $\partial C = P - Q$. The upshot of all this is that the easy statement of the fundamental theorem of calculus will look like all the other statements of the fundamental theorem in higher dimensions. We can simply write it as

$$\int_C df = \int_{\partial C} f$$

Let's interpret this equation. The left-hand side is

$$\int_C df = \int_C \frac{\partial f}{\partial x}\,dx + \frac{\partial f}{\partial y}\,dy + \frac{\partial f}{\partial z}\,dz.$$

While the right-hand side, assuming C goes from Q to P, is

$$\int_{\partial C} f = f(P) - f(Q).$$

If $f(x,y,z) = x^2 e^y z$ as above, and C is the path parametrized by $\gamma(t) = (t, 3t, t+1)$ for $0 \leq t \leq 1$, so starting at $(0,0,1)$ and ending at $(1,3,2)$, then

$$\int_C df = \int_{\partial C} f = f(1,3,2) - f(0,0,1) = 1^2 e^3 2 - 0^2 e^0 1 = 2e^3.$$

Another example of this use is to compute a path integral by computing the antiderivative, provided it exists, of course. For example, suppose C is the straight line from $(0,0,0)$ to $(1,2,3)$, and we want to compute

$$\int_C y\,dx + x\,dy + 2z\,dz.$$

If we can find an f whose total derivative is the form above, then we are done. If f exists, then $\frac{\partial f}{\partial x} = y$, so $f = xy + g(y,z)$ for some function g. Taking the derivative with respect to y gets us $\frac{\partial f}{\partial y} = x + \frac{\partial g}{\partial y}$, so g is independent of y. Taking the derivative with respect to z we find $2z = \frac{\partial f}{\partial z} = \frac{\partial g}{\partial z}$, so $g = z^2$ (plus a constant, but we only need one antiderivative). So $f = xy + z^2$. In other words:

$$\int_C y\,dx + x\,dy + 2z\,dz = \int_C df = \int_{\partial C} f = f(1,2,3) - f(0,0,0) = 1 \cdot 2 + 3^2 = 11.$$

2-forms

So far we have only justified notation that you have seen before. Let us move to the surface integral (the flux integral) and how to frame it in terms of differential forms. For 2-forms, we need to be a bit more careful with orientation, and we need to keep track of it on the form side of things. For this purpose, we introduce a new object, the so-called *wedge* or *wedge product*. It is a way to put together forms. The wedge product takes two 1-forms ω and η and gets a 2-form $\omega \wedge \eta$. We start by wedging together dx, dy, and dz. We write

$$dx \wedge dy, \qquad dy \wedge dz, \qquad dz \wedge dx.$$

We define that

$$dx \wedge dy = -dy \wedge dx, \qquad dy \wedge dz = -dz \wedge dy, \qquad dz \wedge dx = -dx \wedge dz.$$

Finally, a wedge of something with itself is just zero:

$$dx \wedge dx = 0, \qquad dy \wedge dy = 0, \qquad dz \wedge dz = 0.$$

This is true for any 1-form: $\omega \wedge \omega = 0$.

An arbitrary 2-form is an expression of the form

$$\omega = f\, dy \wedge dz + g\, dz \wedge dx + h\, dx \wedge dy.$$

If any other wedges appear, we can (if we really want to) use the rules above to convert them to this form. For example,

$$x^2\, dy \wedge dz + y\, dx \wedge dz + z^2\, dx \wedge dx = x^2\, dy \wedge dz - y\, dz \wedge dx.$$

We also impose some further algebra rules on this product. Anything we would call a "product" had better be what we call bilinear: If ω, η, and γ are 1-forms, then

$$\begin{aligned} (\omega + \eta) \wedge \gamma &= \omega \wedge \gamma + \eta \wedge \gamma, \\ \omega \wedge (\eta + \gamma) &= \omega \wedge \eta + \omega \wedge \gamma. \end{aligned}$$

