Several Complex Variables are Better than One

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Abstract

Complex analysis in one complex variable is the calculus student's dream come true. No annoying pathological cases, no annoying extra hypotheses to theorems, things just work, sometimes even better than one thinks they should. And it gets better in several variables, including even more surprising results. In this talk I will talk about the Hartogs Phenomenon, a theorem with a coolness factor at least double that of the Maximum Principle. In particular it will tell us something about how the zero sets of holomorphic (analytic) functions behave when we have more than one complex variable. It says that holomorphic functions simply extend across compact sets (no more isolated singularities). I will define holomorphic functions (both in one and several variables), give you enough to understand the theorem and even give a sketch of the proof (the d-bar equation makes an appearance). Only very basic knowledge of analysis will be required to enjoy the talk.

1 Introduction

The field of Several Complex Variables (SCV) lies at the intersection of several other fields. There are deep connections to Partial Differential Equations, Functional Analysis, Complex Dynamics, Algebraic Geometry (both real and complex), Pluripotential Theory, Differential Geometry, and of course One Complex Variable. There are even connections to Combinatorics, Commutative Algebra, or Representation Theory. Basically any time you talk about several variables, you often find they can be treated as complex numbers, and we arrive at SCV.

Whereas in basic real analysis one sees things go wrong in almost every way imaginable, complex analysis deals with incredibly well-behaved objects. Often it is more likely that in complex analysis you are surprised that some simple minded idea works all too well, rather than being surprised that an entirely reasonable idea breaks spectacularly. One can vaguely think of complex analysis as the "study of polynomials of infinite degree," and we all know how wonderful polynomials are.

We will go over a very fundamental theorem that will illustrate how several complex variables is a very different field from one complex variable. Let us however first start with one complex variable from the point of view of an SCV person…

2 Holomorphic Functions in One Variable

In complex analysis, we study *holomorphic* (or *analytic*) functions. We identify $\mathbb{C}$ with $\mathbb{R}^2$ as $z = x+iy$. The complex conjugate is then $\bar{z} = x-iy$. We start by defining the following formal differential operators (the so-called Wirtinger operators)

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right), \qquad \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right).$$

A function on $\mathbb{C}$ is really a function defined on $\mathbb{R}^2$ as identified above and hence it is a function of x and y. Writing $x = \frac{z+\bar{z}}{2}$ and $y = \frac{z-\bar{z}}{2i}$, we can think of it as a function of two complex variables z and $\bar{z}$. Here we pretend for a moment as if $\bar{z}$ did not depend on z. The formal differential operators work as if z and $\bar{z}$ really were independent variables. For example:

$$\frac{\partial}{\partial z}\left[z^2\bar{z}^3 + z^{10}\right] = 2z\bar{z}^3 + 10z^9 \qquad \text{and} \qquad \frac{\partial}{\partial \bar{z}}\left[z^2\bar{z}^3 + z^{10}\right] = z^2(3\bar{z}^2) + 0.$$

A holomorphic function is a function not depending on $\bar{z}$.

Definition 1. Let $U \subset \mathbb{C}$ be open and $f : U \to \mathbb{C}$ continuously differentiable. We say f is *holomorphic* if it satisfies the *Cauchy-Riemann equations*:

$$\frac{\partial f}{\partial \bar{z}} = 0.$$

Examples of holomorphic functions are polynomials in z or elementary functions such as sin, cos, or exp. Combining holomorphic functions algebraically or composing holomorphic functions yields again holomorphic functions.

Holomorphic functions have a power series representation in z at each point a:

$$f(z) = \sum_{j=0}^{\infty} c_j (z-a)^j.$$

Notice there is no $\bar{z}$ necessary there since $\frac{\partial f}{\partial \bar{z}} = 0$.

A basic problem in complex analysis of one variable is the classification of the *isolated singularities* of holomorphic functions. That is, you can have a holomorphic function defined everywhere but a single point, for example:

$$f(z) = \frac{1}{z}, \qquad \text{or even worse,} \qquad g(z) = e^{1/z}.$$

A related problem is to study sets where $f(z)$ is zero. A holomorphic function is locally just

$$(z-a)^k h(z)$$

where h is nonzero and holomorphic. In other words in one dimension, all zeros are isolated.

3 Several Variables

The questions of singularities and zeros get a lot more interesting in several variables. We first need to define what do we mean by holomorphic functions. The definition could not be simpler. In $\mathbb{C}^n$, we use the notation $z = (z_1, z_2, \ldots, z_n)$.

