Severi's theorem for codimension two CR singular submanifolds of $\mathbb{C}^3$

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joint work with Alan Noell and Sivaguru Ravisankar

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Complexification

Let $\mathbb{R}^n \subset \mathbb{C}^n$ be the natural embedding (that is $\operatorname{Im} z = 0$).

Suppose $M \subset \mathbb{R}^n$ is a domain and $f\colon M \to \mathbb{C}$ is real-analytic.

$\Rightarrow \exists$ a domain $V \subset \mathbb{C}^n$, $M \subset V$, and $F\colon V \to \mathbb{C}$ holomorphic such that $F|_M = f$. (We say f *extends holomorphically*)

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May not work if M is another submanifold. Two examples:

(a) Consider $M = \{z \in \mathbb{C}^2 \mid \operatorname{Im} z_2 = 0\}$, $f\colon M \to \mathbb{C}$ given by $f(z) = \operatorname{Re} z_1$.
$\Rightarrow$ f does not extend holomorphically.

(b) Consider $M = \{z \in \mathbb{C}^2 \mid z_2 = |z_1|^2\}$, $f\colon M \to \mathbb{C}$ given by $f(z) = \bar{z}_1$.
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Note: all my submanifolds are embedded, all issues considered are local, and everything is real-analytic.

CR vectors

Let $M \subset \mathbb{C}^n$ be a submanifold, write

$$T_p^{0,1}M = \left(\mathbb{C} \otimes T_pM\right) \cap \operatorname{span}_{\mathbb{C}} \left\{ \frac{\partial}{\partial \bar{z}_1}\bigg|_p, \ldots, \frac{\partial}{\partial \bar{z}_n}\bigg|_p \right\}$$

Def.: M is CR if

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$\therefore$ if M is CR, $\Rightarrow L(F|_M) = 0 \; \forall L \in \Gamma(T^{0,1} M)$.

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Theorem (Severi)

Suppose $M \subset \mathbb{C}^n$ *is a real-analytic CR submanifold and* $f\colon M \to \mathbb{C}$ *is a real-analytic CR function.*
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Step 5) Profit!

Example:

Suppose $M \subset \mathbb{C}^2$ is a real-analytic real hypersurface.

Write M as

$$\bar{w} = \Phi(z, \bar{z}, w),$$

and consider a real-analytic CR function $f(z, \bar{z}, w, \bar{w})$.
Treat $\bar{z}$ and $\bar{w}$ as independent.

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Write

$$F(z, \bar{z}, w) = f(z, \bar{z}, w, \Phi(z, \bar{z}, w))$$

Find CR vector field:

$$L = \frac{\partial}{\partial \bar{z}} + \frac{\partial \Phi}{\partial \bar{z}} \frac{\partial}{\partial \bar{w}}$$

$LF = 0 \qquad \Rightarrow$

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Done! (there's a technicality or two in there, but overall that's the idea)

CR singular submanifolds

Def.: If M is not CR, then M is CR singular.

Def.: $f\colon M \to \mathbb{C}$ is CR if $Lf = 0$ for all vector fields L that are CR ($L|_p \in T_p^{0,1} M \ \forall p \in M$).

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There are no CR vector fields,
and you can't easily solve for $\bar{z}$ and $\bar{w}$ in terms of z and w:

$$\bar{z} = \frac{w}{z}, \qquad \bar{w} = w$$

Extra conditions needed on f here (codimension 2 in $\mathbb{C}^2$).

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We will switch to $\mathbb{C}^3$, where there is an actual CR vector field.

Some previous work

Harris ('78) provides a complete (but difficult to apply) criterion for f on an arbitrary CR singular M to be a restriction of a holomorphic function in C^ω case.

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In '11 we (L.–Minor–Shroff–Son–Zhang) proved that if a real-analytic CR singular manifold $M = \varphi(N)$ for a real-analytic CR map

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of a CR submanifold N, and φ is a diffeomorphism onto $\varphi(N) = M$, then there exists a real-analytic CR function on M that does not extend holomorphically.

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In '16 we (L.–Noell–Ravisankar) proved that a real-analytic codimension 2 real-analytic CR singular manifold in $\mathbb{C}^n$ $(n \geq 3)$ that is flat (subset of $\mathbb{C}^{n-1} \times \mathbb{R}$) and nondegenerate has the extension property.