If f is a function, then

$$(f\omega) \wedge \eta = f(\omega \wedge \eta) = \omega \wedge (f\eta).$$

Let's see these rules on an example:

$$\begin{aligned} (x^2 y\, dx + z^2\, dz) \wedge (e^z\, dy + 8\, dz) &= x^2 y\, dx \wedge (e^z\, dy + 8\, dz) + z^2\, dz \wedge (e^z\, dy + 8\, dz) \\ &= x^2 y e^z\, dx \wedge dy + 8x^2 y\, dx \wedge dz + z^2 e^z\, dz \wedge dy + 8z^2\, dz \wedge dz \\ &= -z^2 e^z\, dy \wedge dz - 8x^2 y\, dz \wedge dx + x^2 y e^z\, dx \wedge dy. \end{aligned}$$

In general,

$$\begin{aligned} (f\, dx + g\, dy + h\, dz) \wedge (a\, dx + b\, dy + c\, dz) &= fa\, dx \wedge dx + fb\, dx \wedge dy + fc\, dx \wedge dz \\ &\quad + ga\, dy \wedge dx + gb\, dy \wedge dy + gc\, dy \wedge dz \\ &\quad + ha\, dz \wedge dx + hb\, dz \wedge dy + hc\, dz \wedge dz \\ &= (gc - hb)\, dy \wedge dz + (ha - fc)\, dz \wedge dx + (fb - ga)\, dx \wedge dy. \end{aligned}$$

You should recognize the formula for the cross product. That is, the result is a 2-form whose coefficients are $\langle f, g, h\rangle \times \langle a, b, c\rangle$. The wedge product is always the right product in the right context.

OK, now that we know what 2-forms are, what do we do with them. First, let's see how to differentiate 1-forms to get 2-forms, with the d operator. We want the derivative to be linear, so that in particular $d(\omega + \eta) = d\omega + d\eta$. When we have an expression such as $f\,dx$, we define

$$d(f\,dx) = df \wedge dx.$$

Similarly for dy and dz. Let's compute the derivative of any 1-form:

$$\begin{aligned} d(f\,dx + g\,dy + h\,dz) &= df \wedge dx + dg \wedge dy + dh \wedge dz \\ &= \left(\frac{\partial f}{\partial x}\,dx + \frac{\partial f}{\partial y}\,dy + \frac{\partial f}{\partial z}\,dz\right) \wedge dx \\ &\qquad + \left(\frac{\partial g}{\partial x}\,dx + \frac{\partial g}{\partial y}\,dy + \frac{\partial g}{\partial z}\,dz\right) \wedge dy \\ &\qquad + \left(\frac{\partial h}{\partial x}\,dx + \frac{\partial h}{\partial y}\,dy + \frac{\partial h}{\partial z}\,dz\right) \wedge dz \\ &= \frac{\partial f}{\partial x}\,dx \wedge dx + \frac{\partial f}{\partial y}\,dy \wedge dx + \frac{\partial f}{\partial z}\,dz \wedge dx \\ &\qquad + \frac{\partial g}{\partial x}\,dx \wedge dy + \frac{\partial g}{\partial y}\,dy \wedge dy + \frac{\partial g}{\partial z}\,dz \wedge dy \\ &\qquad + \frac{\partial h}{\partial x}\,dx \wedge dz + \frac{\partial h}{\partial y}\,dy \wedge dz + \frac{\partial h}{\partial z}\,dz \wedge dz \\ &= \left(\frac{\partial h}{\partial y} - \frac{\partial g}{\partial z}\right) dy \wedge dz + \left(\frac{\partial f}{\partial z} - \frac{\partial h}{\partial x}\right) dz \wedge dx + \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dy \wedge dx. \end{aligned}$$

You should recognize the formula for the curl. That is, if the functions f, g, h are coefficients of a vector field, then the coefficients of the derivative of the 1-form are the coefficients of the curl of the vector field. Again the wedge product and d gets us the right thing in the right context. And to do computations with d and the wedge is much easier to do because one only needs to follow a couple of simple rules.

