Singular Levi-flat hypersurfaces (1)

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$z = (z_1, z_2, \ldots, z_n) \in \mathbb{C}^n$ and $\mathbb{C}^n \cong \mathbb{R}^n \times \mathbb{R}^n = \mathbb{R}^{2n}$ via

$$z = x + iy, \quad \bar{z} = x - iy, \quad x, y \in \mathbb{R}^n, \quad i = \sqrt{-1}.$$
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Define the Wirtinger operators

\[ \frac{\partial}{\partial z_\ell} \text{ def } \frac{1}{2} \left( \frac{\partial}{\partial x_\ell} - i \frac{\partial}{\partial y_\ell} \right), \quad \frac{\partial}{\partial \bar{z}_\ell} \text{ def } \frac{1}{2} \left( \frac{\partial}{\partial x_\ell} + i \frac{\partial}{\partial y_\ell} \right). \]
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These are determined by being the dual bases of \( dz \) and \( d\bar{z} \)

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dz_k \left( \frac{\partial}{\partial z_\ell} \right) = \delta^k_\ell, \quad \dz_k \left( \frac{\partial}{\partial \bar{z}_\ell} \right) = 0, \quad \dz_k \left( \frac{\partial}{\partial z_\ell} \right) = 0, \quad \dz_k \left( \frac{\partial}{\partial \bar{z}_\ell} \right) = \delta^k_\ell.
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$f : U \subset \mathbb{C}^n \to \mathbb{C}$ is holomorphic if $f$ satisfies

$$\frac{\partial f}{\partial \bar{z}_\ell} = 0 \quad \text{for } \ell = 1, 2, \ldots, n \quad \text{(the Cauchy–Riemann (CR) equations)}. $$
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Write $\mathcal{O}(U)$ for set of holomorphic functions on $U$. 
We write a smooth ($C^\infty$) function $f : U \subset \mathbb{C}^n \rightarrow \mathbb{C}$ as $f(z, \bar{z})$. 

If $f$ is a polynomial (in $x$ and $y$), write $x = z + \bar{z}$, $y = z - \bar{z}$, and it really does become a polynomial in $z$ and $\bar{z}$.

E.g., $2x^1 + 2y^1 + 4y^2 = (1 - i)z^1 + (1 + i)\bar{z}^1 - z^2 + 2z^2\bar{z}^2 - \bar{z}^2$.

$f$ is holomorphic if it does not depend on $\bar{z}$.

If $f$ is real-analytic (has a power series in $x$ and $y$), then $f$ has a power series in $z$ and $\bar{z}$.

$f$ is holomorphic if the power series only has $z$ terms.

Treat $z$ and $\bar{z}$ as separate variables.

$f(z, \bar{z})$ becomes $f(z, \zeta)$. This is called complexification.

We must worry about convergence! More on all this later.
We write a smooth ($C^\infty$) function $f: U \subset \mathbb{C}^n \to \mathbb{C}$ as $f(z, \bar{z})$.

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We must worry about convergence! More on all this later.
Theorem (Hartogs)

Let $U \subset \mathbb{C}^n$, $n \geq 2$, be a domain, and $K \subset\subset U$ be compact with $U \setminus K$ connected. If $f \in \mathcal{O}(U \setminus K)$, then there exists a unique $F \in \mathcal{O}(U)$ such that $F|_{U\setminus K} = f$. 

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If $U, V \subset \mathbb{C}^n$ and $f : U \rightarrow V$ is holomorphic (every component is holomorphic), bijective, and $f^{-1}$ is holomorphic, then $f$ is a **biholomorphism** and $U$ and $V$ are **biholomorphic**.

**Remark:** $f^{-1}$ is automatically holomorphic.

Suppose $f$ extends past the boundary of $U$. Then biholomorphic invariants of the boundary of $U$ are invariants of the boundary of $V$.

