Levi-flat Plateau Problem

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In $\mathbb{R}^3$ a minimal surface is an isometric immersion of a Riemann surface using harmonic functions. (That sounds like complex analysis is involved !)

Problem: Given $M \subset \mathbb{C}^m = \mathbb{R}^{2m}$ of real dimension $2p - 1$, find a complex manifold (or variety) H of complex dimension p such that the boundary of H is M ...

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Harvey–Lawson '75: Not possible in general, but in the right sense (in the sense of currents) and under some natural condition on M, it is true.

Simple example: Analytic disc with smooth boundary

Consider a smooth

$$f\colon S^1 \to \mathbb{C}^m$$

Is there an analytic disc with boundary $f(S^1)$? That is, is there

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We solve the Dirichlet problem, and for F to be holomorphic we need all the negative Fourier coefficients of f to be zero:

$$\int_{S^1} f(z) z^k \, dz = 0$$

for all $k = 0, 1, 2, 3, \ldots$

So $M = f(S^1)$ was given as an image of a subset of $\mathbb{C}$ and by extending the function to all of $\overline{\mathbb{D}}$ we found that $H = F(\overline{\mathbb{D}})$ is our solution.

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Singularities might crop up even if M is not singular:

$$f(z) = F(z) = (z^2, z^3)$$

Then $M = f(S^1)$ is a nice smooth curve, but $F(\mathbb{D})$ is a cusp.

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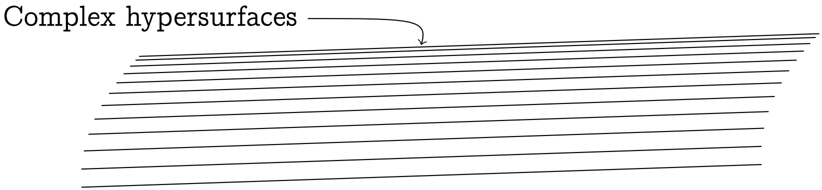

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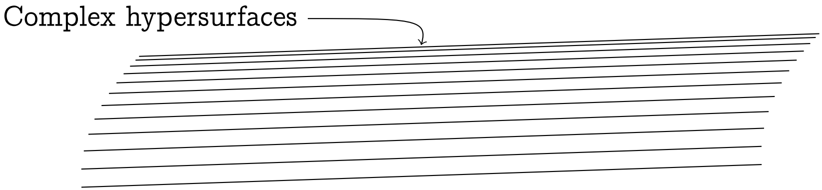

A simple example: $\mathbb{C}^n \times \mathbb{R} \subset \mathbb{C}^{n+1}$.

Given $M \subset \mathbb{C}^{n+1}$ a compact real codimension 2 submanifold, is there a Levi-flat hypersurface H with boundary M?

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In $n \geq 2$, Dolbeault–Tomassini–Zaitsev ('05 and '11) found a possibly singular solution given some conditions on M (elliptic CR singular points, nowhere minimal at CR points).

The nowhere-minimality is necessary, the ellipticity is not.

If $M \subset \mathbb{C}^{n+1}$ is a real submanifold, the CR vectors are

$$T_p^{0,1} M = \operatorname{span}_{\mathbb{C}} \left\{ \frac{\partial}{\partial \bar{z}_k}, \frac{\partial}{\partial \bar{w}} \right\} \cap \mathbb{C} \otimes T_p M$$

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Real hypersurfaces in $\mathbb{C}^{n+1}$ are always CR submanifolds of CR dimension n.

Real codimension two submanifolds generically have isolated CR singularities.

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A hypersurface is Levi-flat iff it is nowhere minimal.
The CR orbits then give a foliation by complex hypersurfaces.

CR boundaries of Levi-flats

Suppose $M = \partial H \subset \mathbb{C}^{n+1}$ for a Levi-flat hypersurface H.

If M is CR, it is of CR dimension n (M is complex) or $n - 1$.