For example,

$$\begin{aligned} d(x\,dx + y^2\,dz) &= 1\,dx \wedge dx + 0\,dy \wedge dx + 0\,dz \wedge dz + 0\,dx \wedge dz + 2y\,dy \wedge dz + 0\,dz \wedge dz \\ &= 2y\,dy \wedge dz. \end{aligned}$$

Of course, we do not need to do this in excruciating detail; we know which derivatives will end up zero and which wedge products will end up zero. We need only to look at those that end up nonzero. For instance,

$$\begin{aligned} d(xy\,dx + z^2\,dy + y^2\,dz) &= x\,dy \wedge dx + 2z\,dz \wedge dy + 2y\,dy \wedge dz \\ &= (2y - 2z)\,dy \wedge dz - x\,dx \wedge dy. \end{aligned}$$

The formula $\nabla \times \nabla f = \vec{0}$ appears in the fact that

$$d(df) = 0.$$

This formula is a general feature of the d operator, and it is sometimes written as $d^2 = 0$.

Now that we have the derivative, we want to integrate 2-forms. As 2-forms are "two dimensional," they are integrated over surfaces. Let S be an oriented surface, where $\hat{\mathbf{n}}$ is the unit normal that gives the orientation. Suppose S is a graph of $z = \varphi(x, y)$ and $\hat{\mathbf{n}}$ is the upward unit normal. We could define

$$\int_S f\, dy \wedge dz + g\, dz \wedge dx + h\, dx \wedge dy = \iint_S \langle f, g, h \rangle \cdot \hat{\mathbf{n}}\, dS.$$

We use only one integral sign for integrals of forms by convention, even though it is a surface integral. The definition works for any surface integral, cutting any surface into pieces that can be written as a graph, and figuring out the right orientation in case it is not z over x, y as above.

Again, we have only defined a new notation for something we knew how to compute already, the flux integral. But using this notation, a way to compute surface integrals is suggested by the change of variables formula from multivariable calculus. It is really the right way of defining the integral. Moreover, it gives an easy way to compute the integral for any parametrized surface. Denote

$$\frac{\partial(x, y)}{\partial(u, v)} = \det\left(\begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix}\right) = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}.$$

This expression is the determinant of the derivative from the change of variables formula for 2 dimensional integrals. This formula is called the Jacobian determinant. Let S be parametrized by (u, v) ranging over a domain D, where the ordering u and then v gives the orientation of S via the right-hand rule. That is, if we curl the fingers on our right hand, from the u direction to the v direction, then our thumb would be the unit normal giving the orientation. So x, y, and z are functions of (u, v). Then

$$\int_S f\, dy \wedge dz + g\, dz \wedge dx + h\, dx \wedge dy = \iint_D \left(f \frac{\partial(y, z)}{\partial(u, v)} + g \frac{\partial(z, x)}{\partial(u, v)} + h \frac{\partial(x, y)}{\partial(u, v)} \right) du\, dv.$$

Compare this to how we computed 1-form integrals above in equation (1), and it will feel very familiar.

For example, let $\omega = x\, dy \wedge dz + y\, dz \wedge dx + z\, dx \wedge dy$ be the 2-form, and let S be the surface given by the graph $z = x^2 + y^2$ where x and y lie in the unit square $0 \leq x, y \leq 1$. We have $x = u$, $y = v$, $z = u^2 + v^2$. The domain D is the unit square $0 \leq u, v \leq 1$. Then

$$\begin{aligned} \int_S \omega = \int_S x\, dy \wedge dz + y\, dz \wedge dx + z\, dx \wedge dy \\ = \int_0^1 \int_0^1 \Big(u(-2u) + v(-2v) + (u^2 + v^2) \Big)\, du\, dv \\ = \int_0^1 \int_0^1 (-u^2 - v^2)\, du\, dv = \frac{-2}{3}. \end{aligned}$$

For another example, suppose $\eta = xz\, dy \wedge dz$, and let the surface S be the tube of radius 1 around the z-axis for $0 \le z \le 1$ (the curved part of the surface of the cylinder). Orient S with the normal outwards (away from the z-axis). Let us compute $\int_S \eta$.