**Example:** $U = B(0, 2) \setminus B(0, 1)$. The outer (convex) and the inner (concave) boundaries have very different properties. In fact it is a form of "convexity" that we need to study to understand boundaries.
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Take the real tangent space $T_p \mathbb{C}^n = T_p \mathbb{R}^{2n}$. Write

$$\mathbb{C} \otimes T_p \mathbb{C}^n = \text{span}_\mathbb{C} \left\{ \frac{\partial}{\partial x_1} \bigg|_p, \frac{\partial}{\partial y_1} \bigg|_p, \ldots, \frac{\partial}{\partial x_n} \bigg|_p, \frac{\partial}{\partial y_n} \bigg|_p \right\}.$$
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\[ \frac{\partial}{\partial z_k} \bigg|_p, \frac{\partial}{\partial \bar{z}_k} \bigg|_p \in \mathbb{C} \otimes T_p \mathbb{C}^n \]
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Define

$$T_p^{(1,0)} \mathbb{C}^n \overset{\text{def}}{=} \text{span}_\mathbb{C} \left\{ \frac{\partial}{\partial z_1} |_p, \ldots, \frac{\partial}{\partial z_n} |_p \right\} \quad (\text{holomorphic vectors}),$$

$$T_p^{(0,1)} \mathbb{C}^n \overset{\text{def}}{=} \text{span}_\mathbb{C} \left\{ \frac{\partial}{\partial \bar{z}_1} |_p, \ldots, \frac{\partial}{\partial \bar{z}_n} |_p \right\} \quad (\text{antiholomorphic vectors}).$$
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Then
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\mathbb{C} \otimes T_p \mathbb{C}^n = T_p^{(1,0)} \mathbb{C}^n \oplus T_p^{(0,1)} \mathbb{C}^n.
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Decompose $\mathbb{C} \otimes T_pM$ as

$$\mathbb{C} \otimes T_pM = T_p^{(1,0)}M \oplus T_p^{(0,1)}M \oplus B_p.$$
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$B_p \cong \mathbb{C} \otimes T_p M \big/ T_p^{(1,0)} M \oplus T_p^{(0,1)} M$ is a one-dimensional space.
More explicitly,

\[
X_p = \sum_{k=1}^{n} a_k \frac{\partial}{\partial z_k} \big|_p + b_k \frac{\partial}{\partial \bar{z}_k} \big|_p \in \mathbb{C} \otimes T_pM \iff \sum_{k=1}^{n} a_k \frac{\partial r}{\partial z_k} \big|_p + b_k \frac{\partial r}{\partial \bar{z}_k} \big|_p = 0.
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More explicitly,

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Example: \( \text{Im} z_n = \frac{z_n - \bar{z}_n}{2i} = 0 \) defines \( M = \mathbb{C}^{n-1} \times \mathbb{R} \subset \mathbb{C}^n \).

\( T^{(1,0)}_p M = \text{span}_\mathbb{C} \left\{ \frac{\partial}{\partial z_1} |_p, \ldots, \frac{\partial}{\partial z_{n-1}} |_p \right\} \)

\( T^{(0,1)}_p M = \text{span}_\mathbb{C} \left\{ \frac{\partial}{\partial \bar{z}_1} |_p, \ldots, \frac{\partial}{\partial \bar{z}_{n-1}} |_p \right\} \)

\( B_p = \text{span}_\mathbb{C} \left\{ \frac{\partial}{\partial x_n} |_p \right\} = \text{span}_\mathbb{C} \left\{ \frac{\partial}{\partial (\text{Re} z_n)} |_p \right\} = \text{span}_\mathbb{C} \left\{ \frac{\partial}{\partial z_n} |_p + \frac{\partial}{\partial \bar{z}_n} |_p \right\} \)
If $M \subset \mathbb{C}^n$ is a smooth real submanifold (any dimension), do the same:

$$T_p^{(1,0)} M \overset{\text{def}}{=} (\mathbb{C} \otimes T_p M) \cap (T_p^{(1,0)} \mathbb{C}^n),$$

and

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\]

Now

\[
\mathbb{C} \otimes T_p M = T_p^{(1,0)} M \oplus T_p^{(0,1)} M \oplus B_p.
\]

If $T_p^{(1,0)} M$ and $T_p^{(0,1)} M$ have constant dimension as $p$ ranges over $M$, then $M$ is called a CR submanifold.
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**Remark:** Every hypersurface is a CR submanifold (next slide).
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**Remark:** Every hypersurface is a CR submanifold (next slide).

