Sphere maps and polynomials constant on a hyperplane (or a plane . . . or a line)

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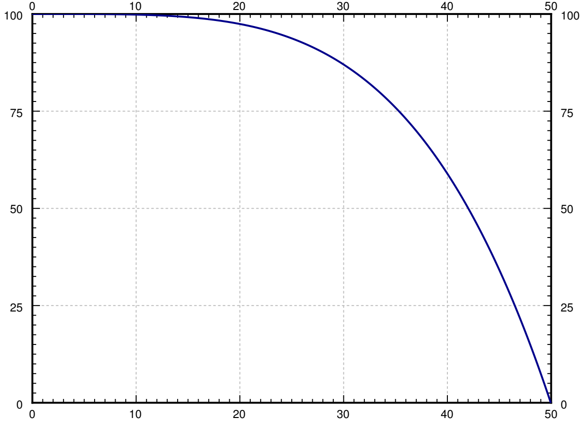
Plan for the talk

The plan is for the following proportion of the audience awake:



Plan for the talk (the content)

- Motivation from complex analysis.
- Degree estimates for polynomials constant on a line (or plane, or hyperplane).
- Proofs.
  - In dimension 2.
  - A 2 player board game arising from the proof in 2 dimensions.
  - In dimension 3.

Sphere maps in $\mathbb{C}^n$

$$z = (z_1, z_2, \ldots, z_n) \in \mathbb{C}^n,$$

$$\mathbb{B}_n = \text{unit ball in } \mathbb{C}^n = \{z : \|z\| < 1\},$$

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Consider rational maps of z, such as this $f \colon \mathbb{C}^2 \to \mathbb{C}^3$ (except where the denominator is zero):

$$f(z_1, z_2) = \left(\frac{z_1}{z_2^2 + 1}, \frac{z_1 z_2}{z_2^2 + 1}, z_2^2 + 1 \right).$$

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Natural question

Classify rational mappings from $\mathbb{C}^n$ to $\mathbb{C}^N$ such that

$$f(\partial\mathbb{B}_n) \subset \partial\mathbb{B}_N$$

A sample of what is known

If $n = N = 1$, then f is a finite Blaschke product. That is,

$$z \mapsto e^{i\theta} \prod_{j=1}^{k} \frac{z - a_j}{1 - \bar{a}_j z}.$$

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Theorem (Alexander / Pincuk '77 (complicated history))

If $n = N$, $n \geq 2$, then f is an automorphism of $\mathbb{B}_n$.

An automorphism of the ball is a linear fractional transformation:

$$F(z) = U \frac{w - Lz}{1 - \langle z, w \rangle}$$

for a fixed $w \in \mathbb{B}_n \subset \mathbb{C}^n$, a unitary map U, and a linear map L.

A sample of what is known II

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Suppose a rational f takes $\partial\mathbb{B}_n$ to $\partial\mathbb{B}_N$. Then degree of f is bounded by a constant $D(n, N)$.

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What's the degree? Write

$$f(z) = \frac{(f_1(z), f_2(z), \dots, f_N(z))}{g(z)}$$

$$\deg f = \max\{\deg f_1, \deg f_2, \dots, \deg f_N, \deg g\}.$$

e.g. $\quad f(z_1, z_2) = \left(\frac{z_1}{z_2^2 + 1}, \frac{z_1 z_2}{z_2^2 + 1}, z_2^2 + 1 \right) = \frac{(z_1, z_1 z_2, (z_2^2 + 1)^2)}{z_2^2 + 1}$

so $\deg f = 4$.

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so $\deg f = 4$.

A conjecture of D'Angelo is that

$$\deg f \leq \begin{cases} 2N-3 & \text{if } n = 2, \\ \frac{N-1}{n-1} & \text{if } n \geq 3. \end{cases}$$

And we get to monomial maps

Theorem (Faran '82)

A rational f takes $\partial\mathbb{B}_2$ to $\partial\mathbb{B}_3$. Then f is equivalent to

- ☛ $(z, w) \mapsto (z, w, 0)$
- ☛ $(z, w) \mapsto (z, zw, w^2)$
- ☛ $(z, w) \mapsto (z^2, \sqrt{2}\, zw, w^2)$
- ☛ $(z, w) \mapsto (z^3, \sqrt{3}\, zw, w^3)$

The equivalence is up to automorphisms of $\mathbb{B}_2$ and $\mathbb{B}_3$.

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The equivalence is up to automorphisms of $\mathbb{B}_2$ and $\mathbb{B}_3$.

So the degree bound conjecture holds when $n = 2$ and $N = 3$.

Furthermore, all of Faran's maps are *monomial* maps
(each component is a single monomial).

More ...

Theorem (Webster,Faran,Huang,etc.)