First, we parametrize S. Let (u, v) map to $(\cos u, \sin u, v)$ for $0 \le u \le 2\pi$ and $0 \le v \le 1$. We check that the right-hand rule, curling our fingers around the u direction (going in horizontal circles around the z axis) followed by the v direction (same as the z direction) gets us the outward normal. If it didn't, we could just swap u and v. So

$$\begin{aligned}\int_S xz\, dy \wedge dz &= \int_0^1 \int_0^{2\pi} \underbrace{(\cos u)}_{x} \underbrace{v}_{z} \underbrace{(\cos u)}_{\frac{\partial(y,z)}{\partial(u,v)}}\, du\, dv \\ &= \int_0^1 \int_0^{2\pi} (\cos u)^2 v\, du\, dv = \pi.\end{aligned}$$

This is a good way to remember how to integrate parametrized surfaces. Another advantage is that you do not always have to put everything in the normal form. Perhaps in the last example, you swap dy and dz (and so introduce a negative sign) and write the integral as $\int_S -xz\, dz \wedge dy$. We can just compute the integral that way:

$$\begin{aligned}\int_S -xz\, dz \wedge dy &= \int_0^1 \int_0^{2\pi} -\underbrace{(\cos u)}_{x} \underbrace{v}_{z} \underbrace{(-\cos u)}_{\frac{\partial(z,y)}{\partial(u,v)}}\, du\, dv \\ &= \int_0^1 \int_0^{2\pi} (\cos u)^2 v\, du\, dv = \pi.\end{aligned}$$

We computed $\frac{\partial(z,y)}{\partial(u,v)}$ because we had $dz \wedge dy$. Before, we computed $\frac{\partial(y,z)}{\partial(u,v)}$ because we had $dy \wedge dz$. That's what we meant when we said the wedge product keeps track of orientation. It keeps track of how you are supposed to integrate a 2-form, no matter how you write it.

Stokes' Theorem

The classical Stokes' Theorem can now be stated. Let S be an oriented surface and ∂S be the boundary curve of S oriented according to the right-hand rule as we have for the classical Stokes Theorem. Let ω be a 1-form. Then Stokes' Theorem in terms of differential forms is

$$\int_S d\omega = \int_{\partial S} \omega.$$

If $\omega = f\, dx + g\, dy + h\, dz$, then $d\omega$, as we saw above, is really the 2-form whose coefficients are the components of $\nabla \times \langle f, g, h \rangle$. So the left-hand side is

$$\int_S d\omega = \iint_S \nabla \times \langle f, g, h \rangle \cdot \hat{\mathbf{n}}\, dS.$$

The right-hand side is the integral

$$\int_{\partial S} \omega = \int_{\partial S} \langle f, g, h \rangle \cdot \hat{\mathbf{t}} \, ds.$$

That is, we have the classical Stokes'. Notice how the expression

$$\int_S d\omega = \int_{\partial S} \omega$$

is now the same for both the Stokes' Theorem and the Fundamental Theorem of Calculus. The only difference is that S is now a surface and not a curve, and ω is a 1-form and not a 0-form (function).

3-forms and the Divergence Theorem

If we take one more wedge, we find that the only forms that survive our rules, namely that $dx \wedge dx = dy \wedge dy = dz \wedge dz = 0$, are the ones that look like

$$f \, dx \wedge dy \wedge dz.$$

Notice that

$$\begin{aligned} dx \wedge dy \wedge dz = dz \wedge dx \wedge dy = dy \wedge dz \wedge dx \\ = -dy \wedge dx \wedge dz = -dx \wedge dz \wedge dy = -dz \wedge dy \wedge dx. \end{aligned}$$

Integrating 3-forms is easy. Write the 3-form as $f \, dx \wedge dy \wedge dz$ and then, given a region R in 3-space, we have

$$\int_R f \, dx \wedge dy \wedge dz = \iiint_R f \, dV,$$

where dV is the volume measure. We also put orientation on R, and the above is for positive orientation. If orientation is not mentioned, we always mean the positive orientation. If R were oriented negatively, then we would define the integral to be the negative of the integral for positive orientation. Let us not worry about it, and just do positively oriented regions in 3-space.

