Singular Levi-flat hypersurfaces (5)

Jiří Lebl

Departemento pri Matematiko de Oklahoma Ŝtata Universitato

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A real (resp. complex) subvariety X of an open set $U \subset \mathbb{R}^n$ (resp. $\mathbb{C}^n$) is a set locally given by vanishing of a set of real-analytic (resp. holomorphic) functions

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Question: Did anybody get the “one defining function” exercise?

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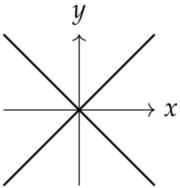

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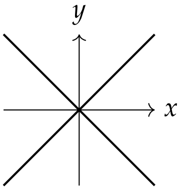

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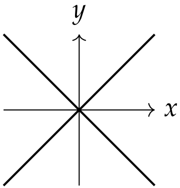

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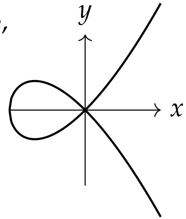

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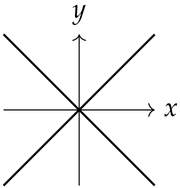

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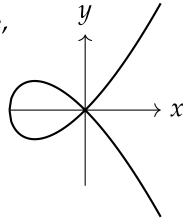

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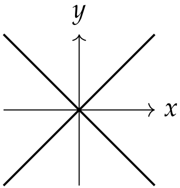

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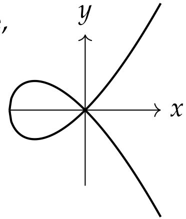

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Remark: For complex subvarieties, X_{reg} being connected is equivalent to being irreducible. Not so for real subvarieties (example above).

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Exercise: Prove that $x^2 - y^3$ is a defining function for the cusp at every point.

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The issues surrounding complexification are extremely subtle. Mainly, the complexification at one point may not be used at another point (an example coming up).

Example: (Whitney umbrella)

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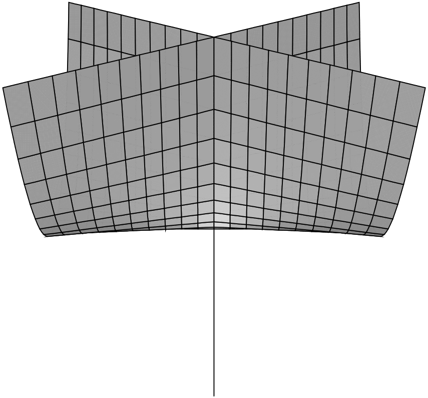

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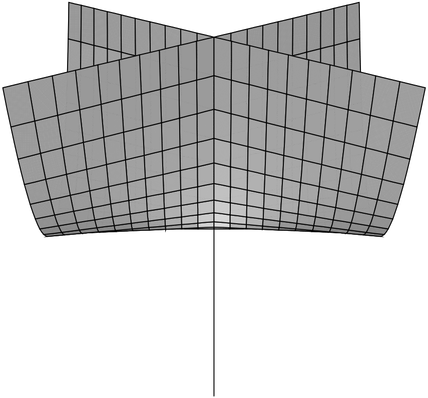

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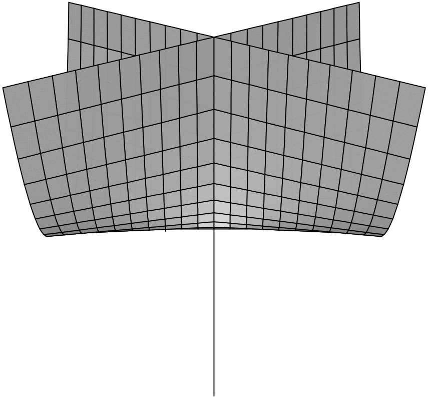

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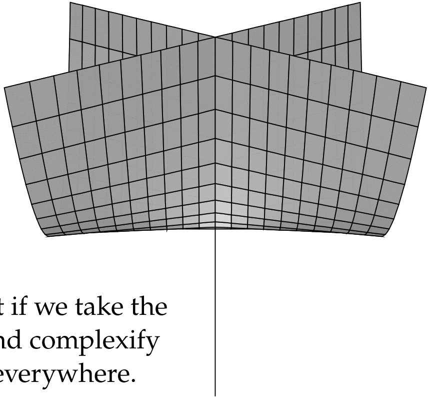