**Example 1:** $M = \mathbb{R}^2 \subset \mathbb{C}^2$.

$$T_p^{(1,0)} M = \{0\}, \quad T_p^{(0,1)} M = \{0\}, \quad B_p = \mathbb{C} \otimes T_p M.$$
If $M \subset \mathbb{C}^n$ is a smooth real submanifold (any dimension), do the same:

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T_p^{(1,0)} M \overset{\text{def}}{=} (\mathbb{C} \otimes T_p M) \cap (T_p^{(1,0)} \mathbb{C}^n), \quad \text{and}
\]
\[
T_p^{(0,1)} M \overset{\text{def}}{=} (\mathbb{C} \otimes T_p M) \cap (T_p^{(0,1)} \mathbb{C}^n).
\]

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\[
\mathbb{C} \otimes T_p M = T_p^{(1,0)} M \oplus T_p^{(0,1)} M \oplus B_p.
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If $T_p^{(1,0)} M$ and $T_p^{(0,1)} M$ have constant dimension as $p$ ranges over $M$, then $M$ is called a CR submanifold.

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**Example 1:** $M = \mathbb{R}^2 \subset \mathbb{C}^2$.

\[T_p^{(1,0)} M = \{0\}, \quad T_p^{(0,1)} M = \{0\}, \quad B_p = \mathbb{C} \otimes T_p M.\]

**Example 2:** $M = \mathbb{C} \times \{0\} \subset \mathbb{C}^2$.

\[T_p^{(1,0)} M = \text{span}_\mathbb{C} \left\{ \frac{\partial}{\partial z_1} \bigg|_p \right\}, \quad T_p^{(0,1)} M = \text{span}_\mathbb{C} \left\{ \frac{\partial}{\partial \overline{z}_1} \bigg|_p \right\}, \quad B_p = \{0\}.\]
Suppose $M \subset \mathbb{C}^n$ is a smooth real hypersurface, $p \in M$. After a translation and rotation via a unitary matrix, $p = 0$ and near the origin $M$ is written in variables $(z, w) \in \mathbb{C}^{n-1} \times \mathbb{C} (w = z_n)$ as

$$\text{Im } w = \varphi(z, \bar{z}, \text{Re } w),$$

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Consequently

$$T^{(1,0)}_0 M = \text{span}_\mathbb{C} \left\{ \left. \frac{\partial}{\partial z_1} \right|_0, \ldots, \left. \frac{\partial}{\partial z_{n-1}} \right|_0 \right\},$$

$$T^{(0,1)}_0 M = \text{span}_\mathbb{C} \left\{ \left. \frac{\partial}{\partial \bar{z}_1} \right|_0, \ldots, \left. \frac{\partial}{\partial \bar{z}_{n-1}} \right|_0 \right\},$$

$$B_0 = \text{span}_\mathbb{C} \left\{ \left. \frac{\partial}{\partial (\text{Re } w)} \right|_0 \right\}.$$

In particular, $\dim_\mathbb{C} T^{(1,0)}_p M = \dim_\mathbb{C} T^{(0,1)}_p M = n - 1$ and $\dim_\mathbb{C} B_p = 1$. 
Suppose $M = \{r = 0\}$ as before, and $p \in M$. 

Write the (full) Hessian of $r$ at $p$ as the Hermitian matrix $H_p = 
\begin{pmatrix}
\frac{\partial^2 r}{\partial \bar{z}_1 \partial z_1}
p \cdots \\
\frac{\partial^2 r}{\partial \bar{z}_1 \partial \bar{z}_1}
p \cdots \\
\vdots \ & \ddots \ & \ddots \\
\frac{\partial^2 r}{\partial \bar{z}_n \partial z_1}
p \cdots \\
\frac{\partial^2 r}{\partial \bar{z}_n \partial \bar{z}_1}
p \cdots \\
\frac{\partial^2 r}{\partial z_1 \partial z_1}
p \cdots \\
\vdots \ & \ddots \ & \ddots \\
\frac{\partial^2 r}{\partial z_n \partial \bar{z}_1}
p \cdots \\
\frac{\partial^2 r}{\partial z_n \partial \bar{z}_n}
p \cdots \\
\end{pmatrix} = L_p Z_p L_p^t.
Suppose $M = \{ r = 0 \}$ as before, and $p \in M$.