Example: Let R be the region defined by $-1 < x < 2$, $2 < y < 3$, $0 < z < 1$. Then

$$\begin{aligned} \int_R x^2 y e^z \, dx \wedge dy \wedge dz &= \int_{-1}^{2} \int_{2}^{3} \int_{0}^{1} x^2 y e^z \, dz \, dy \, dx = \int_{-1}^{2} \int_{2}^{3} x^2 y (e-1) \, dy \, dx \\ &= \int_{-1}^{2} x^2 \left(\frac{3^2}{2} - \frac{2^2}{2} \right) (e-1) \, dx = \left(\frac{2^3}{3} - \frac{(-1)^3}{3} \right) \left(\frac{3^2}{2} - \frac{2^2}{2} \right) (e-1). \end{aligned}$$

Next, how do we differentiate 2-forms to get 3-forms? We apply essentially the same formula as before:

$$d(f \, dy \wedge dz + g \, dz \wedge dx + h \, dx \wedge dy) = df \wedge dy \wedge dz + dg \wedge dz \wedge dx + dh \wedge dx \wedge dy.$$

Let us carry this through. For example, let's start with the first term:

$$\begin{aligned} df \wedge dy \wedge dz &= \left(\frac{\partial f}{\partial x}\,dx + \frac{\partial f}{\partial y}\,dy + \frac{\partial f}{\partial z}\,dz\right) \wedge dy \wedge dz \\ &= \frac{\partial f}{\partial x}\,dx \wedge dy \wedge dz + \frac{\partial f}{\partial y}\,dy \wedge dy \wedge dz + \frac{\partial f}{\partial z}\,dz \wedge dy \wedge dz = \frac{\partial f}{\partial x}\,dx \wedge dy \wedge dz. \end{aligned}$$

In the second term, it is only the $\frac{\partial g}{\partial y}$ term to survive, and in the third term it is only the $\frac{\partial h}{\partial z}$ term. All in all we find that for $\omega = f\,dy \wedge dz + g\,dz \wedge dx + h\,dx \wedge dy$,

$$d\omega = d(f\,dy \wedge dz + g\,dz \wedge dx + h\,dx \wedge dy) = \left(\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z}\right) dx \wedge dy \wedge dz.$$

And again, notice the expression for the divergence pops up. We are then not surprised that the Divergence Theorem

$$\iiint_R \nabla \cdot \langle f, g, h\rangle\,dV = \iint_{\partial R} \langle f, g, h\rangle \cdot \hat{\mathbf{n}}\,dS,$$

where R is oriented positively and $\hat{\mathbf{n}}$ is the outward unit normal on the boundary ∂R, takes the form

$$\int_R d\omega = \int_{\partial R} \omega.$$

Remark: In the plane, that is, in two dimensions, as there is no z, there is also no dz. So all 3-forms are zero because every which way we take a basic wedge of dx and dy to get a 3-form, we always get zero: $dx \wedge dx \wedge dx = dx \wedge dy \wedge dx = dy \wedge dx \wedge dx = dx \wedge dy \wedge dy = dy \wedge dx \wedge dy = dy \wedge dy \wedge dx = dy \wedge dy \wedge dy = 0$. We could then say that there are no 3-forms in the plane as there are no nontrivial ones.

Similarly, we could try to keep going in 3-space and define 4-forms, but we notice that no matter how we wedge together dx, dy, and dz four times, at least one of them will appear twice and we will get 0. So we could say that 4-forms in 3-space exist but are zero. Further, we could just say that all k-forms are just zero for all $k \geq 4$ in 3-space.

Generalized Stokes' Theorem

The formula

$$\int_\Omega d\omega = \int_{\partial\Omega} \omega$$

is called the *Generalized Stokes' Theorem*. Here ω is a $(k-1)$-form and Ω is a k-dimensional geometric object over which to integrate. In 3-space, ω is a 0-, 1-, or 2-form, and Ω is a path (1-dimensional), a surface (2-dimensional), or a region (3-dimensional).