Main theorem setup

A CR singular submanifold of codimension 2 in $\mathbb{C}^3$ is written as (after a rotation by a unitary)

$$\begin{aligned} w &= \rho(z, \bar{z}) \\ &= Q(z, \bar{z}) + E(z, \bar{z}) \\ &= z^* A z + \overline{z^t B z} + z^t C z + E(z, \bar{z}), \end{aligned}$$

$(z, w) \in \mathbb{C}^2 \times \mathbb{C}$, $\quad \rho$ is $O(\|z\|)^2$, $\quad E$ is $O(\|z\|^3)$.

A, B, C, 2×2 complex matrices,
z column vector,
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A, B, C, 2×2 complex matrices,
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B, C symmetric.

M can be parametrized by z (and $\bar{z}$)

Normal forms

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This might be a good place to note that normal forms for codimension 2 CR singular manifolds has a long history:

$\mathbb{C}^2$: Bishop '65, Moser–Webster '83, Moser '85, Kenig–Webster '82, Gong '94, Huang–Krantz '95, Huang–Yin '09, Slapar '16, etc...

$\mathbb{C}^n$ $(n \geq 3)$ Dolbeault–Tomassini–Zaitsev '05, '11, Huang–Yin '09, '16, '17, Burcea '13, Gong–L. '15, Fang–Huang '18.

Coffman's table

Table 1. Normal forms for Theorem 7.1

N	P			
$\begin{pmatrix}1&0\\0&e^{i\theta}\end{pmatrix}$ $0<\theta<\pi$	$\begin{pmatrix}a&b\\b&d\end{pmatrix}$	5	$a>0, d>0, b\sim -b\in\mathbb{C}$	$+-0$
	$\begin{pmatrix}0&b\\b&d\end{pmatrix}$	3	$b\geq 0, d\geq 0$	$+-0$
	$\begin{pmatrix}a&b\\b&0\end{pmatrix}$	3	$a>0, b\geq 0$	$+-0$
$\begin{pmatrix}1&0\\0&1\end{pmatrix}$	$\begin{pmatrix}a&0\\0&d\end{pmatrix}$	2	$0\leq a\leq d$	$+-0$
$\begin{pmatrix}1&0\\0&-1\end{pmatrix}$	$\begin{pmatrix}a&0\\0&d\end{pmatrix}$	2	$0\leq a\leq d$	$+-0$
	$\begin{pmatrix}0&b\\b&0\end{pmatrix}$	1	$b>0$	$+$
	$\begin{pmatrix}1&1\\1&1\end{pmatrix}$	0		$+$
$\begin{pmatrix}0&1\\1&0\end{pmatrix}$	$\begin{pmatrix}0&b\\b&1\end{pmatrix}$	1	$b>0$	$+0$
	$\begin{pmatrix}1&0\\0&d\end{pmatrix}$	2	$\mathrm{Im}(d)>0$	$+$
$\begin{pmatrix}0&1\\\tau&0\end{pmatrix}$ $0<\tau<1$	$\begin{pmatrix}a&b\\b&d\end{pmatrix}$	5	$b>0, \lvert a\rvert=1, (a,d)\sim(-a,-d)$	$+-0$
	$\begin{pmatrix}0&b\\b&d\end{pmatrix}$	3	$b>0, \lvert d\rvert=1, d\sim -d$	$+-0$
	$\begin{pmatrix}0&b\\b&0\end{pmatrix}$	2	$b>0$	$+-0$
	$\begin{pmatrix}1&0\\0&d\end{pmatrix}$	3	$d\in\mathbb{C}$	$+0$
	$\begin{pmatrix}0&0\\0&1\end{pmatrix}$	1		$+$
	$\begin{pmatrix}0&0\\0&0\end{pmatrix}$	1		$+$
$\begin{pmatrix}0&1\\0&0\end{pmatrix}$	$\begin{pmatrix}a&b\\b&1\end{pmatrix}$	3	$b>0, a\in\mathbb{C}$	$+-0$
	$\begin{pmatrix}1&b\\b&0\end{pmatrix}$	1	$b>0$	$+-0$
	$\begin{pmatrix}0&b\\b&0\end{pmatrix}$	1	$b>0$	$+-0$
	$\begin{pmatrix}a&0\\0&1\end{pmatrix}$	1	$a\geq 0$	$+0$
	$\begin{pmatrix}1&0\\0&0\end{pmatrix}$	0		0
	$\begin{pmatrix}0&0\\0&0\end{pmatrix}$	0		0
$\begin{pmatrix}0&1\\1&i\end{pmatrix}$	$\begin{pmatrix}a&0\\0&d\end{pmatrix}$	3	$a>0, d\in\mathbb{C}$	$+-0$
	$\begin{pmatrix}0&b\\b&0\end{pmatrix}$	1	$b>0$	$+0$
	$\begin{pmatrix}0&0\\0&d\end{pmatrix}$	1	$d\geq 0$	$+$
$\begin{pmatrix}1&0\\0&0\end{pmatrix}$	$\begin{pmatrix}a&0\\0&1\end{pmatrix}$	1	$a\geq 0$	$+-0$
	$\begin{pmatrix}0&1\\1&0\end{pmatrix}$	0		$+$
	$\begin{pmatrix}a&0\\0&0\end{pmatrix}$	1	$a\geq 0$	0
$\begin{pmatrix}0&0\\0&0\end{pmatrix}$	$\begin{pmatrix}1&0\\0&1\end{pmatrix}$	0		$+$
	$\begin{pmatrix}1&0\\0&0\end{pmatrix}$	0		0
	$\begin{pmatrix}0&0\\0&0\end{pmatrix}$	0		0