If M is compact, it cannot be complex.

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$\Rightarrow$

M is nowhere minimal.

Question

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Note that when $n = 1$ every codimension 2 real submanifold is nowhere minimal (it is totally real).

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Near a CR singular point? Yes ($n \geq 2$) if the CR singularity is nondegenerate (or an exceptional case), Fang–Huang '17.

In $n = 1$, not always. Yes if the CR singularity is e.g. elliptic. (e.g. Bishop '65, Moser–Webster '82, Moser '85, Huang–Yin '09 ... lots of others)

CR singularity

A codimension 2 CR singular submanifold M is locally

$$w = \rho(z, \bar{z}) = A(z, \bar{z}) + B(z, z) + \overline{B(z, z)} + O(\|z\|^3)$$

$(z, w) \in \mathbb{C}^n \times \mathbb{C}$, A sesquilinear, B bilinear.

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To be locally boundary of a Levi-flat hypersurface, we need to have, after a change of variables, A to be real-valued (Hermitian) and also the "$O(\|z\|^3)$" to be real valued.

A theorem

Theorem (L.–Noell-Ravisankar '17, '18)

Let $\Omega \subset \mathbb{C}^n \times \mathbb{R}$, $n \geq 2$, be a bounded domain with connected real-analytic boundary such that $\partial\Omega$ has only A-nondegenerate CR singularities. Let $\Sigma \subset \partial\Omega$ be the set of CR singularities of $\partial\Omega$. Let $f : \partial\Omega \to \mathbb{C}^{n+1}$ be a real-analytic embedding that is CR at CR points of $\partial\Omega$ and takes CR points of $\partial\Omega$ to CR points of $f(\partial\Omega)$.

Then, there exists a real-analytic CR map $F : \overline{\Omega} \to \mathbb{C}^{n+1}$ such that $F|_{\partial\Omega} = f$ and $F|_{\overline{\Omega}\setminus\Sigma}$ is an immersion.

In other words, $F(\overline{\Omega})$ is the solution of the Levi-flat Plateau problem for $f(\partial\Omega)$.

Proof? (simplified)

Work along “leaves”, extend $f(\cdot, s)$ using Hartogs–Bochner (really Martinelli), or Severi and then Hartogs.

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Prove regularity at CR points and at the CR singularities of M.

The Jacobian of F vanishes on too large of a set contradicting f being a diffeomorphism.

A better result via Fang–Huang

We get a better result if $f(\partial\Omega)$ also has only nondegenerate singularities by applying Fang–Huang.

Corollary

Let $\Omega \subset \mathbb{C}^n \times \mathbb{R}$, $n \geq 2$, be a bounded domain with connected real-analytic boundary such that $\partial\Omega$ has only A-nondegenerate CR singularities, and let $f \colon \partial\Omega \to \mathbb{C}^{n+1}$ be a real-analytic embedding that is CR at CR points of $\partial\Omega$. Assume that $f(\partial\Omega)$ has only A-nondegenerate CR singularities. Further assume that either $n \geq 3$ or no CR singularity of $f(\partial\Omega)$ is the exceptional case (every CR singularity has an elliptic direction).

Then, there exists a real-analytic CR map $F \colon \overline{\Omega} \to \mathbb{C}^{n+1}$ such that $F|_{\partial\Omega} = f$ and F is an immersion on $\overline{\Omega}$.

(exceptional case:
$w = |z_1|^2 - |z_2|^2 + \lambda(z_1^2 + \bar{z}_1^2) + \lambda(z_2^2 + \bar{z}_2^2) + O(\|z\|^3)$, $\lambda \geq \frac{1}{2}$)

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$F(z, s) = (z, zs)$ is even worse

($F(N)$ is degenerate in both cases)

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This is impossible when $n \geq 2$, and $F(N)$ is A-nondegenerate.