Another thing to notice is the following diagram:

$$\text{0-forms} \xrightarrow{d} \text{1-forms} \xrightarrow{d} \text{2-forms} \xrightarrow{d} \text{3-forms}$$

corresponds to the diagram

$$\text{functions} \xrightarrow{\nabla} \text{vector fields} \xrightarrow{\nabla\times} \text{vector fields} \xrightarrow{\nabla\cdot} \text{functions}.$$

We mentioned above that $\nabla \times \nabla f = \vec{0}$ is the formula $d(df) = 0$ for a function (0-form) f. Similarly, $\nabla \cdot \nabla \times \vec{F} = 0$ is the formula $d(d\omega) = 0$ for a 1-form ω. It is always true that using the d operator on an output of a d operator, that is a d-derivative of a d-derivative, is 0. In other words,

$$d(d\omega) = 0$$

for all differential forms ω. It is sometimes shortened to $d^2 = 0$.

Applying in the plane

In the plane, think of everything as if it were in 3-space but with no z dependence, so no dz. So there are only 0-forms, 1-forms and 2-forms. The only 2-form that appears is $f\,dx \wedge dy$, since the other possible wedge product gets you $dy \wedge dx = -dx \wedge dy$. The derivative of a 1-form is

$$\begin{aligned} d(f\,dx + g\,dy) &= df \wedge dx + dg \wedge dy \\ &= \left(\frac{\partial f}{\partial x}\,dx + \frac{\partial f}{\partial y}\,dy\right) \wedge dx + \left(\frac{\partial g}{\partial x}\,dx + \frac{\partial g}{\partial y}\,dy\right) \wedge dy \\ &= \frac{\partial f}{\partial y}\,dy \wedge dx + \frac{\partial g}{\partial x}\,dx \wedge dy \\ &= \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right)\,dx \wedge dy. \end{aligned}$$

If R is a region in the plane and ∂R is its boundary, the Generalized Stokes' Theorem says:

$$\int_{\partial R} f\,dx + g\,dy = \int_R d(f\,dx + g\,dy) = \int_R \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right)\,dx \wedge dy.$$

And you will recognize Green's Theorem.

Changing coordinates

Differential forms take care of changing coordinates easily. The trick is to know that dx, dy, dz are the derivatives of the x, y, and z coordinate functions. Suppose we wish to write down everything in terms of dr, $d\theta$, dz of cylindrical coordinates. Consider

$$x = r\cos\theta, \quad y = r\sin\theta, \quad z = z.$$

Then

$$\begin{aligned} dx &= d(r\cos\theta) = \cos\theta\,dr - r\sin\theta\,d\theta, \\ dy &= d(r\sin\theta) = \sin\theta\,dr + r\cos\theta\,d\theta, \\ dz &= d(z) = dz. \end{aligned}$$

Consider the 1-form

$$\omega = (x^2 + y^2)\, dx + z\, dy + dz.$$

Let us change this 1-form into cylindrical coordinates.

$$\begin{aligned}\omega = r^2\, dx + z\, dy + dz &= r^2(\cos\theta\, dr - r\sin\theta\, d\theta) + z(\sin\theta\, dr + r\cos\theta\, d\theta) + dz \\ &= (r^2\cos\theta + z\sin\theta)\, dr + (-r^3\sin\theta + z\cos\theta)\, d\theta + dz.\end{aligned}$$

Now suppose we wish to find

$$\int_C \omega,$$

where C is the spiral given in cylindrical coordinates by $r = 1$, $\theta = t$, $z = t$ for $0 \le t \le 2\pi$. So

$$dr = \frac{dr}{dt}\, dt = 0, \qquad d\theta = \frac{d\theta}{dt}\, dt = dt, \qquad dz = \frac{dz}{dt}\, dt = dz.$$

And so plugging it in we compute

$$\begin{aligned}\int_C \omega &= \int_C (r^2\cos\theta + z\sin\theta)\, dr + (-r^3\sin\theta + z\cos\theta)\, d\theta + dz \\ &= \int_0^{2\pi} (-\sin t + t\cos t + 1)\, dt = 2\pi.\end{aligned}$$

Changing variables for 2-forms and 3-forms is exactly the same idea, since they are constructed out of dx, dy, dz.