Write the (full) Hessian of $r$ at $p$ as the Hermitian matrix

$$ H_p = \begin{bmatrix} \frac{\partial^2 r}{\partial z_1 \partial \bar{z}_1} |_p & \cdots & \frac{\partial^2 r}{\partial z_1 \partial z_n} |_p & \frac{\partial^2 r}{\partial z_1 \partial \bar{z}_1} |_p & \cdots & \frac{\partial^2 r}{\partial z_1 \partial \bar{z}_n} |_p \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 r}{\partial z_n \partial \bar{z}_1} |_p & \cdots & \frac{\partial^2 r}{\partial z_n \partial z_n} |_p & \frac{\partial^2 r}{\partial z_n \partial \bar{z}_1} |_p & \cdots & \frac{\partial^2 r}{\partial z_n \partial \bar{z}_n} |_p \\ \frac{\partial^2 r}{\partial z_1 \partial \bar{z}_1} |_p & \cdots & \frac{\partial^2 r}{\partial z_1 \partial z_n} |_p & \frac{\partial^2 r}{\partial z_1 \partial \bar{z}_1} |_p & \cdots & \frac{\partial^2 r}{\partial z_1 \partial \bar{z}_n} |_p \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 r}{\partial z_n \partial \bar{z}_1} |_p & \cdots & \frac{\partial^2 r}{\partial z_n \partial z_n} |_p & \frac{\partial^2 r}{\partial z_n \partial \bar{z}_1} |_p & \cdots & \frac{\partial^2 r}{\partial z_n \partial \bar{z}_n} |_p \end{bmatrix} = \begin{bmatrix} L_p & Z_p \\ Z_p & L^t_p \end{bmatrix} $$
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$$H_p = \begin{bmatrix}
\frac{\partial^2 r}{\partial z_1 \partial z_1} |_p & \cdots & \frac{\partial^2 r}{\partial z_1 \partial z_n} |_p \\
\frac{\partial^2 r}{\partial z_1 \partial \bar{z}_1} |_p & \cdots & \frac{\partial^2 r}{\partial z_1 \partial \bar{z}_n} |_p \\
\vdots & \ddots & \vdots \\
\frac{\partial^2 r}{\partial z_n \partial z_1} |_p & \cdots & \frac{\partial^2 r}{\partial z_n \partial z_n} |_p \\
\frac{\partial^2 r}{\partial z_n \partial \bar{z}_1} |_p & \cdots & \frac{\partial^2 r}{\partial z_n \partial \bar{z}_n} |_p \\
\frac{\partial^2 r}{\partial \bar{z}_1 \partial z_1} |_p & \cdots & \frac{\partial^2 r}{\partial \bar{z}_1 \partial z_n} |_p \\
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\frac{\partial^2 r}{\partial \bar{z}_n \partial \bar{z}_1} |_p & \cdots & \frac{\partial^2 r}{\partial \bar{z}_n \partial \bar{z}_n} |_p
\end{bmatrix} = \begin{bmatrix}
L_p & \bar{Z}_p \\
Z_p & L^t_p
\end{bmatrix}$$

$M$ is (strictly if inequality strict) convex at $p$ (really one side of $M$ is) if

$$X^*_p H_p X_p \geq 0 \quad \text{for all } X_p \in \mathbb{C} \otimes T_p M.$$
Suppose \( M = \{ r = 0 \} \) as before, and \( p \in M \).

Write the (full) Hessian of \( r \) at \( p \) as the Hermitian matrix

\[
H_p = \begin{bmatrix}
\frac{\partial^2 r}{\partial z_1 \partial z_1} | p & \cdots & \frac{\partial^2 r}{\partial z_1 \partial z_n} | p & \frac{\partial^2 r}{\partial \bar{z}_1 \partial z_1} | p & \cdots & \frac{\partial^2 r}{\partial \bar{z}_1 \partial z_n} | p \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\frac{\partial^2 r}{\partial z_n \partial z_1} | p & \cdots & \frac{\partial^2 r}{\partial z_n \partial z_n} | p & \frac{\partial^2 r}{\partial \bar{z}_n \partial z_1} | p & \cdots & \frac{\partial^2 r}{\partial \bar{z}_n \partial z_n} | p \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\frac{\partial^2 r}{\partial z_n \partial z_1} | p & \cdots & \frac{\partial^2 r}{\partial z_n \partial z_n} | p & \frac{\partial^2 r}{\partial \bar{z}_n \partial z_1} | p & \cdots & \frac{\partial^2 r}{\partial \bar{z}_n \partial z_n} | p
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\( M \) is (strictly if inequality strict) convex at \( p \) (really one side of \( M \) is) if

\[
X_p^* H_p X_p \geq 0 \quad \text{for all } X_p \in \mathbb{C} \otimes T_p M.
\]

