Proper maps of balls and annuli and the general hyperplane rank

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Theorem (Alexander, Pinchuk circa '77 (complicated history...))

If $f : \mathbb{B}_n \to \mathbb{B}_n$ $(n \geq 2)$ is a proper holomorphic map, then $f \in \mathrm{Aut}(\mathbb{B}_n)$.

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Theorem (Forstnerič '89)

Suppose $2 \leq n \leq N$. If a proper holomorphic $f\colon \mathbb{B}_n \to \mathbb{B}_N$ extends smoothly up to the boundary, then f is rational, and its degree is bounded in terms of n and N.

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H_d doesn't just take the unit sphere to unit sphere.

If $N = \binom{n+d-1}{d}$ is the target dimension, then for all $r > 0$,

$$H_d(rS^{2n-1}) \subset r^d S^{2N-1}$$

Theorem (Rudin '84)

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There exist maps where $k_f < N-1$ even if the embedding dimension is N.

Example: If g, h are rational proper maps of balls, then the *juxtaposition* $f = \sqrt{1-t}\, g \oplus \sqrt{t}\, h$ is a proper map of balls with $k_f \leq N-2$.

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Suppose $n \geq 2$ and $f \colon \mathbb{B}_n \longrightarrow \mathbb{B}_N$ is a proper holomorphic map such that $k_f = N - 1$ and $f(rS^{2n-1}) \subset RS^{2N-1}$ for some $r, R \in (0, 1)$. Then there exists a unitary map U and an integer d such that $f = UH_d$.

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There exist degree 2 rational proper ball maps that take a sphere to a sphere as above and are not constructed out of homogeneous maps.

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Theorem (see Helal-L.-Nandi '25)

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Moreover, $f(S^{2n-1}) \subset S^{2N-1}$ *and* $f(rS^{2n-1}) \subset RS^{2N-1}$ *(so* f *extends to a proper map of* $\mathbb{B}_n \to \mathbb{B}_N$ *and also* $r\mathbb{B}_n \to R\mathbb{B}_N$*).*

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Conversely, if $f\colon \mathbb{C}^n \dashrightarrow \mathbb{C}^N$ is rational, $f(S^{2n-1}) \subset S^{2N-1}$, and $f(rS^{2n-1}) \subset RS^{2N-1}$,
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Remark: Not true if $n = 1$ and $N > 1$.

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Proof hint: $\log\left|f(z)\right|$ is harmonic and extends to the boundary.

Our theorem on homogeneous maps can be restated for maps of annuli.

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Suppose $n \geq 2$ *and* $f \colon \mathbb{A}_{n,r} \to \mathbb{A}_{N,R}$ *is a proper holomorphic map.*
Then $k_f = N - 1$ *if and only if* $N = \binom{n+d-1}{d}$, $R = r^d$, *and*

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Conjecture: For map of degree d, is seems that $R \geq r^d$.

Combining with Faran's classification of proper maps from $\mathbb{B}_2 \to \mathbb{B}_3$ gets:

Theorem (Helal-L.-Nandi)

Suppose $f : \mathbb{A}_{2,r} \to \mathbb{A}_{3,R}$ is a proper holomorphic map. Then f is unitarily equivalent to exactly one of the following two maps:

1. *The affine embedding*

$$(z_1, z_2) \mapsto \left(\sqrt{\frac{1-R^2}{1-r^2}}\, z_1, \sqrt{\frac{1-R^2}{1-r^2}}\, z_2, \sqrt{\frac{R^2-r^2}{1-r^2}} \right),$$

   where $R \geq r$ and $k_f = 1$.

2. *The homogeneous map H_2*

$$(z_1, z_2) \mapsto \left(z_1^2, \sqrt{2}\, z_1 z_2, z_2^2 \right),$$

   where $R = r^2$ and $k_f = 2$.

Up to unitaries, the only degree 1 maps are the identity if $N = n$, or the affine embedding if $N > n$: $\sqrt{\frac{1-R^2}{1-r^2}}\ \mathrm{id} \oplus \sqrt{\frac{R^2-r^2}{1-r^2}} \oplus 0$

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For degree 2, we get the following complete normal form:

Suppose $f\colon \mathbb{A}_{n,r} \to \mathbb{A}_{N,R}$ is a proper rational map of degree 2, $n \geq 2$, such that the image is not contained in any affine subspace.

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1) If $R = r^2$, then $N = \binom{n+1}{2}$ and f is unitarily equivalent to H_2.

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Then $R \geq r^2$ and:

1) If $R = r^2$, then $N = \binom{n+1}{2}$ and f is unitarily equivalent to H_2.

2) If $r^2 < R < r$ and f is polynomial, then $N = \binom{n+1}{2} + n$ and f is unitarily equivalent to a map of the form $\sqrt{1-t}\,H_1 \oplus \sqrt{t}\,H_2$ for exactly one $t \in (0,1)$.

. . .

3) If $r^2 < R < r$ and f is not polynomial, then $N = \binom{n+1}{2} + n$ and f is unitarily equivalent to a map of the form (where $b = \frac{1-R^2}{1-r^2}$)

$$\left(\frac{\sqrt{1-b+Qr^2}+\frac{(1-b)a}{\sqrt{1-b+Qr^2}}z_1}{1+az_1}, \frac{\sqrt{ba^2+Q}\,z_1^2+\frac{ba}{\sqrt{ba^2+Q}}z_1}{1+az_1}, \frac{\sqrt{b-Q(1+r^2)}+\frac{ba}{\sqrt{b-Q(1+r^2)}}z_1}{1+az_1} \otimes (z_2,\ldots,z_n), \right.$$
$$\left. \frac{\sqrt{ba^2+2Q-\frac{b^2a^2}{ba^2+2Q}}\,z_1}{1+az_1} \otimes (z_2,\ldots,z_n), \frac{\sqrt{Q}}{1+az_1}H_2'(z_2,\ldots,z_n) \right) \tag{1}$$

for exactly one pair a and Q, where $a > 0$,

$$a \leq \sqrt{\frac{br^4+(b^2-b+1)r^2-b+1-2r\sqrt{(b-1)b(1+r^2)(br^2-b+1)}}{b^2r^6+2br^4+r^2}} \tag{2}$$

and Q is either one of

$$Q = \frac{-a^2br^4+(2(1-a^2)b+a^2-1)r^2+b-1\pm\sqrt{a^4r^4(br^2+1)^2-2a^2r^2(br^2(r^2+b-1)+1+r^2-b)+(1+r^2-b)^2}}{2(r^4+r^2)}. \tag{3}$$

Moreover, for each such pair a and Q, a map exists.

4) If $R = r$, then $N = \binom{n+1}{2} + n$ and f is unitarily equivalent to a map of the form (1) for exactly one $a \in \left(0, \frac{1}{\sqrt{1+r^2}}\right)$ and $Q = \frac{1}{1+r^2} - a^2$. Moreover, for each such a, a map exists.

4) If $R = r$, then $N = \binom{n+1}{2} + n$ and f is unitarily equivalent to a map of the form (1) for exactly one $a \in \left(0, \frac{1}{\sqrt{1+r^2}}\right)$ and $Q = \frac{1}{1+r^2} - a^2$. Moreover, for each such a, a map exists.

5) If $r < R < 1$, then $N = \binom{n+1}{2} + n$ and f is unitarily equivalent to a map of the form (1) for exactly one $a \in (0, 1)$ where Q is the larger value of (3). Moreover, for each such a, a map exists.

Thanks for listening!