Main theorem

$M \subset \mathbb{C}^3$, $(z, w) \in \mathbb{C}^2 \times \mathbb{C}$

$M : w = \rho(z, \bar{z}) = Q(z, \bar{z}) + E(z, \bar{z}) = z^* Az + \overline{z^t Bz} + z^t Cz + E(z, \bar{z})$

Theorem (L.–Noell–Ravisankar)

Suppose

$$\text{null } A^* \cap \text{null } B = \{0\}$$

If $f(z, \bar{z})$ is real-analytic CR function defined near the origin, then f extends holomorphically near the origin. That is, $\exists$ $F(z, w)$ such that

$$f(z, \bar{z}) = F(z, \rho(z, \bar{z})).$$

The quadric

Theorem (L.–Noell–Ravisankar)

Suppose $M \subset \mathbb{C}^3$ *is a quadric given by*

$$w = Q(z, \bar{z}) = z^* A z + \overline{z^t B z} + z^t C z$$

Assume $\bar{\partial} Q \not\equiv 0$. *TFAE:*

(a) $\operatorname{null} A^* \cap \operatorname{null} B = \{0\}$

(b) *For every CR polynomial* $f(z, \bar{z})$, $\exists!$ *holomorphic polynomial* $F(z, w)$ *such that* $f(z, \bar{z}) = F(z, Q(z, \bar{z}))$.
If f *is homogeneous, then* F *is weighted homogeneous.*

(c) *Every CR real-linear* $h(z, \bar{z})$ *is holomorphic (does not depend on* $\bar{z}$*).*

The difficulty

Consider $f(z, \bar{z})$ on

$$w = z_1^2 + z_2^2 + \bar{z}_1^2 + \bar{z}_2^2 \qquad (B = I, A = 0, E = 0)$$

Solve for say $\bar{z}_1$:

$$\bar{z}_1 = \pm\sqrt{w - z_1^2 + z_2^2 + \bar{z}_2^2}$$

We can get rid of all but the first power of $\bar{z}_1$:

$$f = \alpha(z_1, z_2, w, \bar{z}_2) + \bar{z}_1 \beta(z_1, z_2, w, \bar{z}_2)$$

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$Lf = 0$ (CR vector field) must get rid of not only the dependence on $\bar{z}_2$ in α,
but also force $\beta \equiv 0$.

Optimal conditions

Fix Q, $\bar{\partial} Q \not\equiv 0$, and suppose null $A^* \cap$ null $B \neq \{0\}$.

Let

$$f = \bar{v}_2 \bar{z}_1 - \bar{v}_1 \bar{z}_2,$$

where (v_1, v_2) is a nonzero vector in null $A^* \cap$ null B.

Then f is not a restriction to M of a holomorphic function (in any neighborhood of the origin)

As M is a graph of w over z:

$$w = \rho(z, \bar{z}),$$

write everything on M in terms of z.

A function on M is a function $f(z, \bar{z})$.

