Singular Levi-flat hypersurfaces (2)

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Small review:

$M$ given by $\{r = 0\}$.

The full Hessian is

$$H_p = \begin{bmatrix}
\frac{\partial^2 r}{\partial z_1 \partial z_1} & \cdots & \frac{\partial^2 r}{\partial z_1 \partial z_n} & \frac{\partial^2 r}{\partial \bar{z}_1 \partial z_1} & \cdots & \frac{\partial^2 r}{\partial \bar{z}_1 \partial z_n} \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
\frac{\partial^2 r}{\partial z_n \partial z_1} & \cdots & \frac{\partial^2 r}{\partial z_n \partial z_n} & \frac{\partial^2 r}{\partial \bar{z}_n \partial z_1} & \cdots & \frac{\partial^2 r}{\partial \bar{z}_n \partial z_n} \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
\frac{\partial^2 r}{\partial z_n \partial \bar{z}_1} & \cdots & \frac{\partial^2 r}{\partial z_n \partial \bar{z}_n} & \frac{\partial^2 r}{\partial \bar{z}_n \partial \bar{z}_1} & \cdots & \frac{\partial^2 r}{\partial \bar{z}_n \partial \bar{z}_n}
\end{bmatrix} = \begin{bmatrix}
L_p & \bar{Z}_p \\
Z_p & L_p^t
\end{bmatrix}$$

$L_p$ is the complex Hessian.

$X_p^* L_p X_p$ for $X_p \in T_p^{(1,0)} M$ is the Levi form.
We are mostly interested in biholomorphic invariants. So what happens under a biholomorphism?
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Suppose $f : V \to V'$ is biholomorphic, $r : V' \to \mathbb{R}$ a defining function for $M \subset V'$ and $r \circ f$ the defining function for $f^{-1}(M) \subset V$. 

\[
\frac{\partial^2}{\partial z_k \partial z_\ell} (r \circ f) = \frac{\partial}{\partial z_k} \left( \sum_{m,n} \frac{\partial^2 r}{\partial \zeta_m \partial \bar{\zeta}_n} (f(z), \bar{f}(\bar{z})) \frac{\partial f_m}{\partial z_\ell} z \right) + \frac{\partial^2}{\partial \bar{\zeta}_m \partial \zeta_\nu} \left( \sum_{n} \frac{\partial^2 r}{\partial \zeta_n \partial \bar{\zeta}_\nu} (f(z), \bar{f}(\bar{z})) \frac{\partial f_n}{\partial z_k} \right).
\]
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First the \( Z \) matrix:

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\frac{\partial^2 (r \circ f)}{\partial z_k \partial z_\ell}
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\frac{\partial^2 (r \circ f)}{\partial z_k \partial z_\ell} = \sum_{m=1}^{n} \left( \frac{\partial r}{\partial \zeta_m} \bigg|_{(f(z), \bar{f}(\bar{z}))} \frac{\partial f_m}{\partial z_\ell} \bigg|_{z} + \frac{\partial r}{\partial \zeta_m} \bigg|_{(f(z), \bar{f}(\bar{z}))} \frac{\partial \bar{f}_m}{\partial z_\ell} \bigg|_{\bar{z}} \right)
$$
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\]

\[
= \sum_{m, \nu=1}^{n} \left( \frac{\partial^2 r}{\partial \zeta_\nu \partial \zeta_m} \Big|_{(f(z), \bar{f}(\bar{z}))} \frac{\partial f_\nu}{\partial z_k} \Big|_z \frac{\partial f_m}{\partial z_\ell} \Big|_z + \frac{\partial^2 r}{\partial \bar{\zeta}_\nu \partial \zeta_m} \Big|_{(f(z), \bar{f}(\bar{z}))} \frac{\partial \bar{f}_\nu}{\partial z_k} \Big|_{\bar{z}} \frac{\partial \bar{f}_m}{\partial z_\ell} \Big|_{\bar{z}} \right)
\]

\[
+ \sum_{m=1}^{n} \frac{\partial r}{\partial \zeta_m} \Big|_{(f(z), \bar{f}(\bar{z}))} \frac{\partial^2 f_m}{\partial z_k \partial z_\ell} \Big|_z
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$$

$$
= \sum_{m,v=1}^{n} \left( \frac{\partial^2 r}{\partial \zeta_v \partial \zeta_m} \bigg|_{(f(z), \bar{f}(\bar{z}))} \frac{\partial f_v}{\partial z_k} \bigg|_{z} \frac{\partial f_m}{\partial z_\ell} \bigg|_{z} + \frac{\partial^2 r}{\partial \bar{\zeta}_v \partial \zeta_m} \bigg|_{(f(z), \bar{f}(\bar{z}))} \frac{\partial \bar{f}_v}{\partial z_k} \bigg|_{\bar{z}} \frac{\partial \bar{f}_m}{\partial z_\ell} \bigg|_{\bar{z}} \right) + \sum_{m=1}^{n} \frac{\partial r}{\partial \zeta_m} \bigg|_{(f(z), \bar{f}(\bar{z}))} \frac{\partial^2 f_m}{\partial z_k \partial z_\ell} \bigg|_{z}
$$

$$
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$$

does not transform nicely
Now the $L_p$ matrix:

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$$\Rightarrow \quad L_p \text{ changes by } \ast\text{-congruence:}$$
Now the $L_p$ matrix:

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$\Rightarrow$ $L_p$ changes by $\ast$-congruence:

$\Rightarrow$ inertia of the Levi-form is a biholomorphic invariant!
If for all $X_p \in T_{p}^{(1,0)} M$, $X_p = \sum_{k=1}^{n} a_k \frac{\partial}{\partial z_k} |_p$,

$$X_p^* L_p X_p = \sum_{k=1,\ell=1}^{n} \bar{a}_k a_\ell \frac{\partial^2 r}{\partial \bar{z}_k \partial z_\ell} |_p \geq 0,$$

then $M$ is said to be pseudoconvex at $p$. 

Note the similarity of the definition to classical convexity. Really, it is one side of the hypersurface that is pseudoconvex.

If $U \subset \mathbb{C}^n$ is a domain with smooth boundary, $U = \{r < 0\}$, and $dr \neq 0$ near $\partial U$, then $U$ is pseudoconvex if $\partial U = \{r = 0\}$ is pseudoconvex.

Pseudoconvex domains are the natural domains of definition for holomorphic functions.
If for all $X_p \in T_p^{(1,0)} M$, $X_p = \sum_{k=1}^{n} a_k \frac{\partial}{\partial z_k} |_{p}$,

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strictly or strongly pseudoconvex if $X_p^* L_p X_p > 0$ for $X_p \neq 0$. 

Note the similarity of the definition to classical convexity. Really, it is one side of the hypersurface that is pseudoconvex. If $U \subset \mathbb{C}^n$ is a domain with smooth boundary, $U = \{ r < 0 \}$, and $dr \neq 0$ near $\partial U$, then $U$ is pseudoconvex if $\partial U = \{ r = 0 \}$ is pseudoconvex. Pseudoconvex domains are the natural domains of definition for holomorphic functions.
If for all \( X_p \in T_p^{(1,0)} M \), \( X_p = \sum_{k=1}^{n} a_k \frac{\partial}{\partial z_k} \big|_p \),

\[
X_p^* L_p X_p = \sum_{k=1, \ell=1}^{n} \bar{a}_k a_\ell \frac{\partial^2 r}{\partial \bar{z}_k \partial z_\ell} \big|_p \geq 0,
\]

then \( M \) is said to be \textit{pseudoconvex} at \( p \).

\textit{Strictly} or \textit{strongly pseucodonvex} if \( X_p^* L_p X_p > 0 \) for \( X_p \neq 0 \).

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If for all $X_p \in T_p^{(1,0)} M$, $X_p = \sum_{k=1}^{n} a_k \frac{\partial}{\partial z_k} \bigg|_p$, 

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If $U \subset \mathbb{C}^n$ is a domain with smooth boundary, $U = \{r < 0\}$, and $dr \neq 0$ near $\partial U$, then $U$ is pseudoconvex if $\partial U = \{r = 0\}$ is pseudoconvex.
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Pseudoconvex domains are the natural domains of definition for holomorphic functions.
Example 1: $U = \mathbb{B}_2 \subset \mathbb{C}^2$, then $r = |z_1|^2 + |z_2|^2 - 1$.
At $p = (1, 0)$, $X_p \in T_p^{(1,0)} \partial U$ means $X_p = a_2 \frac{\partial}{\partial z_2} |_p$ (so $a_1 = 0$)
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The Levi-form (at \( p \)) is
\[
\sum_{k=1, \ell=1}^{2} \bar{a}_k a_\ell \frac{\partial^2 r}{\partial \bar{z}_k \partial z_\ell} \bigg|_p = \bar{a}_1 a_1 + \bar{a}_2 a_2 = \bar{a}_2 a_2 = |a_2|^2 \geq 0
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So $\mathbb{B}_2$ is pseudoconvex (strongly in fact) at $p$
(and similarly at all $p \in \partial \mathbb{B}_2$).
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**Example 2:** In \((z, w) \in \mathbb{C}^2\), the set the domain \( H_+ = \{ \text{Im} w > 0 \} \) is pseudoconvex at all \( p \in M = \partial H_+ = \mathbb{C} \times \mathbb{R} \).
**Example 1:** $U = \mathbb{B}_2 \subset \mathbb{C}^2$, then $r = |z_1|^2 + |z_2|^2 - 1$.

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So $\mathbb{B}_2$ is pseudoconvex (strongly in fact) at $p$ (and similarly at all $p \in \partial \mathbb{B}_2$).

**Example 2:** In $(z, w) \in \mathbb{C}^2$, the set the domain $H_+ = \{\text{Im } w > 0\}$ is pseudoconvex at all $p \in M = \partial H_+ = \mathbb{C} \times \mathbb{R}$.