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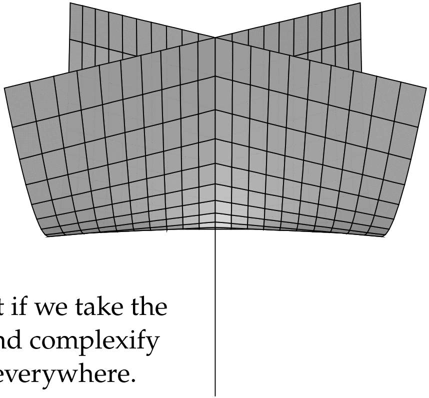

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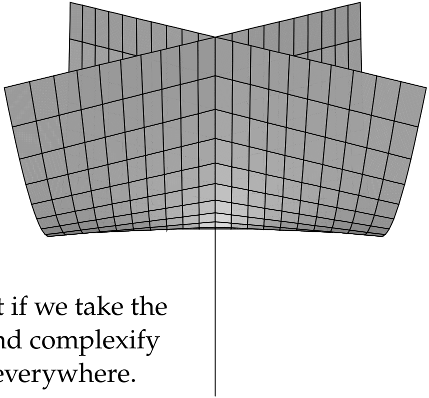

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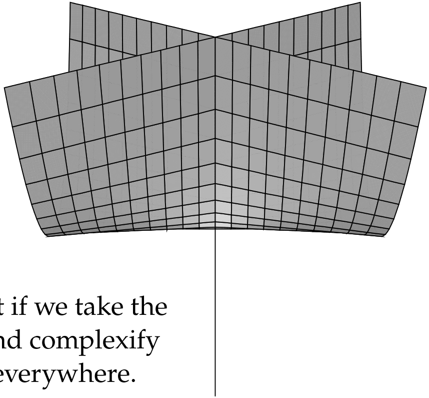

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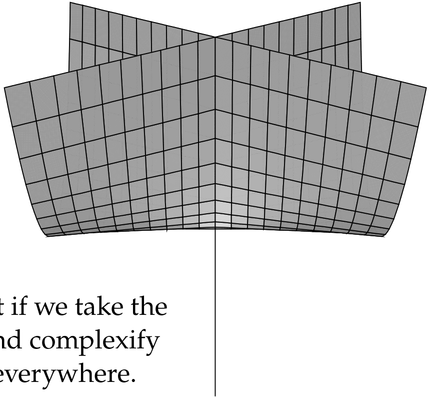

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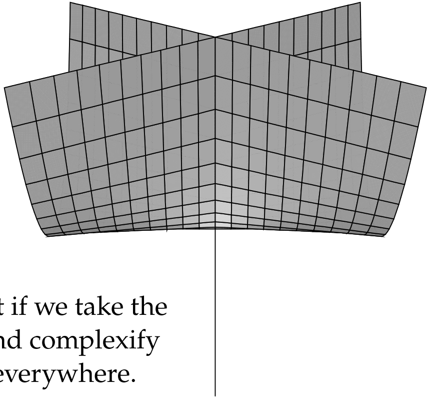

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Complex subvarieties do not have such issues and are coherent: Near every point we can find a set of defining functions that also work at all nearby points. Depending on U, perhaps even one global set of defining functions.

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Let $X \subset U \subset \mathbb{C}^n$ be a real-analytic subvariety and $p \in X$.

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Suppose U is small enough so that the defining functions $\rho(z, \bar{z})$ have a convergent series on $U \times U^*$ (that is as $\rho(z, \xi)$ for $(z, \xi) \in U \times U^*$).

Define the Segre variety (depends on U (and a priori on ρ)) as

$$\Sigma_q(X, U) = \{z \in U : \rho(z, \bar{q}) = 0\}.$$

Alternatively, let $\mathcal{X} \subset U \times U^*$ be the smallest complex subvariety containing the image $\iota(X)$ in the diagonal where $\iota(z) = (z, \bar{z})$. (For a small enough U, $\mathcal{X} = \{(z, \xi) \in U \times U^* : \rho(z, \xi) = 0\}$.)

Then

$$\Sigma_q(X, U) = \left\{z \in U : (z, \bar{q}) \in \mathcal{X}\right\}.$$

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Put another way: $T_p^{(1,0)}X \oplus T_p^{(0,1)}X = \mathbb{C} \otimes T_p\Sigma_p(X, U)$.

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Example: $X \subset \mathbb{C}^2$ given by $|z|^2 - |w|^2 = 0$ (a complex cone) is Segre degenerate at the origin: The defining function can be written as $z\bar{z} - w\bar{w} = 0$.

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Exercise: Prove that X is Levi-flat at regular points (outside the origin).