Tasty Bits of Several Complex Variables (3)

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Complexification (traditional):

If \( U \subset \mathbb{C}^n \) is a domain, \( U \cap \mathbb{R}^n \neq \emptyset \), \( f, g \in \mathcal{O}(U) \), and \( f = g \) on \( U \cap \mathbb{R}^n \).

\( \Rightarrow \ f \equiv g \)
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$\Rightarrow f \equiv g$

Goes the other way too: If $V \subset \mathbb{R}^n, f : V \to \mathbb{R}$ is real-analytic,

$\Rightarrow \exists U \subset \mathbb{C}^n$ open, $V \subset U, F \in \mathcal{O}(U), F|_V = f$.

*Proof:* Given real power series $\sum_{\alpha} c_n (x - p)^n$, plug in complex numbers: $\sum_{\alpha} c_n (z - p)^n$. 
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$$f(x, y) = \sum_{m=0}^{\infty} f_m(x, y) = \sum_{m=0}^{\infty} f_m \left( \frac{z + \bar{z}}{2}, \frac{z - \bar{z}}{2i} \right)$$
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So (at any point) \( f \) equals

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\sum_{\alpha, \beta} c_{\alpha, \beta} (z - a)^\alpha (\bar{z} - \bar{a})^\beta.
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Let $U \subset \mathbb{C}^n \times \mathbb{C}^n$ be a domain and $f, g \in \mathcal{O}(U)$ so that $f = g$ on the diagonal

$$U \cap D = U \cap \{ (z, \zeta) \in \mathbb{C}^n \times \mathbb{C}^n : \zeta = \bar{z} \}.$$
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We identify $\mathbb{C}^n$ and $D \subset \mathbb{C}^n \times \mathbb{C}^n$ with $\iota(z) = (z, \bar{z})$. 
Example: \( f(z, \bar{z}) = \frac{1}{1 + |z|^2} = \frac{1}{1 + z \bar{z}} \) is real-analytic in \( \mathbb{C} \).
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    u(z, \bar{z}) = \frac{f(z) + \bar{f}(\bar{z})}{2}, \quad \text{WLOG } f(0) = 0 \quad \Rightarrow \quad f(z) = 2u(z, 0).
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**Remark:** There is no good control of the neighborhood to which $f$ extends. Even in 1D: Given any interval $(a, b)$ and any neighborhood $U$ of $(a, b)$, there is an $F \in \mathcal{O}(U)$ that does not extend past any boundary point of $U$. So $f = F|_{(a, b)}$ also cannot extend further.
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Suppose $M \subset \mathbb{C}^n$ is a hypersurface, then $f : M \to \mathbb{C}$ is a CR function if $X_pf = 0$ for all $X_p \in T_p^{(0,1)}M$ for all $p \in M$. 
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\zbar = \Phi(z, \zbar, w),
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\( \Phi, \frac{\partial \Phi}{\partial z_k}, \frac{\partial \Phi}{\partial \zbar_k} \) vanish at \( 0 \) and \( w = \bar{\Phi}(\zbar, z, \Phi(z, \zbar, w)) \).
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\( \Phi, \frac{\partial \Phi}{\partial z_k}, \frac{\partial \Phi}{\partial \zeta_k} \) vanish at 0 and \( w = \bar{\Phi}(\zeta, z, \Phi(z, \zeta, w)) \). A basis for \( T^{(0,1)} M \):

\[
\left( \frac{\partial}{\partial \bar{z}_k} + \frac{\partial \Phi}{\partial \bar{z}_k} \frac{\partial}{\partial \bar{w}} \right) = \left( \frac{\partial}{\partial \bar{z}_k} + \frac{\partial \Phi}{\partial \bar{\zeta}_k} \frac{\partial}{\partial \bar{w}} \right), \quad k = 1, \ldots, n - 1.
\]
So: $M$ is $\bar{w} = \Phi(z, \bar{z}, w)$, $T^{(0,1)}M$ is given by $\frac{\partial}{\partial \bar{z}_k} + \frac{\partial \Phi}{\partial \bar{z}_k} \frac{\partial}{\partial \bar{w}}$. 

Example: Consider $M \subset \mathbb{C}^2$ given by $\text{Im} w = |z|^2$, that is, $w - \bar{w} = 2iz\bar{z}$, or in other words, $M$ is given by $\omega = -2iz\bar{z} + w$, and the CR vector field by $\frac{\partial}{\partial \bar{z}} - 2iz \frac{\partial}{\partial \bar{w}}$. 