$N \subset \mathbb{C}^2 \times \mathbb{R}$: $\qquad s = z_1 + \bar{z}_1 + |z_2|^2$

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The singularity of $F(N)$ is degenerate!

Examples...

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In $(\xi, \sigma) \in \mathbb{C}^n \times \mathbb{C}$,

$F(N)$: $\qquad \sigma = \|\xi\|^4$

$F(N)$ is CR singular and degenerate in every sense.

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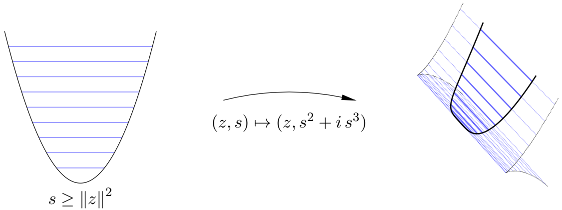
$F(N)$: $\sigma = \|\xi\|^4$ and $\tau = \|\xi\|^6$



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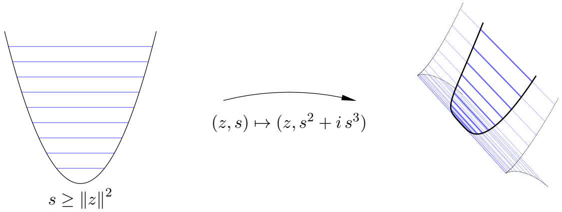

$F(N)$ is degenerate, and the singular $\{\sigma^3 = \tau^2\}$ is the unique Levi-flat that contains $F(N)$.

$F(N)$ is an example of the necessity of nondegeneracy in Fang–Huang.

Examples... (now think globally)

$\Omega \subset \mathbb{C}^n \times \mathbb{R}$: $\quad \|z\|^2 + (s+\epsilon)^2 < 1$

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In $(\xi, \sigma) \in \mathbb{C}^n \times \mathbb{R}$,
$F(\partial\Omega)$ is $4\epsilon^2\sigma = (1 - \epsilon^2 - \|\xi\|^2 - \sigma)^2$

$F|_{\partial\Omega}$ is a diffeomorphism,

$F(\partial\Omega)$ has CR singularities at

$\xi = 0, \quad 4\epsilon^2\sigma = (1 - \epsilon^2 - \sigma)^2$ (isolated)

$\sigma = 0$ and $\|\xi\|^2 = 1 - \epsilon^2$ (not isolated)

but ...

Examples... (now think globally)

$\Omega \subset \mathbb{C}^n \times \mathbb{R}$: $\qquad \|z\|^2 + (s+\epsilon)^2 < 1$

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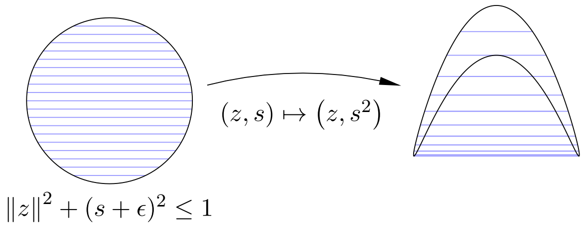
but ... F is not 1-1 on Ω!



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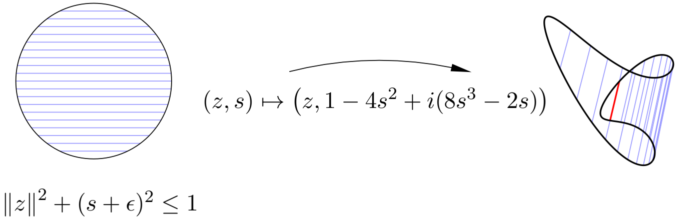

A final example ...

Let H in $(\xi, \eta) \in \mathbb{C}^2$ be defined by ($\epsilon > 0$ small)

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H is **not** an image of a domain in $\mathbb{C} \times \mathbb{R}$!

(There is noting special about $\mathbb{C}^2$ here).

Thank you