Definition 2. Let $U \subset \mathbb{C}^n$ be open. A function $f\colon U \to \mathbb{C}$ is *holomorphic* if it is holomorphic in each variable separately. That is, if

$$\zeta \mapsto f(z_1, \ldots, z_{k-1}, \zeta, z_{k+1}, \ldots, z_n)$$

is holomorphic wherever it makes sense. Or in other words if for all k

$$\frac{\partial f}{\partial \bar{z}_k} = 0.$$

Examples are similar (polynomials, etc...) except now we allow n variables instead of just one. We also have the power series representation of functions. For example in two variables, a power series at the origin looks like

$$\sum_{j,k=0}^{\infty} a_{jk} {z_1}^j {z_2}^k$$

where a_{jk} are complex numbers.

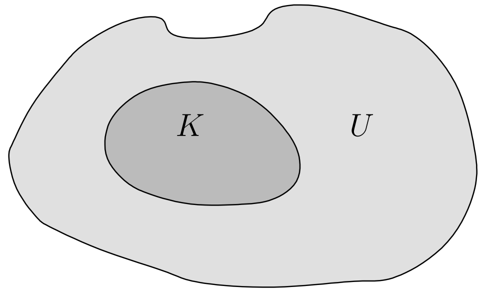
An important property of holomorphic functions (one or several variables) arising from the power series representation is that if a holomorphic function $f\colon U \to \mathbb{C}$ is zero on an open subset of U then the function is identically zero everywhere in U, provided U is connected.

We say $f\colon U \to \mathbb{C}$ has a holomorphic extension $\widetilde{f}$ to a larger domain $\widetilde{U}$ where $U \subset \widetilde{U}$ if $\widetilde{f}\colon \widetilde{U} \to \mathbb{C}$ is holomorphic and is equal to f on U. By the uniqueness property above, this extension is unique provided $\widetilde{U}$ is connected. To see this fact, note that if we have another extension on $\widetilde{U}$, say $\widehat{f}$, then $\widehat{f} - \widetilde{f}$ is holomorphic on $\widetilde{U}$ and identically zero on U. This uniqueness will be important in a moment.

4 Hartogs' Phenomenon

The Hartogs' phenomenon theorem tells us something even more fascinating about extending functions, and also it tells us a fundamental thing about the domains we use to define holomorphic functions.

Theorem 1. *Let $n \geq 2$, $U \subset \mathbb{C}^n$ be an open connected set (a domain), and let $K \subset U$ be a compact set (closed and bounded) such that $U \setminus K$ is connected. Every holomorphic $f\colon U \setminus K \to \mathbb{C}$ extends uniquely to a holomorphic function on U.*



The setup for Hartogs Phenomenon.

The result is not true in one complex dimension: Let $U = \mathbb{C}$ and $K = \{0\}$. Then let $f(z) = \frac{1}{z}$. Clearly f does not extend through the origin (it blows up).

The theorem tells us something about the zero set of a holomorphic function: in several variables the set $\{z \in \mathbb{C}^n : f(z) = 0\}$ cannot be compact. If it were, we could look at the function $\frac{1}{f(z)}$ and try to extend this function. Therefore the zero set must "go all the way to the boundary," or perhaps "off to infinity." So in some sense zero sets of holomorphic functions in more than one variable are in some sense "large." Similarly singular sets cannot be compact.

A more fundamental, and perhaps surprising, consequence of the theorem is that not every domain is a natural domain for a holomorphic function if $n \geq 2$. In one variable, on any domain we can cook up a function which does not extend to anything larger. In two or more variables, there are domains (for example domains with compact holes) where every holomorphic function extends to a larger domain.

It is the geometry of the boundary that makes the difference. Intuitively what happens is that near the "hole" (think K being a closed ball or similar) the domain must be "concave" somewhere. Therefore convexity enters the picture, although in a weaker and more complicated form, we call it *pseudoconvexity*, let us not define that concept here however.

Let us sketch the general outline of the proof. We solve the differential equation

$$\bar{\partial}\psi = g,$$

and use the solution to construct the needed extension. The equation is called the *inhomogeneous* $\bar{\partial}$ (pronounced d-bar) equation. Well I am cheating, I did not define the notation. The ψ is some smooth function, and g is a differential form, but we can simply think of it being n different smooth functions: g_1 through g_n. The equation above is really n equations

$$\frac{\partial \psi}{\partial \bar{z}_k} = g_k,$$

where the functions g_k satisfy the compatibility conditions

$$\frac{\partial g_k}{\partial \bar{z}_\ell} = \frac{\partial g_\ell}{\partial \bar{z}_k}.$$

The compatibility conditions simply mean that partial derivatives are supposed to commute, and the equation cannot be solved without them. Note that where $g = 0$, ψ is holomorphic. So for a general g, what we are doing is finding a function that is not holomorphic in a very specific way.