If $f(z, w, \bar{z}, \bar{w})$ is a CR function, the holomorphic extension is $f(z, w, \bar{z}, -2iz\bar{z} + w)$, the $\bar{z}$ will cancel.
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**Example:** Consider \( M \subset \mathbb{C}^2 \) given by \( \text{Im} \ w = |z|^2 \), that is, \( \frac{w-\bar{w}}{2i} = z\bar{z} \), or in other words, \( \mathcal{M} \) is given by \( \omega = -2iz\zeta + w \), and the CR vector field by \( \frac{\partial}{\partial z} - 2iz \frac{\partial}{\partial \bar{w}} \).
So: $M$ is $\bar{w} = \Phi(z, \bar{z}, w)$, $T^{(0,1)}M$ is given by $\frac{\partial}{\partial \bar{z}_k} + \frac{\partial \Phi}{\partial \bar{z}_k} \frac{\partial}{\partial \bar{w}}$.

Define the complexification $\mathcal{M} \subset \mathbb{C}^{2n}$ by $\omega = \Phi(z, \zeta, w)$

Complexify $f(z, w, \bar{z}, \bar{w})$ to $f(z, w, \zeta, \omega)$. Now the trick: Define

$$F(z, w, \zeta) = f(z, w, \zeta, \Phi(z, \zeta, w)).$$

As $f$ is a CR function, it is killed by $\frac{\partial}{\partial \bar{z}_k} + \frac{\partial \Phi}{\partial \bar{z}_k} \frac{\partial}{\partial \bar{w}}$ on $M$. So

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So $F$ is a function of $z$ and $w$ only $\Rightarrow$ $F$ is holomorphic in $\mathbb{C}^n$. □

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If $f(z, w, \bar{z}, \bar{w})$ is a CR function, the holomorphic extension is $f(z, w, \bar{z}, -2iz\bar{z} + w)$, the $\bar{z}$ will cancel.
What if $f$ is only smooth?

Proposition:
Suppose $U \subset \mathbb{C}^n$ is open with smooth boundary and $f: U \rightarrow \mathbb{C}$ is smooth, holomorphic on $U$. Then $f|_{\partial U}$ is a smooth CR function.

Proof:
Each $X_p \in T(0,1)_p \partial U$ is a limit of $T(0,1)_\mathbb{C}^n$ vectors from inside.

□

Proposition:
Suppose $U \subset \mathbb{C}^n$ is a domain with smooth boundary and $f: U \rightarrow \mathbb{C}$ is smooth, holomorphic on $U$ and $f|_{\partial U}$ is zero on a nonempty open subset. Then $f \equiv 0$.

Proof:
Use Radó's theorem to extend $as 0 outside ($g$ in the picture), then use identity.

□

Theorem (Radó):
If $U \subset \mathbb{C}^n$ is open and $g: U \rightarrow \mathbb{C}$ continuous and holomorphic on $U'$, then $g \in O(U)$.
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**Example:** Suppose $M = \mathbb{R} \subset \mathbb{C}$. Define $f : M \to \mathbb{C}$:

$$f(x) = \begin{cases} 
  e^{-x^2} & \text{if } x \neq 0, \\
  0 & \text{if } x = 0. 
\end{cases}$$

Then $f$ is CR (trivially), but is not a restriction nor boundary value (from either side) of a holomorphic function continuous up to 0.
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**Example:** Define the function $f \in \overline{\mathbb{B}}_2 \to \mathbb{C}$ by

$$f(z_1, z_2) = \begin{cases} e^{-1/\sqrt{z_1+1}} & \text{if } z_1 \neq -1, \\ 0 & \text{if } z_1 = -1. \end{cases}$$

Then $f$ is smooth on $\overline{\mathbb{B}}_2$, holomorphic on $\mathbb{B}_2$, but near $(-1, 0)$ is not a restriction of a holomorphic function (only one sided extension).
A neat technique for extension is to approximate by polynomials.

Theorem (Baouendi–Trèves):
Suppose $M \subset \mathbb{C}^n$ is a smooth real hypersurface, $p \in M$. Then there exists a compact neighborhood $K \subset M$ of $p$, such that for every CR function $f : M \to \mathbb{C}$, there exists a sequence $\{p_\ell\}$ of polynomials in $z$ such that $p_\ell(z) \to f(z)$ uniformly in $K$.

Example: The $K$ depends only on $M$, but can't always be all of $M$: E.g., $M = S^1$ and $f = \bar{z}$.

The proof is based on the standard proof of Weierstrass theorem: If $f : [0, 1] \to \mathbb{R}$ is continuous, then it is approximated on $[0, 1]$ by the entire functions $f_\ell(z) = \int_0^1 c_\ell e^{-\ell(z-t)^2} f(t) \, dt$ for properly chosen $c_\ell$. Then just take partial sums of the powers series.

