Levi-flat Plateau Problem

Jiří Lebl
joint work with Alan Noell and Sivaguru Ravisankar

Department of Mathematics, Oklahoma State University

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In $\mathbb{R}^3$ a minimal surface is an isometric immersion of a Riemann surface using harmonic functions. (That sounds like complex analysis is involved !)

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Then we ask for

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We solve the Dirichlet problem, and for F to be holomorphic we need all the negative Fourier coefficients of f to be zero:

$$\int_{S^1} f(z) z^k \, dz = 0$$

for all $k = 0, 1, 2, 3, \ldots$

Levi-flat as a "minimal surface"

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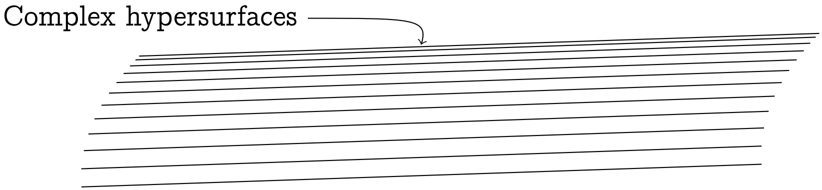
Levi-flat H:
1) Levi-form zero
2) Solution to a Monge-Ampère like equation
3) Foliated by complex hypersurfaces:



A simple example: $\mathbb{C}^n \times \mathbb{R} \subset \mathbb{C}^{n+1}$.

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In $n \geq 2$, Dolbeault–Tomassini–Zaitsev ('05 and '11) found a possibly singular solution given some conditions on M (elliptic CR singular points, CR orbits of codimension 1 so nowhere minimal).

The nowhere-minimality is necessary, the ellipticity is not.

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If $\dim T_p^{0,1} M$ is not constant in any neighborhood of q, q is a *CR singularity*.

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Real codimension two: generically isolated CR singularities

A CR M is *nowhere minimal* if through each point $p \in M$ there is a smaller CR submanifold $P_p \subset M$ with the same CR dimension.

CR boundaries of Levi-flats

Suppose $M = \partial H \subset \mathbb{C}^{n+1}$ for a Levi-flat hypersurface H.

If M is CR, it is of CR dimension n (M is complex) or $n-1$.

As M is compact, it is not complex.

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$\Rightarrow$ M is nowhere minimal.

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Near a CR singular point? Yes ($n \geq 2$) if the CR singularity is nondegenerate (or an exceptional case), Fang–Huang '17.

In $n = 1$, not always. Yes if the CR singularity is e.g. elliptic. (e.g. Bishop '65, Moser–Webster '82, Moser '85, Huang–Yin '09 ... lots of others)

A codimension 2 CR singular submanifold M is locally

$$w = \rho(z, \bar{z}) = A(z, \bar{z}) + B(z, z) + \overline{B(z, z)} + O(\|z\|^3)$$

$(z, w) \in \mathbb{C}^n \times \mathbb{C}$, A sesquilinear, B bilinear.

CR singularity

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To be locally boundary of a Levi-flat hypersurface, we need to have, after a change of variables, A to be real-valued (Hermitian) and also the "$O(\|z\|^3)$" to be real-valued.

A theorem

Theorem (L.–Noell-Ravisankar ’17, ’18)

Let $\Omega \subset \mathbb{C}^n \times \mathbb{R}$, $n \geq 2$, be a bounded domain with connected real-analytic boundary such that $\partial\Omega$ has only A-nondegenerate CR singularities. Let $\Sigma \subset \partial\Omega$ be the set of CR singularities of $\partial\Omega$. Let $f \colon \partial\Omega \to \mathbb{C}^{n+1}$ be a real-analytic embedding that is CR at CR points of $\partial\Omega$ and takes CR points of $\partial\Omega$ to CR points of $f(\partial\Omega)$.

Then, there exists a real-analytic CR map $F \colon \overline{\Omega} \to \mathbb{C}^{n+1}$ such that $F|_{\partial\Omega} = f$ and $F|_{\overline{\Omega}\setminus\Sigma}$ is an immersion.

In other words, $H = F(\overline{\Omega})$ is the solution of the Levi-flat Plateau problem for $M = f(\partial\Omega)$.

Proof? (simplified)

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Prove regularity at CR points and at the CR singularities of M.

If the Jacobian determinant of F is zero somewhere, then it vanishes on too large of a set (if $n \geq 2$) contradicting f being a diffeomorphism.

A better result via Fang–Huang

We get a better result if $f(\partial\Omega)$ also has only nondegenerate singularities by applying Fang–Huang.

