Singular Levi-flat hypersurfaces (2)

Jiří Lebl

Departemento pri Matematiko de Oklahoma Ŝtata Universitato

Small review:

M given by $\{r = 0\}$.

The full Hessian is

$$H_p = \begin{bmatrix} \frac{\partial^2 r}{\partial \bar{z}_1 \partial z_1}\Big|_p & \cdots & \frac{\partial^2 r}{\partial \bar{z}_1 \partial z_n}\Big|_p & \frac{\partial^2 r}{\partial \bar{z}_1 \partial \bar{z}_1}\Big|_p & \cdots & \frac{\partial^2 r}{\partial \bar{z}_1 \partial \bar{z}_n}\Big|_p \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 r}{\partial \bar{z}_n \partial z_1}\Big|_p & \cdots & \frac{\partial^2 r}{\partial \bar{z}_n \partial z_n}\Big|_p & \frac{\partial^2 r}{\partial \bar{z}_n \partial \bar{z}_1}\Big|_p & \cdots & \frac{\partial^2 r}{\partial \bar{z}_n \partial \bar{z}_n}\Big|_p \\ \frac{\partial^2 r}{\partial z_1 \partial z_1}\Big|_p & \cdots & \frac{\partial^2 r}{\partial z_1 \partial z_n}\Big|_p & \frac{\partial^2 r}{\partial z_1 \partial \bar{z}_1}\Big|_p & \cdots & \frac{\partial^2 r}{\partial z_1 \partial \bar{z}_n}\Big|_p \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 r}{\partial z_n \partial z_1}\Big|_p & \cdots & \frac{\partial^2 r}{\partial z_n \partial z_n}\Big|_p & \frac{\partial^2 r}{\partial z_n \partial \bar{z}_1}\Big|_p & \cdots & \frac{\partial^2 r}{\partial z_n \partial \bar{z}_n}\Big|_p \end{bmatrix} = \begin{bmatrix} L_p & \overline{Z_p} \\ Z_p & L_p^t \end{bmatrix}$$

L_p is the complex Hessian.

$X_p^* L_p X_p$ for $X_p \in T_p^{(1,0)} M$ is the Levi form.

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$\Rightarrow$ L_p changes by $*$-congruence:

$\Rightarrow$ inertia of the Levi-form is a biholomorphic invariant!

If for all $X_p \in T_p^{(1,0)}M$, $X_p = \sum_{k=1}^{n} a_k \frac{\partial}{\partial z_k}\Big|_p$,

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Pseudoconvex domains are the natural domains of definition for holomorphic functions.

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So $M = \mathbb{C} \times \mathbb{R} = \{\operatorname{Im} w = 0\}$ is pseudoconvex from both sides
(we call that Levi-flat).

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Exercise: Work out that this definition gives a form that has the same inertia as the previous definition.

Write an M as before as ($T\colon \mathbb{C}^{n-1} \to \mathbb{C}$ is linear)

$$\operatorname{Im} w = \varphi(z, \bar{z}, \operatorname{Re} w) = q(z, \bar{z}) + (\operatorname{Re} w)(Tz + \overline{Tz}) + a(\operatorname{Re} w)^2 + O(3),$$

Change variables in z to make $Tz = \epsilon z_1$ where $\epsilon = 0$ or 1.

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Diagonalizing A and rescaling

$$\operatorname{Im} w = \lambda_1 |z_1|^2 + \cdots + \lambda_{n-1} |z_{n-1}|^2 + O(3) \quad \text{where } \lambda_k = 0 \text{ or } \pm 1.$$

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M and N are *CR diffeomorphic* if there is a diffeomorphism $f: M \to N$ such that f and f^{-1} are CR.