The CR vector field in terms of z as a parameter on M is

$$L = \rho_{\bar{z}_2} \frac{\partial}{\partial \bar{z}_1} - \rho_{\bar{z}_1} \frac{\partial}{\partial \bar{z}_2}$$

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Normally to complexify in $\mathbb{C}^3$: we consider $(z, \bar{z}, w, \bar{w})$ in $\mathbb{C}^6$.
But we only need to complexify into $\mathbb{C}^5$ and consider $(z, \bar{z}, w)$.

$\bar\partial Q \equiv 0$

If $\bar\partial Q \equiv 0$, things can be complicated.

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So for some E we may have extension.

$E = \|z\|^4$) M is given by

$$w = \|z\|^4 = \left(|z_1|^2 + |z_2|^2\right)^2$$

and

$$f(z, \bar{z}) = \|z\|^2 = |z_1|^2 + |z_2|^2$$

is CR but equal to $\sqrt{w}$ on M, so does not extend.
So for some E we do not have extension.

null $A^* \cap$ null $B \neq \{0\}$

Suppose null $A^* \cap$ null $B \neq \{0\}$.

$E \equiv 0$) Extension does not hold. E.g. if

$$w = \bar{z}_1 z_2$$

then $\bar{z}_1$ is CR as the CR vector field is $L = -z_2 \frac{\partial}{\partial \bar{z}_2}$

Note: The theorem is an if-and-only-if when $E \equiv 0$.

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Note: The theorem is an if-and-only-if when $E \equiv 0$.

$E \not\equiv 0$) Extension may or may not hold depending on E. E.g. if

$$w = \bar{z}_1 z_2 + \bar{z}_2^3$$

then extension holds (explicit computation), but if

$$w = \bar{z}_1 z_2 + \bar{z}_1^3$$

then extension does not hold ($\bar{z}_1$ again).

Proof outline

The proof has the following outline:

Step 1) Prove theorem for homogeneous polys. and quadrics.

Step 2) Prove a formal extension theorem.

Step 3) Prove that in $\mathbb{C}^2$ a formal solution is convergent.

Step 4) Use this to prove convergence of F in $\mathbb{C}^3$.

Proof sketch for the quadrics I

Suppose $\bar{\partial} Q \not\equiv 0$ and M is a quadric:
$w = Q(z, \bar{z}) = z^* A z + \overline{z^t B z} + z^t C z.$

Suppose $f(z, \bar{z})$ is a homogeneous polynomial that is CR.

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There are two cases:

$B \neq 0$) Let's tackle that one first.

$B = 0$) Special case, needs to be handled differently.

Proof sketch for the quadrics II ($B \neq 0$)

If $B \neq 0$, then it can be diagonalized by a transformation in z:

$$Q(z, \bar{z}) = z^* A z + \bar{z}_1^2 + \epsilon \bar{z}_2^2 + z^t C z$$

where $\epsilon = 0, 1$.

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Weierstrass division algorithm (using $\bar{z}_1$) says

$$f(z, \bar{z}) = h(z, \bar{z}, w)\big(Q(z, \bar{z}) - w\big) + \alpha(z, \bar{z}_2, w) + \beta(z, \bar{z}_2, w)\bar{z}_1.$$

The remainder in Weierstrass is unique:
Any equality on M, as long $\bar{z}_1$ appears up to first power holds everywhere.

Proof sketch for the quadrics II ($B \neq 0$)

If $B \neq 0$, then it can be diagonalized by a transformation in z:

$$Q(z, \bar{z}) = z^* Az + \bar{z}_1^2 + \epsilon \bar{z}_2^2 + z^t Cz$$

where $\epsilon = 0, 1$.

Weierstrass division algorithm (using $\bar{z}_1$) says

$$f(z, \bar{z}) = h(z, \bar{z}, w)\big(Q(z, \bar{z}) - w\big) + \alpha(z, \bar{z}_2, w) + \beta(z, \bar{z}_2, w)\bar{z}_1.$$

The remainder in Weierstrass is unique:
Any equality on M, as long $\bar{z}_1$ appears up to first power holds everywhere.

f equals a holomorphic polynomial $g(z, w)$ if and only if $\alpha + \beta \bar{z}_1 - g \equiv 0$, or in other words if

$$\alpha_{\bar{z}_2} \equiv 0, \qquad \text{and} \qquad \beta \equiv 0.$$

Proof sketch for the quadrics III ($B \neq 0$)

Solving $Lf = 0$ we get differential equations for α and β, in fact a single equation for β.

