CR functions at CR singularities: approximation, extension, and hulls

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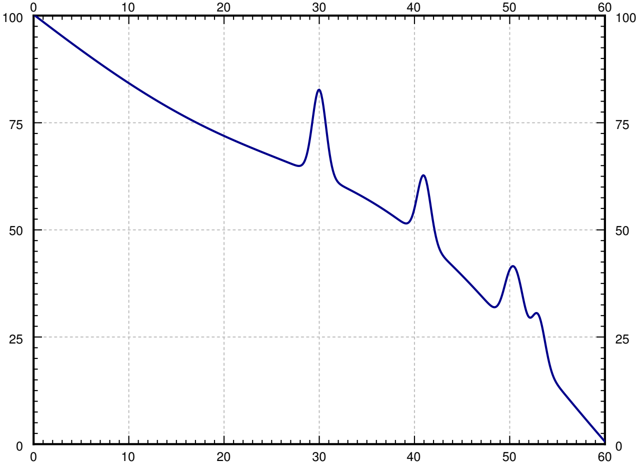
joint work with Alan Noell and Sivaguru Ravisankar

Department of Mathematics, Oklahoma State University

Plan for the talk

The plan for the percentage of the audience that is still awake is the following:



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Holomorphic functions are analytic:

$$f(z) = \sum_\alpha c_\alpha (z-p)^\alpha = \sum_\alpha c_\alpha (z_1 - p_1)^{\alpha_1} \cdots (z_n - p_n)^{\alpha_n} \quad \text{near any point } p$$

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where $\quad \frac{\partial}{\partial z_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j}\right) \quad$ and $\quad \frac{\partial}{\partial \bar{z}_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} + i\frac{\partial}{\partial y_j}\right)$

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Question: Is every CR function a restriction of a holomorphic function?
Answer: Yes if real-analytic, Not quite if smooth. (Just think $n = 1$)

Theorem (Severi '31)

Suppose $M \subset \mathbb{C}^n$ is a real-analytic CR submanifold and $f : M \to \mathbb{C}$ is a real-analytic CR function. $\Rightarrow$ *f extends holomorphically to some neighborhood of M.*

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Step 5) Profit!

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Write M as

$$\bar{w} = \Phi(z, \bar{z}, w),$$

and consider a real-analytic CR function $f(z, \bar{z}, w, \bar{w})$.

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Done! (there's a technicality or two in there, but overall that's the idea)

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Theorem (Baouendi–Trèves '81)

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$\Rightarrow$ $\exists$ *a compact neighborhood $K \subset M$ of p, such that for each continuous CR function $f \colon M \to \mathbb{C}$, $\exists$ a sequence $\{p_\ell\}$ of polynomials in z such that*

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Based on the standard Weierstrass: If $f : [0, 1] \to \mathbb{R}$ is continuous, then it is approximated on $[0, 1]$ by the entire functions

$$f_\ell(z) = \int_0^1 c_\ell e^{-\ell(z-t)^2} f(t)\, dt$$

for properly chosen c_ℓ. Then just take partial sums of the power series.

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Key feature: If f is holomorphic in a neighborhood of $M \cup \varphi(\overline{\mathbb{D}})$, then

$$|f(z_0)| \leq \sup_{z\in M} |f(z)| \quad \text{for any } z_0 \in \varphi(\overline{\mathbb{D}}).$$

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Key feature: If f is holomorphic in a neighborhood of $M \cup \varphi(\overline{\mathbb{D}})$, then

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$$w = z\bar{z} + \lambda(z^2 + \bar{z}^2) + E(z, \bar{z}).$$

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Coffman (’09) normalizes codim-2 M in $\mathbb{C}^3$ to
$w = z^*Nz + \operatorname{Re}(z^tPz) + E(z, \bar{z})$ where N and P are:

TABLE 1. Normal forms for Theorem 7.1

N	P			
$\begin{pmatrix} 1 & 0 \\ 0 & e^{i\theta} \end{pmatrix}$ $0<\theta<\pi$	$\begin{pmatrix} a & b \\ b & d \end{pmatrix}$	5	$a>0, d>0, b\sim -b\in\mathbb{C}$	$+-0$
	$\begin{pmatrix} 0 & b \\ b & d \end{pmatrix}$	3	$b\geq 0, d\geq 0$	$+-0$
	$\begin{pmatrix} a & b \\ b & 0 \end{pmatrix}$	3	$a>0, b\geq 0$	$+-0$
$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$	2	$0\leq a\leq d$	$+-0$
$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$	$\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$	2	$0\leq a\leq d$	$+-0$
	$\begin{pmatrix} 0 & b \\ b & 0 \end{pmatrix}$	1	$b>0$	$+$
	$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$	0		$+$
$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & b \\ b & 1 \end{pmatrix}$	1	$b>0$	$+0$
	$\begin{pmatrix} 1 & 0 \\ 0 & d \end{pmatrix}$	2	$\mathrm{Im}(d)>0$	$+$
$\begin{pmatrix} 0 & 1 \\ \tau & 0 \end{pmatrix}$ $0<\tau<1$	$\begin{pmatrix} a & b \\ b & d \end{pmatrix}$	5	$b>0, \lvert a\rvert=1, (a,d)\sim(-a,-d)$	$+-0$
	$\begin{pmatrix} 0 & b \\ b & d \end{pmatrix}$	3	$b>0, \lvert d\rvert=1, d\sim -d$	$+-0$
	$\begin{pmatrix} 0 & b \\ b & 0 \end{pmatrix}$	2	$b>0$	$+-0$
	$\begin{pmatrix} 1 & 0 \\ 0 & d \end{pmatrix}$	3	$d\in\mathbb{C}$	$+0$
	$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$	1		$+$
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$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$	$\begin{pmatrix} a & b \\ b & 1 \end{pmatrix}$	3	$b>0, a\in\mathbb{C}$	$+-0$
	$\begin{pmatrix} 1 & b \\ b & 0 \end{pmatrix}$	1	$b>0$	$+-0$
	$\begin{pmatrix} 0 & b \\ b & 0 \end{pmatrix}$	1	$b>0$	$+-0$
	$\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}$	1	$a\geq 0$	$+0$
	$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$	0		0
	$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$	0		0
$\begin{pmatrix} 0 & 1 \\ 1 & i \end{pmatrix}$	$\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$	3	$a>0, d\in\mathbb{C}$	$+-0$
	$\begin{pmatrix} 0 & b \\ b & 0 \end{pmatrix}$	1	$b>0$	$+0$
	$\begin{pmatrix} 0 & 0 \\ 0 & d \end{pmatrix}$	1	$d\geq 0$	$+$
$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$	$\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}$	1	$a\geq 0$	$+-0$
	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$	0		$+$
	$\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$	1	$a\geq 0$	0
$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	0		$+$
	$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$	0		0
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Try 4: Functions that are locally uniform limits of polynomials in z, w.
We get holomorphic extension for real-analytic CR functions,
and extensions to CR functions in $w \in \mathbb{R}$, $w \geq |z|^2$ for smooth.