For Hartogs' theorem, let us assume that all the g_k have compact support. Then these equations always have a solution ψ (many solutions in fact), and if $n \geq 2$, then we can find ψ that has compact support. We obtain a solution explicitly in fact:

$$\psi(z) = \frac{1}{2\pi i} \int_{\mathbb{C}} \frac{g_1(\zeta, z_2, \ldots, z_n)}{\zeta - z_1} d\zeta \wedge d\bar{\zeta} = \frac{1}{2\pi i} \int_{\mathbb{C}} \frac{g_1(\zeta + z_1, z_2, \ldots, z_n)}{\zeta} d\zeta \wedge d\bar{\zeta}. \quad (1)$$

Here $d\zeta \wedge d\bar{\zeta}$ is just a fancy way to write the area measure on $\mathbb{C}$ up to a funny constant and we really think of $\mathbb{C}$ as $\mathbb{R}^2$ for the integral. Note the singularity, and think why the integral still makes sense. Hint: because we integrate in $\mathbb{R}^2$, use polar coordinates and note the r in $r\,dr\,d\theta$.

To check we have the solution we have to do a bit of work. Let us just give some hints for the intrepid reader to fill in. Using the compatibility condition we get

$$\begin{aligned} \frac{\partial \psi}{\partial \bar{z}_j}(z) &= \frac{1}{2\pi i} \int_{\mathbb{C}} \frac{\frac{\partial g_1}{\bar{z}_j}(\zeta + z_1, z_2, \ldots, z_n)}{\zeta} d\zeta \wedge d\bar{\zeta} \\ &= \frac{1}{2\pi i} \int_{\mathbb{C}} \frac{\frac{\partial g_j}{\bar{z}_1}(\zeta + z_1, z_2, \ldots, z_n)}{\zeta} d\zeta \wedge d\bar{\zeta} = g_j(z). \end{aligned}$$

The last equality follows by the generalized Cauchy formula (itself a version of the generalized Stokes theorem). Let φ be any smooth function (not necessarily holomorphic), then

$$\varphi(z) = \frac{1}{2\pi i} \int_{|\zeta| = R} \frac{\varphi(\zeta)}{\zeta - z} d\zeta + \frac{1}{2\pi i} \int_{|\zeta| \leq R} \frac{\frac{\partial \varphi}{\partial \bar{z}}(\zeta)}{\zeta - z} d\zeta \wedge d\bar{\zeta}.$$

The computation above uses $\varphi(\cdot) = g_j(\cdot, z_2, \ldots, z_n)$ and a large enough R so that φ is zero when $|\zeta| = R$.

That ψ has compact support follows because g_1 has compact support and analytic continuation. In particular note that ψ is holomorphic for very large z and it is identically zero when $z_2, \ldots, z_n$ are large.

Let us move onto the actual proof of the theorem. First find a smooth function φ that is 1 in a neighbourhood of K and is compactly supported in U. Let $f_0 = (1 - \varphi) f$ on $U \setminus K$ and $f_0 = 0$ on K. The function f_0 is smooth on U and it is holomorphic and equal to f near the boundary of U, where φ is 0. We let $g = \bar{\partial} f_0$, that is $g_k = \frac{\partial f_0}{\partial \bar{z}_k}$. Let's see why g_k is compactly supported. The only place to check is on $U \setminus K$ as elsewhere we have 0 automatically. Note that f is holomorphic and compute

$$\frac{\partial f_0}{\partial \bar{z}_k} = \frac{\partial}{\partial \bar{z}_k}\big((1-\varphi)f\big) = \frac{\partial f}{\partial \bar{z}_k} - \varphi\frac{\partial f}{\partial \bar{z}_k} - \frac{\partial \varphi}{\partial \bar{z}_k}f = -\frac{\partial \varphi}{\partial \bar{z}_k}f.$$

And $\frac{\partial \varphi}{\partial \bar{z}_k}$ must be compactly supported in U. And now we apply the above solution (1) to find a compactly supported function ψ such that $\bar{\partial}\psi = g$. We set $\widetilde{f} = f_0 - \psi$. Let's check that $\widetilde{f}$ is the desired extension. Let's check it's holomorphic:

$$\frac{\partial \widetilde{f}}{\partial \bar{z}_k} = \frac{\partial f_0}{\partial \bar{z}_k} - \frac{\partial \psi}{\partial \bar{z}_k} = g_k - g_k = 0.$$

It almost feels like we're cheating doesn't it? OK, but does it extend f. A bit of thought and the fact that $U \setminus K$ is connected reveals that ψ must be compactly supported in U. This means that $\widetilde{f}$ agrees with f near the boundary (in particular on an open set) and thus everywhere in U since U is connected.

And that is how using sheep's bladders can prevent earthquakes![6]

...any questions?

References

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