Then it is an almost-undergraduate-first-order-DE computation to find for which coefficients in A (and B) do we get a polynomial solution.

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QED!

Proof sketch for the quadrics IV ($B = 0$)

Let L be the CR vector field.

Proof outline.

Step 1) For each degree d, compute L as a matrix taking homogeneous polynomials of fixed degree to themselves.

Step 2) Compute the dimension of the kernel of L for each d.

Step 3) Compute the dimension of weighted homogeneous polynomials $F(z, w)$ of degree d.

Step 4) ... the two dimensions match!
(if the nullspace condition is met)

Step 5) QED!

Matrix for L in degree 3

	$\bar z_1^3$	$\bar z_1^2\bar z_2$	$\bar z_1\bar z_2^2$	$\bar z_2^3$	$z_1\bar z_1^2$	$z_1\bar z_1\bar z_2$	$z_1\bar z_2^2$	$z_2\bar z_1^2$	$z_2\bar z_1\bar z_2$	$z_2\bar z_2^2$	$z_1^2\bar z_1$	$z_1^2\bar z_2$	$z_1z_2\bar z_1$	$z_1z_2\bar z_2$	$z_2^2\bar z_1$	$z_2^2\bar z_2$	z_1^3	$z_1^2z_2$	$z_1z_2^2$	z_2^3
$\bar z_1^3$	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
$\bar z_1^2\bar z_2$	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
$\bar z_1\bar z_2^2$	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
$\bar z_2^3$	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
$z_1\bar z_1^2$	.	-1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
$z_1\bar z_1\bar z_2$	.	.	-2	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
$z_1\bar z_2^2$	.	.	.	-3	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
$z_2\bar z_1^2$	3δ	$-\beta$	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
$z_2\bar z_1\bar z_2$	.	2δ	-2β	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
$z_2\bar z_2^2$	.	.	δ	-3β	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
$z_1^2\bar z_1$	.	.	.	.	.	-1	.	.	.	.	.	.	.	.	.	.	.	.	.	.
$z_1^2\bar z_2$	.	.	.	.	.	.	-2	.	.	.	.	.	.	.	.	.	.	.	.	.
$z_1z_2\bar z_1$	.	.	.	.	2δ	$-\beta$	.	.	-1	.	.	.	.	.	.	.	.	.	.	.
$z_1z_2\bar z_2$	.	.	.	.	.	δ	-2β	.	.	-2	.	.	.	.	.	.	.	.	.	.
$z_2^2\bar z_1$	.	.	.	.	.	.	.	2δ	$-\beta$	.	.	.	.	.	.	.	.	.	.	.
$z_2^2\bar z_2$	.	.	.	.	.	.	.	.	δ	-2β	.	.	.	.	.	.	.	.	.	.
z_1^3	.	.	.	.	.	.	.	.	.	.	.	-1	.	.	.	.	.	.	.	.
$z_1^2z_2$	.	.	.	.	.	.	.	.	.	.	δ	$-\beta$	.	-1	.	.	.	.	.	.
$z_1z_2^2$	.	.	.	.	.	.	.	.	.	.	.	.	δ	$-\beta$	.	-1	.	.	.	.
z_2^3	.	.	.	.	.	.	.	.	.	.	.	.	.	.	δ	$-\beta$	.	.	.	.