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Other definitions are possible too. See e.g., Nacinovich–Porten '24 for a class sitting somewhere between CR_H and CR_P.

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L.–Minor–Shroff–Son–Zhang ('11): If a real-analytic CR singular manifold $M = \varphi(N)$ for a real-analytic CR map

$$\varphi \colon N \subset \mathbb{C}^n \to \mathbb{C}^n$$

of a CR submanifold N, and φ is a diffeomorphism onto $\varphi(N) = M$, then there exists $f \in CR^\omega(M)$ that does not extend holomorphically.

Suppose $M \subset \mathbb{C}^{n+1}$, $n \geq 2$, is real codimension 2.
In particular, $(z, w) \in \mathbb{C}^n \times \mathbb{C}$ and $w = z^* A z + \overline{z^t B z} + z^t C z + E(z, \bar{z})$

Theorem (L.–Noell–Ravisankar '21)

Suppose

$$\operatorname{rank} \begin{bmatrix} A^* \\ B \end{bmatrix} \geq 2.$$

If $f(z, \bar{z})$ is real-analytic CR function defined near the origin ($CR^\omega(M)$), then f extends holomorphically near the origin.
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$w = \bar{z}_2 z_2$ does not have this extension property, but $w = \bar{z}_2 z_2 + \bar{z}_2^3$ does.
In both cases, $\operatorname{rank} \begin{bmatrix} A^* \\ B \end{bmatrix} = 1$.

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Following is L.–Noell–Ravisankar '24.

In general $\quad CR^k(M) \supset CR^k_P(M) \supset CR_H(M), \quad$ for $k = 1, 2, \ldots, \infty, \omega$.

Theorem: For $w = z^2 + \bar{z}^2 + \lambda(z^2 + \bar{z}^2) + E(z, \bar{z})$, $E \in \mathbb{R}$, $0 \le \lambda < \frac{1}{2}$,

$CR^k(M) \supsetneq CR^k_P(M)$ for all k.

$CR_H(M)$ functions do **not** all extend to a fixed neighborhood.

$CR^\omega_P(M) = CR_H(M)$ and $CR^\omega_P(M)$ functions can be approximated on a fixed K (like Baouendi–Trèves).

$CR^\infty_P(M) \supsetneq CR_H(M)$ and $CR^\infty_P(M)$ functions can **not** be approximated on a fixed K.

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E.g., for M given by $w = \bar{z}_1 z_2$, using DH^3 works.
In fact for M, $CR_P^k(M) = CR_H(M)$ for all k.

If our discs come in families, we can use Kontinuitätssatz:

Suppose $U \subset \mathbb{C}^n$ *is open and there exists a sequence of analytic discs* φ_k *converging (pointwise) to an analytic disc* φ *such that* $\varphi_k(\overline{\mathbb{D}}) \subset U$ *and* $\varphi(\partial\mathbb{D}) \subset U$*. Then every* f *holomorphic in* U *can be analytically continued to every point of* $\varphi(\mathbb{D})$*.*

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For $\epsilon > 0$ let X_ϵ denote the ϵ-neighborhood of X. Define

$$\begin{aligned} SADH_q(X) = \{z \in \mathbb{C}^n : \ & \text{for each } \epsilon > 0, \exists \text{ a continuous family of discs } \varphi_t : \overline{\mathbb{D}} \to \mathbb{C}^n, \\ & t \in [0,1],\ z = \varphi_1(0),\ \varphi_t(\partial\mathbb{D}) \subset X_\epsilon\ \forall t \in [0,1],\ \varphi_0 \equiv q, \text{ and} \\ & \|\varphi_t(0) - q\| \text{ is a strictly increasing function of } t\}, \end{aligned}$$

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Remark: Without the shrinking hypothesis the result fails.

Fun little fact (appeared in Minsker '76):
Every continuous function on the closed unit disc $\overline{\mathbb{D}}$ can be written as a uniform limit of polynomials in z and $\bar{z}^2$, that is, $P_\ell(z, \bar{z}^2)$.
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This is like showing that $CR = CR_P$ on $w = \bar{z}^2$.

On the other hand, $\bar{z}$ is not the limit of polynomials of the form $P_\ell(z, z\bar{z})$.
($CR \neq CR_P$ on $w = |z|^2$).

Thanks for listening!