If $f\colon \partial\mathbb{B}_n \to \partial\mathbb{B}_{2n-2}$ *($n \geq 2$) is a rational map of spheres. Then* f *is equivalent to the linear embedding* $z \mapsto (z, 0)$.

Theorem (Faran,Huang,etc.)

If $f\colon \partial\mathbb{B}_n \to \partial\mathbb{B}_{2n-1}$ *($n \geq 2$) is a rational map of spheres. Then* f *of degree at most 2.*

Theorem (L., '11)

If $f\colon \partial\mathbb{B}_n \to \partial\mathbb{B}_N$ *($n \geq 2$) is a rational degree 2 map of spheres. Then* f *is equivalent via automorphisms to a monomial map.*

Real geometric setup

Let $f\colon \partial\mathbb{B}_n \to \partial\mathbb{B}_N$ be a rational map of spheres. Then

$$\|f(z)\|^2 = |f_1(z)|^2 + \cdots + |f_N(z)|^2 = 1 \quad \text{if} \quad |z_1|^2 + \cdots + |z_n|^2 = 1$$

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Suppose f is monomial: every component f_k is of the form $c\, z_1^{d_1} z_2^{d_2} \cdots z_n^{d_n}$. Then

$$|f_k(z)|^2 = |c|^2 \left(|z_1|^2\right)^{d_1} \left(|z_2|^2\right)^{d_2} \cdots \left(|z_n|^2\right)^{d_n}.$$

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Replace $x_1 = |z_1|^2$, $x_2 = |z_2|^2$, $\ldots$. Then $\|f(z)\|^2$ becomes a real polynomial $p(x_1, \ldots, x_n)$ with nonnegative coefficients such that

$$p(x_1, \ldots, x_n) = 1 \quad \text{if} \quad x_1 + \cdots + x_n = 1.$$

If all monomials in f are distinct, then N is the number of monomials in $p(x_1, \ldots, x_n)$.

Basic problem in 2-dimensions

Let $p(x, y)$ be a polynomial of degree d such that

$$p(x, y) = 1 \qquad \text{whenever} \qquad x + y = 1.$$

Suppose p has exactly N positive and no negative coefficients.

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They also provided polynomials for which $d = 2N - 3$ for every odd d. Thus the inequality is sharp.

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Let $p(x, y, z)$ be a polynomial of degree d such that

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Theorem (L., Peters '11)

$$d \leq \frac{N-1}{2}.$$

And again there exist polynomials where equality holds.

n-dimensions

Let $p(x_1, \ldots, x_n)$, $n \geq 4$, be a polynomial of degree d such that

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Suppose p has exactly N positive and no negative coefficients.

Theorem (L., Peters, '12)

$$d \leq \frac{N-1}{n-1}.$$

And again there exist polynomials where equality holds. In this case we can classify such polynomials.

One dimension?

Example: $x^d = 1$ whenever $x = 1$.

$N = 1$ and d is arbitrary.

Constructing polynomials

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Unfortunately (or fortunately?) this construction doesn't get everything.

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A very basic result from algebraic geometry is: if $p(x, y) = 1$ when $x + y = 1$, then there exists a polynomial $q(x, y)$ such that

$$p(x, y) - 1 = q(x, y)(x + y - 1).$$

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That -1 in the $(x + y - 1)$ causes makes the thing kind of nonsymmetric so ...

Let's change the setup

Let us simplify the setup a bit to get rid of those pesky -1's.
We will work with homogeneous polynomials in 3 variables.
That is $P(X_0, X_1, X_2)$ is homogeneous if

$$P(tX_0, tX_1, tX_2) = t^d P(X_0, X_1, X_2).$$

I.e., every monomial of P is of degree d. For example,
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Let us start with

$$p(x, y) - 1 = q(x, y)(x + y - 1).$$

Homogenize with t: multiply each monomial by t as many times as necessary to make it degree d. In formulas:

$$t^d p(x/t, y/t) - t^d = t^{d-1} q(x/t, y/t)(x + y - t).$$

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Now replace x with X_1, y with X_2 and t with $-X_0$.

Example

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$$X_1^3 - 3X_1X_2X_0 + X_2^3 + X_0^3 = \\ ({X_2}^2 - X_1X_2 - X_2X_0 + {X_1}^2 - X_1X_0 + X_0^2)(X_1 + X_2 + X_0)$$

The new setup

We will drop the requirement of positive coefficients. Suppose

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Instead of positive coefficients, let us require that P is *indecomposable* in the following sense. P cannot be written as $P = P_1 + P_2$ where P_1 and P_2 are also divisible by $(X_0 + X_1 + X_2)$ and where P_1 and P_2 are nonzero and have distinct monomials.

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Suppose that monomials of P have no common divisor.