A complex linear change of coordinates \( A : \mathbb{C}^n \to \mathbb{C}^n \) acts like

\[
\begin{bmatrix}
A & 0 \\
0 & \bar{A}
\end{bmatrix}^* \begin{bmatrix}
L & \bar{Z} \\
Z & L^t
\end{bmatrix} \begin{bmatrix}
A & 0 \\
0 & \bar{A}
\end{bmatrix} = \begin{bmatrix}
A^* L A & \bar{A}^t Z A \\
A^t Z A & (A^* L A)^t
\end{bmatrix}.
\]
Consider the Hessian $H_p = \begin{bmatrix} L_p & \overline{Z_p} \\ Z_p & L_p^t \end{bmatrix}$ (an $2n \times 2n$ matrix)
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$L_p = \left[ \frac{\partial^2 r}{\partial \bar{z}_k \partial z_\ell} \right]_{k\ell}$ is called the complex Hessian (an $n \times n$ matrix).
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$L_p = \left[ \frac{\partial^2 r}{\partial \bar{z}_k \partial z_{\ell}} \right]_{k\ell}^p$ is called the complex Hessian (an $n \times n$ matrix).

For $X_p \in T_p^{(1,0)}M$ (n − 1 dimensional space),

$$X^*L_pX_p$$

is called the Levi-form at $p$.  

Consider the Hessian $H_p = \begin{bmatrix} L_p & \overline{Z}_p \\ Z_p & L_p^t \end{bmatrix}$ (an $2n \times 2n$ matrix)

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For $X_p \in T_p^{(1,0)} M$ ($n - 1$ dimensional space),

$$X^*_p L_p X_p$$

is called the Levi-form at $p$.

Explicitly, $X_p = \sum_{k=1}^{n} a_k \frac{\partial}{\partial z_k} \bigg|_p \in T_p^{(1,0)} M$ iff $X_p r = \sum_{k=1}^{n} a_k \frac{\partial r}{\partial z_k} \bigg|_p = 0$,

and

$$X^*_p L_p X_p = \sum_{k=1, \ell=1}^{n} \bar{a}_k a_\ell \frac{\partial^2 r}{\partial \bar{z}_k \partial z_\ell} \bigg|_p.$$
Consider the Hessian $H_p = \begin{bmatrix} L_p & Z_p \\ Z_p^t & L_p^t \end{bmatrix}$ \hspace{1em} (an $2n \times 2n$ matrix)

$L_p = \begin{bmatrix} \frac{\partial^2 r}{\partial \bar{z}_k \partial z_\ell} \bigg|_{p} \end{bmatrix}_{k\ell}$ is called the complex Hessian \hspace{1em} (an $n \times n$ matrix).

For $X_p \in T_p^{(1,0)}M$ \hspace{1em} ($n-1$ dimensional space),

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Explicitly, $X_p = \sum_{k=1}^{n} a_k \frac{\partial}{\partial z_k} \bigg|_{p} \in T_p^{(1,0)}M$ \hspace{1em} iff \hspace{1em} $X_pr = \sum_{k=1}^{n} a_k \frac{\partial r}{\partial z_k} \bigg|_{p} = 0$,

and

$$X_p^*L_pX_p = \sum_{k=1,\ell=1}^{n} \bar{a}_k a_\ell \frac{\partial^2 r}{\partial \bar{z}_k \partial z_\ell} \bigg|_{p}.$$ 

Exercise: $H_p$ and $L_p$ depend on the defining function $r$, but their inertia on the tangent space does not change if we change the defining function $r$. (Assume the new $r$ is negative on the same side of $M$).