For example, what about the area measure on the xy-plane in cylindrical (so in polar coordinates if we just restrict ourselves to the plane). In the plane the area measure dA is $dx \wedge dy$, so

$$\begin{aligned}dx \wedge dy &= (\cos\theta\, dr - r\sin\theta\, d\theta) \wedge (\sin\theta\, dr + r\cos\theta\, d\theta) \\ &= (\cos\theta)(r\cos\theta)\, dr \wedge d\theta + (-r\sin\theta)(\sin\theta)\, d\theta \wedge dr \\ &= (\cos\theta)(r\cos\theta)\, dr \wedge d\theta + (r\sin\theta)(\sin\theta)\, dr \wedge d\theta \\ &= r(\cos^2\theta + \sin^2\theta)\, dr \wedge d\theta = r\, dr \wedge d\theta.\end{aligned}$$

We obtain the familiar $r\, dr\, d\theta$ for the integral with respect to area in polar coordinates from multivariable calculus.

Similarly in 3-space, we find the volume form dV in cylindrical coordinates is

$$dx \wedge dy \wedge dz = r\, dr \wedge d\theta \wedge dz.$$

In $\mathbb{R}^n$

In n coordinates, the whole setup is the same except that there are more differential forms. If $x_1, x_2, \ldots, x_n$ are the coordinates, then we have

$$dx_1, \quad dx_2, \quad \ldots, \quad dx_n$$

as our basic 1-forms. Then the 2-forms are expressions such as

$$f_1\, dx_1 \wedge dx_2 + f_2\, dx_2 \wedge dx_3 + \cdots$$

Note now that there are actually a lot more coefficients in 2-forms (as long as n is bigger than 3). In fact, there are $\frac{n(n+1)}{2}$ coefficients as that's how many basic 1-forms $dx_i \wedge dx_j$ there are. The derivative is still computed the same way. That is

$$d(f\, dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_k}) = df \wedge dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_k}.$$

Here $i_1, i_2, \ldots, i_k$ are just some integers between 1 and n so that we get a basic wedge product of the dxs.

Integration is done much like before. Suppose that we have a k-dimensional object Ω that is parametrized in terms of $u_1, u_2, \ldots, u_k$. That is, x_j (for every j) is a function of $u_1, u_2, \ldots, u_k$. Let D be the domain of the $u_1, u_2, \ldots, u_k$, that is, some region in $\mathbb{R}^k$. The orientation of Ω is given by this parametrization. Let's see how a simple k-form

$$f\, dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_k}$$

is integrated. Then using the determinant in the same way as before we write

$$\int_\Omega f\, dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_k} = \int \cdots \int_D f(x(u)) \frac{\partial(x_{i_1}, x_{i_2}, \ldots, x_{i_k})}{\partial(u_1, u_2, \ldots, u_k)}\, du_1 du_2 \cdots du_k,$$

where by $f(x(u))$ we mean $f(x_1(u_1, u_2, \ldots, u_k), x_2(u_1, u_2, \ldots, u_k), \ldots, x_n(u_1, u_2, \ldots, u_k))$, but that would be a lot to write. The determinant is what it was before, just more complicated:

$$\frac{\partial(x_{i_1}, x_{i_2}, \ldots, x_{i_k})}{\partial(u_1, u_2, \ldots, u_k)} = \det\left(\begin{bmatrix} \frac{\partial x_{i_1}}{\partial u_1} & \frac{\partial x_{i_1}}{\partial u_2} & \cdots & \frac{\partial x_{i_1}}{\partial u_k} \\ \frac{\partial x_{i_2}}{\partial u_1} & \frac{\partial x_{i_2}}{\partial u_2} & \cdots & \frac{\partial x_{i_2}}{\partial u_k} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial x_{i_k}}{\partial u_1} & \frac{\partial x_{i_k}}{\partial u_2} & \cdots & \frac{\partial x_{i_k}}{\partial u_k} \end{bmatrix}\right).$$

All in all, everything is the same and the k-form and the parametrization tells you how to write the integral. You will notice that the way we defined the integral in the special cases of 1, 2, and 3 dimensional objects is just a special case of the above formula. If we make the appropriate definitions for boundaries, we again obtain the generalized Stokes theorem, however, let us not get into the details of all of this. At this juncture, it is sufficient to say that the setup generalizes nicely to n dimensions for any n.