If P is of degree d and has N nonzero coefficients and is indecomposable, then we will show that

$$d \leq 2N - 5.$$

If P is decomposable, then there is no degree bound. For example:

$$\begin{aligned} P(X_0, X_1, X_2) &= X_0^k(X_0 + X_1 + X_2) + X_1^k(X_0 + X_1 + X_2) \\ &= X_0^{k+1} + X_0^k X_1 + X_0^k X_2 + X_1^k X_0 + X_1^{k+1} + X_1^k X_2 \end{aligned}$$

The Newton diagram

Take the Q. For example:

$$X_0^3 + X_1^3 + X_2^3 - 3X_1X_2X_0 =$$
$$= \underbrace{(X_0^2 + {X_1}^2 + {X_2}^2 - X_1X_0 - X_1X_2 - X_2X_0)}_{Q}(X_0 + X_1 + X_2)$$

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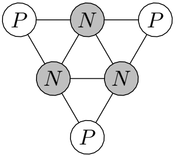
\end{aligned}
$$

Write the table marking the signs of the coefficients of Q for each monomial. For example:

X_1^2	P		
X_1	N	N	
1	P	N	P
	1	X_2	X_2^2

The Newton diagram - nodes

Let us draw the table as a picture and rotate.



Let (a, b, c) be integers. If in Q the monomials

$$X_0^{a-1}X_1^bX_2^c, \quad X_0^aX_1^{b-1}X_2^c, \quad X_0^aX_1^bX_2^{c-1}$$

all have the same sign, then the monomial

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must appear in P with a nonzero coefficient of the same sign.

We call such (a, b, c) a *node*.

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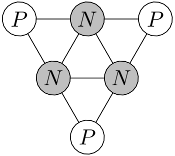

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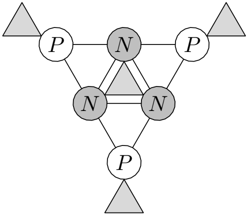
We call such (a, b, c) a *node*.

Note: Not every term in P comes from a node!

The Newton diagram - nodes

Mark nodes with a triangle (each vertex points to one of the three monomials)

Allow some of the three monomials of a node to have zero coefficient (that is, they don't appear in Q)



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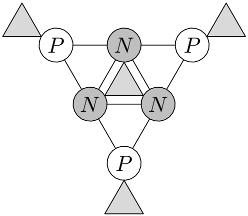
The number of nodes in the diagram, gives a lower bound on the number of nonzero coefficients of P.

The "size" (length of the side) of the diagram is the degree of P.



Filling a diagram

The diagram may not be a triangle (there might have been zeros). It is "connected" if P is indecomposable. We can fill it to a triangle without increasing the number of nodes.

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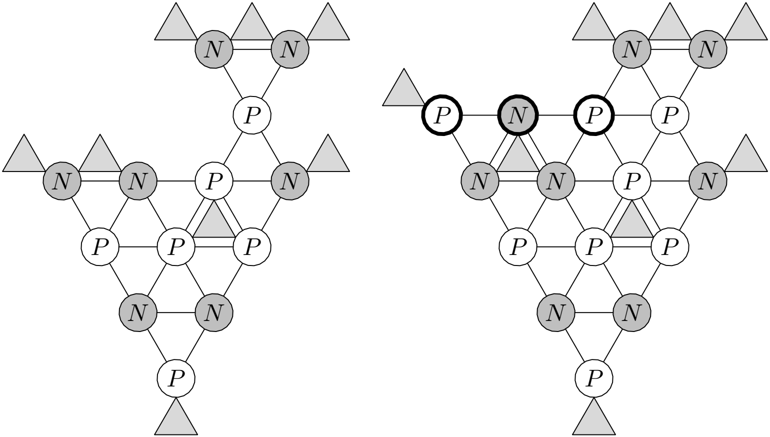
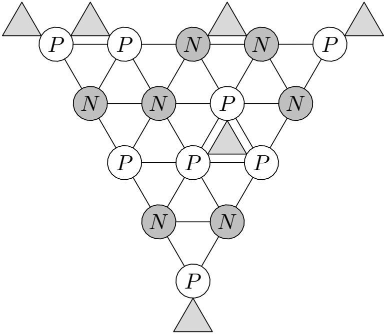


Counting proceeds by induction. Start in the top row and count the nodes above this row.

Now count how many times does the sign change on one row (say c_1), and how many times does the sign change on the row below (say c_2).

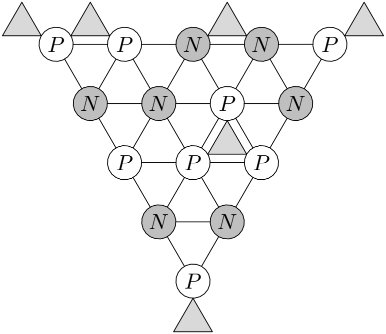
You get at least $\max\left\{\frac{c_1 - c_2}{2}, 0\right\}$ nodes between these rows.