But so is the domain $H_- = \{\text{Im } w < 0\}$.
Example 1: $U = \mathbb{B}_2 \subset \mathbb{C}^2$, then $r = |z_1|^2 + |z_2|^2 - 1$. At $p = (1, 0)$, $X_p \in T_p^{(1,0)} \partial U$ means $X_p = a_2 \frac{\partial}{\partial z_2} |_p$ (so $a_1 = 0$).

The Levi-form (at $p$) is

$$\sum_{k=1, \ell=1}^2 \bar{a}_k a_\ell \frac{\partial^2 r}{\partial \bar{z}_k \partial z_\ell} |_p = \bar{a}_1 a_1 + \bar{a}_2 a_2 = \bar{a}_2 a_2 = |a_2|^2 \geq 0$$

So $\mathbb{B}_2$ is pseudoconvex (strongly in fact) at $p$ (and similarly at all $p \in \partial \mathbb{B}_2$).

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But so is the domain $H_- = \{ \text{Im } w < 0 \}$.

So $M = \mathbb{C} \times \mathbb{R} = \{ \text{Im } w = 0 \}$ is pseudoconvex from both sides (we call that Levi-flat).
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Let \( \pi_p : \mathbb{C} \otimes T_p M \to \mathbb{C} \otimes T_p M / T_p (1,0) M \oplus T_p (0,1) M \cong B_p \)
be the natural projection.

Exercise: Work out that this definition gives a form that has the same
inertia as the previous definition.
The Levi-form is intrinsic.

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be the natural projection.

Extend a vector $X_p \in T_p^{(1,0)} M$ to a vector field $X$ in $T^{(1,0)} M$. 

Then define the intrinsic Levi-form as

$L(X_p, X_p) = \pi_p [X, X]|_p$
The Levi-form is intrinsic.

Let $\pi_p : \mathbb{C} \otimes T_p M \to \mathbb{C} \otimes T_p M \rightarrow \mathbb{C} \otimes T_p M / T_p^{(1,0)} M \oplus T_p^{(0,1)} M \cong B_p$ be the natural projection.

Extend a vector $X_p \in T_p^{(1,0)} M$ to a vector field $X$ in $T^{(1,0)} M$.

Then define the intrinsic Levi-form as

$$ \mathcal{L}(X_p, \bar{X}_p) = \pi_p([X, \bar{X}]|_p) $$
The Levi-form is intrinsic.

Let \( \pi_p : \mathbb{C} \otimes T_p M \to \mathbb{C} \otimes T_p M / T_p^{(1,0)} M \oplus T_p^{(0,1)} M \cong B_p \) be the natural projection.

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**Exercise:** Work out that this definition gives a form that has the same inertia as the previous definition.
Write an $M$ as before as $(T: \mathbb{C}^{n-1} \to \mathbb{C}$ is linear)

$$\text{Im } w = \varphi(z, \bar{z}, \text{Re } w) = q(z, \bar{z}) + (\text{Re } w)(Tz + \bar{Tz}) + a(\text{Re } w)^2 + O(3),$$

Change variables in $z$ to make $Tz = \epsilon z_1$ where $\epsilon = 0$ or $1$. 
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Change variables changing $w$ to $w + iaw^2 + 2i\epsilon wz_1$ to get

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Solve for $\text{Im } w$ (which is $O(2)$) by IVT to get

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Diagonalizing $A$ and rescaling

$$\text{Im } w = \lambda_1|z_1|^2 + \cdots + \lambda_{n-1}|z_{n-1}|^2 + O(3) \quad \text{where } \lambda_k = 0 \text{ or } \pm 1.$$
A smooth function $f : M \to \mathbb{C}$ is a CR function if

$$\nabla f = 0$$

for all vector fields $v \in \Gamma(T^{(0,1)}M)$. 

There are other smooth CR functions. Example: Suppose $M = \{ \text{Im} \, w = 0 \}$, and $f : M \to \mathbb{C}$ is $e^{-\left(\frac{1}{\text{Re} \, w}\right)^2}$ if $\text{Re} \, w \neq 0$ and $0$ if $\text{Re} \, w = 0$. Then $f$ is CR, $C^\infty$, but $f$ is not real-analytic, so not a restriction of a holomorphic function.

We will see that for real-analytic $M$ and $f$, CR functions are restrictions of holomorphic functions.

For two CR submanifolds $M$ and $N$, $f : M \to N$ is a CR mapping if each component of $f$ is a CR function. $M$ and $N$ are CR diffeomorphic if there is a diffeomorphism $f : M \to N$ such that $f$ and $f^{-1}$ are CR.
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**Example:** If $f \in \mathcal{O}(\mathbb{C}^n)$, then $vf = 0$ for all $v \in \Gamma(T^{(0,1)}\mathbb{C}^n)$. So $f|_M$ is CR.
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