One subblock, degree 9

	$z_1^5\bar z_1^4$	$z_1^5\bar z_1^3\bar z_2$	$z_1^5\bar z_1^2\bar z_2^2$	$z_1^5\bar z_1\bar z_2^3$	$z_1^5\bar z_2^4$	$z_1^4z_2\bar z_1^4$	$z_1^4z_2\bar z_1^3\bar z_2$	$z_1^4z_2\bar z_1^2\bar z_2^2$	$z_1^4z_2\bar z_1\bar z_2^3$	$z_1^4z_2\bar z_2^4$	$z_1^3z_2^2\bar z_1^4$	$z_1^3z_2^2\bar z_1^3\bar z_2$	$z_1^3z_2^2\bar z_1^2\bar z_2^2$	$z_1^3z_2^2\bar z_1\bar z_2^3$	$z_1^3z_2^2\bar z_2^4$	$z_1^2z_2^3\bar z_1^4$	$z_1^2z_2^3\bar z_1^3\bar z_2$	$z_1^2z_2^3\bar z_1^2\bar z_2^2$	$z_1^2z_2^3\bar z_1\bar z_2^3$	$z_1^2z_2^3\bar z_2^4$	$z_1z_2^4\bar z_1^4$	$z_1z_2^4\bar z_1^3\bar z_2$	$z_1z_2^4\bar z_1^2\bar z_2^2$	$z_1z_2^4\bar z_1\bar z_2^3$	$z_1z_2^4\bar z_2^4$	$z_2^5\bar z_1^4$	$z_2^5\bar z_1^3\bar z_2$	$z_2^5\bar z_1^2\bar z_2^2$	$z_2^5\bar z_1\bar z_2^3$	$z_2^5\bar z_2^4$
$z_1^6\bar z_1^3$	.	[*]	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
$z_1^6\bar z_1^2\bar z_2$	.	.	[*]	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
$z_1^6\bar z_1\bar z_2^2$	.	.	.	[*]	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
$z_1^6\bar z_2^3$	.	.	.	.	[*]	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
$z_1^5z_2\bar z_1^3$	[*]	?	.	.	.	.	*	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
$z_1^5z_2\bar z_1^2\bar z_2$	.	*	?	.	.	.	.	[*]	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
$z_1^5z_2\bar z_1\bar z_2^2$	.	.	*	?	.	.	.	.	[*]	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
$z_1^5z_2\bar z_2^3$	.	.	.	*	?	.	.	.	.	[*]	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
$z_1^4z_2^2\bar z_1^3$	.	.	.	.	.	[*]	?	.	.	.	.	*	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
$z_1^4z_2^2\bar z_1^2\bar z_2$	.	.	.	.	.	.	[*]	?	.	.	.	.	*	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
$z_1^4z_2^2\bar z_1\bar z_2^2$	.	.	.	.	.	.	.	*	?	.	.	.	.	[*]	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
$z_1^4z_2^2\bar z_2^3$	.	.	.	.	.	.	.	.	*	?	.	.	.	.	[*]	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
$z_1^3z_2^3\bar z_1^3$	.	.	.	.	.	.	.	.	.	.	[*]	?	.	.	.	.	*	.	.	.	.	.	.	.	.	.	.	.	.	.
$z_1^3z_2^3\bar z_1^2\bar z_2$	.	.	.	.	.	.	.	.	.	.	.	[*]	?	.	.	.	.	*	.	.	.	.	.	.	.	.	.	.	.	.
$z_1^3z_2^3\bar z_1\bar z_2^2$	.	.	.	.	.	.	.	.	.	.	.	.	[*]	?	.	.	.	.	*	.	.	.	.	.	.	.	.	.	.	.
$z_1^3z_2^3\bar z_2^3$	.	.	.	.	.	.	.	.	.	.	.	.	.	*	?	.	.	.	.	[*]	.	.	.	.	.	.	.	.	.	.
$z_1^2z_2^4\bar z_1^3$	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	[*]	?	.	.	.	.	*	.	.	.	.	.	.	.	.
$z_1^2z_2^4\bar z_1^2\bar z_2$	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	[*]	?	.	.	.	.	*	.	.	.	.	.	.	.
$z_1^2z_2^4\bar z_1\bar z_2^2$	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	[*]	?	.	.	.	.	*	.	.	.	.	.	.
$z_1^2z_2^4\bar z_2^3$	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	[*]	?	.	.	.	.	*	.	.	.	.	.
$z_1z_2^5\bar z_1^3$	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	[*]	?	.	.	.	.	*	.	.	.
$z_1z_2^5\bar z_1^2\bar z_2$	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	[*]	?	.	.	.	.	*	.	.
$z_1z_2^5\bar z_1\bar z_2^2$	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	[*]	?	.	.	.	.	*	.
$z_1z_2^5\bar z_2^3$	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	[*]	?	.	.	.	.	*
$z_2^6\bar z_1^3$	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	[*]	?	.	.	.
$z_2^6\bar z_1^2\bar z_2$	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	[*]	?	.	.
$z_2^6\bar z_1\bar z_2^2$	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	[*]	?	.
$z_2^6\bar z_2^3$	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	[*]	?

Thanks for listening!