Corollary

Let $\Omega \subset \mathbb{C}^n \times \mathbb{R}$, $n \geq 2$, be a bounded domain with connected real-analytic boundary such that $\partial\Omega$ has only A-nondegenerate CR singularities, and let $f \colon \partial\Omega \to \mathbb{C}^{n+1}$ be a real-analytic embedding that is CR at CR points of $\partial\Omega$. Assume that $f(\partial\Omega)$ has only A-nondegenerate CR singularities. Further assume that either $n \geq 3$ or no CR singularity of $f(\partial\Omega)$ is the exceptional case (every CR singularity has an elliptic direction).

Then, there exists a real-analytic CR map $F \colon \overline{\Omega} \to \mathbb{C}^{n+1}$ such that $F|_{\partial\Omega} = f$ and F is an immersion on $\overline{\Omega}$.

(exceptional case:
$w = |z_1|^2 - |z_2|^2 + \lambda(z_1^2 + \bar{z}_1^2) + \lambda(z_2^2 + \bar{z}_2^2) + O(\|z\|^3)$, $\lambda \geq \frac{1}{2}$)

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$F(z, s) = (z, zs)$ is even worse

($F(N)$ is degenerate in both cases)

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This is impossible when $n \geq 2$ and $F(N)$ is A-nondegenerate.

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$F(z, s) = (z, s^2 + is^3)$

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The singularity of $F(N)$ is degenerate!

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In $(\xi, \sigma) \in \mathbb{C}^n \times \mathbb{C}$,

$F(N)$: $\qquad \sigma = \|\xi\|^4$

$F(N)$ is CR singular and degenerate in every sense.

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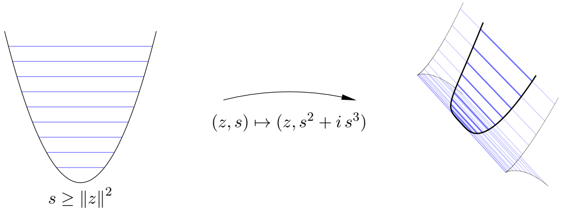

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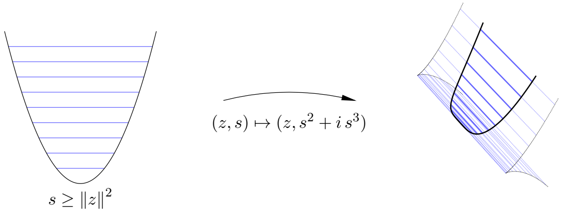

$F(N)$ is degenerate, and the singular $\{\sigma^3 = \tau^2\}$ is the unique Levi-flat that contains $F(N)$.

$F(N)$ is an example of the necessity of nondegeneracy in Fang–Huang.

Examples... (now think globally)

$\Omega \subset \mathbb{C}^n \times \mathbb{R}$: $\quad \|z\|^2 + (s+\epsilon)^2 < 1$

$F(z,s) = (z, s^2)$

In $(\xi, \sigma) \in \mathbb{C}^n \times \mathbb{R}$,
$F(\partial\Omega)$ is $4\epsilon^2\sigma = (1 - \epsilon^2 - \|\xi\|^2 - \sigma)^2$

$F|_{\partial\Omega}$ is a diffeomorphism,

$F(\partial\Omega)$ has CR singularities at

$\xi = 0, \quad 4\epsilon^2\sigma = (1 - \epsilon^2 - \sigma)^2$ (isolated)

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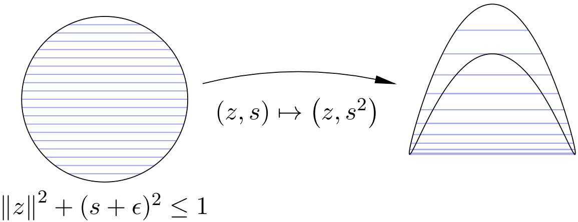
but ... F is not 1-1, nor an immersion!



Examples...

$\Omega \subset \mathbb{C}^n \times \mathbb{R}$: $\qquad \|z\|^2 + (s+\epsilon)^2 < 1$

$F(z,s) = \left(z, 1 - 4s^2 + i(8s^3 - 2s)\right)$

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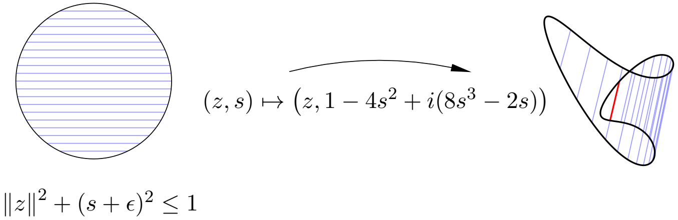

A final example ...

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M has isolated CR singularities at

$\left(0, -\epsilon \pm 1\right), \quad \left(0, \pm i\sqrt{1 - \epsilon^2}\right), \quad \left(\pm i\sqrt{1 - \frac{\epsilon^2}{4}}, \frac{-\epsilon}{2}\right)$

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H is **not** an image of a domain in $\mathbb{C} \times \mathbb{R}$!

(There is nothing special about $\mathbb{C}^2$ here).

Thank you!