Parametrized Surface Integrals

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As Schey only covers integration over surfaces that are graphs, here is a very short writeup on how to do it on a surface parametrized differently, as that is often easier for actual computation. Of course, “easier” depends on whether you pick the right parametrization for your surface.

Suppose a surface S is parametrized by a vector valued function $\mathbf{r}$ of two variables, s and t, where (s,t) lie in some region R in the (s,t)-plane. That is, each (s,t) in R corresponds to a unique point given by $\mathbf{r}(s,t)$ on S and vice-versa. We can write

$$\mathbf{r}(s,t) = r_x(s,t)\mathbf{i} + r_y(s,t)\mathbf{j} + r_z(s,t)\mathbf{k}.$$

For example, the cylinder S given by $x^2 + y^2 = 1$ and $0 \leq z \leq 1$ can be parametrized by

$$\mathbf{r}(s,t) = (\cos s)\mathbf{i} + (\sin s)\mathbf{j} + t\mathbf{k}.$$

where $0 \leq s < 2\pi$ and $0 \leq t \leq 1$, so these inequalities define the region R.

If S is a graph over R, that is, S is given by $z = f(x,y)$ for (x,y) in R, then we could write this as parametrization as

$$\mathbf{r}(s,t) = s\mathbf{i} + t\mathbf{j} + f(s,t)\mathbf{k}.$$

Using this parametrization, everything we did with graphs will work out to be exactly the same if you follow the formulas in this file. So we’re not really defining something new, just extending what we already defined.

To figure out a normal on S, we first find two tangent vectors, which can be gotten by partial derivatives:

$$\frac{\partial \mathbf{r}}{\partial s} = \frac{\partial r_x}{\partial s}\mathbf{i} + \frac{\partial r_y}{\partial s}\mathbf{j} + \frac{\partial r_z}{\partial s}\mathbf{k} \quad \text{and} \quad \frac{\partial \mathbf{r}}{\partial t} = \frac{\partial r_x}{\partial t}\mathbf{i} + \frac{\partial r_y}{\partial t}\mathbf{j} + \frac{\partial r_z}{\partial t}\mathbf{k}.$$

For the cylinder above, these are

$$\frac{\partial \mathbf{r}}{\partial s} = (-\sin s)\mathbf{i} + (\cos s)\mathbf{j} \quad \text{and} \quad \frac{\partial \mathbf{r}}{\partial t} = \mathbf{k}.$$

Then $\frac{\partial \mathbf{r}}{\partial s} \times \frac{\partial \mathbf{r}}{\partial t}$ is normal, so the unit normal is given by

$$\hat{\mathbf{n}} = \frac{\frac{\partial \mathbf{r}}{\partial s} \times \frac{\partial \mathbf{r}}{\partial t}}{\left|\frac{\partial \mathbf{r}}{\partial s} \times \frac{\partial \mathbf{r}}{\partial t}\right|}$$

For the cylinder, the computation yields

$$\hat{\mathbf{n}} = (\cos s)\mathbf{i} + (\sin s)\mathbf{j}.$$

In fact, the product of the two vectors is already unit length, that is,

$$\frac{\partial \mathbf{r}}{\partial s} \times \frac{\partial \mathbf{r}}{\partial t} = (\cos s)\mathbf{i} + (\sin s)\mathbf{j}.$$

And really you didn't need to compute anything, if you think about it, $x\mathbf{i} + y\mathbf{j}$ is clearly the unit normal vector. The only thing to think about is which direction it goes, in or out of the cylinder. Since s is going around the cylinder counterclockwise when looking down the z-axis and t is going up (thinking of the z-axis as the vertical), then it must be that the cross product above must be the outward one.

Let's get back to the general setup. Just as for the graph, the scale you get from going from $dA(s,t) = ds\,dt$ to dS is simply the size of this vector, so the integral is

$$\iint_S G\,dS = \iint_R G(\mathbf{r}(s,t)) \left| \frac{\partial \mathbf{r}}{\partial s} \times \frac{\partial \mathbf{r}}{\partial t} \right| dA(s,t).$$

Similarly for the flux integral, here the size of the normal again cancels out and we find

$$\iint_S \mathbf{F} \cdot \hat{\mathbf{n}}\,dS = \iint_R \mathbf{F}(\mathbf{r}(s,t)) \cdot \left(\frac{\partial \mathbf{r}}{\partial s} \times \frac{\partial \mathbf{r}}{\partial t} \right) dA(s,t).$$

Let's do a short example. Suppose $\mathbf{F} = x\mathbf{i} + zy^2\mathbf{j} + z^2\mathbf{k}$ and let's compute the flux integral over the cylinder S.

$$\begin{aligned}
\iint_S \mathbf{F} \cdot \hat{\mathbf{n}}\,dS &= \int_0^{2\pi} \int_0^1 \big((\cos s)\mathbf{i} + t(\sin^2 s)\mathbf{j} + t^2\mathbf{k}\big) \cdot \big((\cos s)\mathbf{i} + (\sin s)\mathbf{j}\big)\,dt\,ds \\
&= \int_0^{2\pi} \int_0^1 \big((\cos^2 s) + t(\sin^3 s)\big)\,dt\,ds = (\text{calculus goes here}) = \pi.
\end{aligned}$$