The game

Players put down dark and light stones on a triangle in turn, then count the resulting nodes of their color.

To make the game more fair, we count nodes on a corner for $1/3$ of a point and nodes on the side for $1/2$ a point.



Here, white wins with $1/3 + 1/3 + 1/3 + 1/2 + 1 = 2.5$ points to black with $1/2$ point (black played terribly).

The game

We can prove that with that weighting with a triangle of size d, there are at least $1 + \frac{d-1}{2}$ points to be distributed.

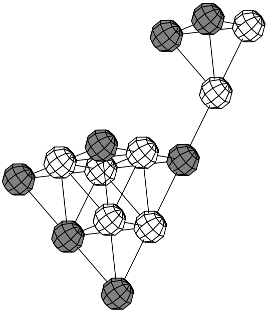
As long as one player doesn't play all three corners, there has to be a winner. In games for $d = 1$, $d = 2$, and $d = 3$ the first player can always win. In larger games? I don't know ...

We will see this weighting make an appearance in the $n = 3$ proof.

Newton diagram in dimension $n = 3$

$$P(X_0, X_1, X_2, X_3) =$$
$$Q(X_0, X_1, X_2, X_3)\,(X_0 + X_1 + X_2 + X_3)$$

Look at the Newton diagram of the quotient Q. Put a black ball for negative coefficient and white ball for positive coefficient.

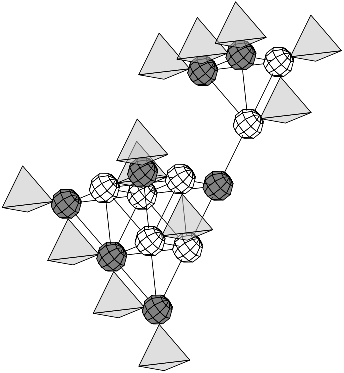


Nodes

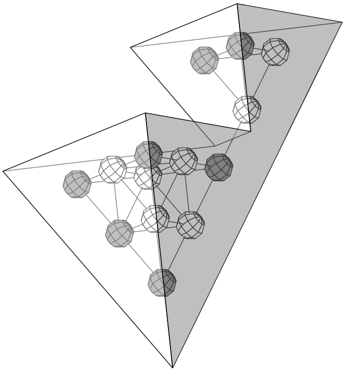
As before, (a, b, c, d) is a node if
$X_0^{a-1} X_1^b X_2^c X_3^d$,
$X_0^a X_1^{b-1} X_2^c X_3^d$,
$X_0^a X_1^b X_2^{c-1} X_3^d$,
$X_0^a X_1^b X_2^c X_3^{d-1}$

all have the same sign,

then
$X_0^a X_1^b X_2^c X_3^d$
must appear in P.

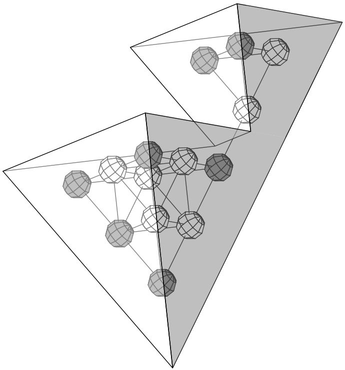


It turns out we only need to count nodes on the faces.



It turns out we only need to count nodes on the faces.

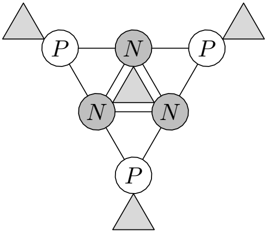
Furthermore count each face independently, counting edge nodes for $1/2$ and corner nodes for $1/3$.



Face counting

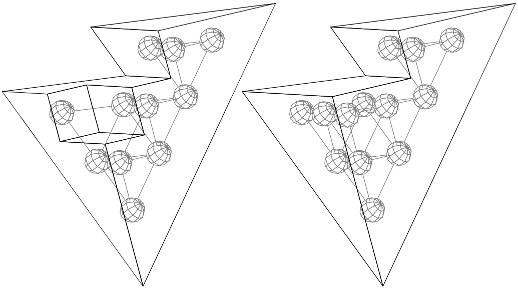
A general version of the two-dimensional argument deals with the fractions.

The example diagram has
$1 + {}^{1}/{}_{3} + {}^{1}/{}_{3} + {}^{1}/{}_{3} = 2$ weighted nodes.



Fill

First we “fill” holes
in the diagram.



And cut

After filling we cut.

Cutting reduces the degree.

Induction does the rest.