Singular Levi-flat hypersurfaces (4)

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Small review:

Theorem of Cartan says that every smooth (nonsingular) real-analytic Levi-flat hypersurface can be *locally* realized as

$$\operatorname{Im} w = 0$$

and the Levi foliation is given by $\{w = t\}$ for $t \in \mathbb{R}$.

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$$z^\alpha = z_1^{\alpha_1} z_2^{\alpha_2} \cdots z_n^{\alpha_n}$$

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Lemma

Let $V \subset \mathbb{C}^n \times \mathbb{C}^n$ *be a domain, let the coordinates be* $(z, \zeta) \in \mathbb{C}^n \times \mathbb{C}^n$*, let*

$$D = \left\{(z, \zeta) \in \mathbb{C}^n \times \mathbb{C}^n : \zeta = \bar{z}\right\},$$

and suppose $D \cap V \neq \emptyset$*. Suppose* $f, g \colon V \to \mathbb{C}$ *are holomorphic functions such that* $f = g$ *on* $D \cap V$*. Then* $f = g$ *on all of* V*.*

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Proof: WLOG, $g = 0$.
$f(z, \bar{z}) = 0$, so applying Wirtinger operators yields zero:

$$0 = \frac{\partial}{\partial \bar{z}_k}\left[f(z, \bar{z}) \right] = \frac{\partial f}{\partial \zeta_k}(z, \bar{z}) \quad \text{and} \quad 0 = \frac{\partial}{\partial z_k}\left[f(z, \bar{z}) \right] = \frac{\partial f}{\partial z_k}(z, \bar{z}).$$

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For all α and β,

$$0 = \frac{\partial^{|\alpha|+|\beta|}}{\partial z^\alpha \partial \bar{z}^\beta} \left[f(z, \bar{z}) \right] = \frac{\partial^{|\alpha|+|\beta|} f}{\partial z^\alpha \partial \zeta^\beta}(z, \bar{z}).$$

So f has a zero power series and is zero by the identity theorem. □

Given a convergent power series

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As long as we are in the domain of convergence, we can treat f as F and treat z and $\bar{z}$ as independent variables.

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The set $\Sigma_p = \Sigma_p(M, U) = \{z \in U : r(z, \bar{p}) = 0\}$ is called the *Segre variety*.

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Exercise: If $dr \neq 0$ as above, show that $z \mapsto r(z, \bar{p})$ is not identically zero.

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In other words, for $p \in M$, the Segre variety Σ_p is precisely the leaf of the Levi-foliation through p.

Let $U \subset \mathbb{R}^k$ (respectively $U \subset \mathbb{C}^k$) be an open set. The set $X \subset U$ is *a real-analytic subvariety* (resp. a *complex-analytic subvariety*) of U if for each point $p \in U$, there exists a neighborhood $V \subset U$ of p and a set of real-analytic (resp. holomorphic) functions $\mathcal{P}(V)$ such that

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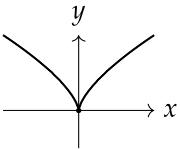
$x^2 - y^3 = 0$ gives



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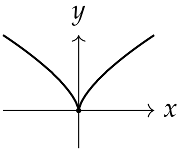

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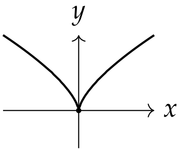

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Example: $\{(0, 0)\}$ is a subvariety of the cusp (defining functions x, y).

Write $X_{reg} \subset X$ be the set of points which are *regular*, that is,

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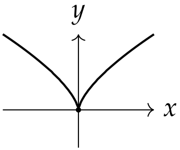
Example: If X is the cusp $x^2 - y^3 = 0$,
$X_{sing} = \{(0, 0)\}$ and
$\dim(X, p) = 1$ for all $p \in X$.



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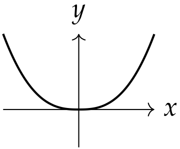

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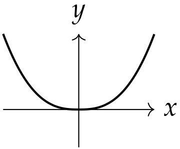

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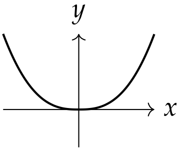

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5) If X is real-analytic, then X_{sing} is "semi-analytic" (defined by equalities and inequalities), not necessarily a subvariety (not the zero set of derivatives).