Baouendi–Trèves uses the same idea on a totally real subset of $M$ and slightly modified version of the above.
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**Theorem (Lewy):** Suppose $M \subset \mathbb{C}^n$ is a smooth real hypersurface and $p \in M$. There exists a neighborhood $U$ of $p$ with the following property. Suppose $r: U \rightarrow \mathbb{R}$ is a smooth defining function for $M \cap U$, denote by $U_- \subset U$ the set where $r$ is negative and $U_+ \subset U$ the set where $r$ is positive. Let $f : M \rightarrow \mathbb{R}$ be a smooth CR function. Then:

(i) If the Levi form with respect to $r$ has a positive eigenvalue at $p$, then $f$ extends to a holomorphic function on $U_-$ continuous up to $M$.

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"Proof of (i):" Write $M$ as

$$\text{Im } w = |z_1|^2 + \sum_{k=2}^{n-1} \epsilon_k |z_k|^2 + E(z_1, z', \bar{z}_1, \bar{z}', \text{Re } w),$$

where $z' = (z_2, \ldots, z_{n-1})$, $\epsilon_k = -1, 0, 1$, and $E$ is $O(3)$. And apply Bauoendi–Trèves to find a $K$. 

we find an analytic disc $\Delta$ "attached" to $K \subset M$ (i.e., $\partial \Delta \subset K$). 

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Apply Baouendi–Trèves to find $p_\ell$ that approximate $f$ uniformly on $K$. 

\[ \{p_\ell\} \text{ is (uniformly) Cauchy on } \partial \Delta \text{ for each disc.} \]

By maximum principle, \[ \{p_\ell\} \text{ is (uniformly) Cauchy on } \Delta. \]

$\Rightarrow \{p_\ell\} \text{ is (uniformly) Cauchy on } U \setminus K \Rightarrow \{p_\ell\} \text{ converges to a holomorphic function on } U \text{ continuous up to the boundary.}$

To see (iii), extend to one side, then use the Tomato can principle to extend to the other side. □

**Example:**
Every CR function on $\text{Im } w = |z_1|^2 - |z_2|^2$ extends to an entire holomorphic function on $\mathbb{C}^3$ and hence must be real-analytic.

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Every CR function on $\text{Im } w = |z_1|^2 + |z_2|^2$ extends to the set $\text{Im } w \geq |z_1|^2 + |z_2|^2$, but not necessarily below.

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These ideas led Lewy to find the example of the unsolvable PDE.
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Another application is a special case of the following theorem:

**Theorem** (Hartogs–Bochner): Suppose $U \subset \mathbb{C}^n$, $n \geq 2$, is bounded open set with smooth boundary and $f : \partial U \to \mathbb{C}$ is a CR function. Then there exists a continuous $F : U \to \mathbb{C}$ holomorphic in $U$ such that $F|_{\partial U} = f$. 

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Example: Every CR function on $S_2^{n-1} \subset \mathbb{C}^n$, $n \geq 2$, is the boundary value of a continuous $F : \mathbb{D}_n \to \mathbb{C}$ that is holomorphic in $\mathbb{D}_n$.

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Example: Similarly, not true in general if $U$ is unbounded. If $U = \mathbb{D} \times \mathbb{C} \subset \mathbb{C}^2$, then $\overline{z}_1$ is a CR function, but does not extend inside for the same reason.
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**Example:** Every CR function on $S^{2n-1} \subset \mathbb{C}^n$, $n \geq 2$, is the boundary value of a continuous $F: \overline{\mathbb{B}_n} \rightarrow \mathbb{C}$ that is holomorphic in $\mathbb{B}_n$.

**Example:** The function $\bar{z}$ on $S^1 \subset \mathbb{C}$ is not the boundary value of a holomorphic function in the disc; it would have a pole.
Another application is a special case of the following theorem:

**Theorem** (Hartogs–Bochner): Suppose $U \subset \mathbb{C}^n$, $n \geq 2$, is bounded open set with smooth boundary and $f : \partial U \to \mathbb{C}$ is a CR function. Then there exists a continuous $F : \overline{U} \to \mathbb{C}$ holomorphic in $U$ such that $F|_{\partial U} = f$.

The special case is if we have at least one positive Levi eigenvalue at each point, and if we can extend through compacts (next lecture).

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**Example:** Every CR function on $S^{2n-1} \subset \mathbb{C}^n$, $n \geq 2$, is the boundary value of a continuous $F : \overline{B}_n \to \mathbb{C}$ that is holomorphic in $B_n$.

**Example:** The function $\bar{z}$ on $S^1 \subset \mathbb{C}$ is not the boundary value of a holomorphic function in the disc; it would have a pole.

**Example:** Similarly, not true in general if $U$ is unbounded. If $U = \mathbb{D} \times \mathbb{C} \subset \mathbb{C}^2$, then $\bar{z}_1$ is a CR function, but does not extend inside for the same reason.