Lewy extension for smooth hypersurfaces in $\mathbb{C}^n \times \mathbb{R}$

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joint work with Alan Noell and Sivaguru Ravisankar

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Hypersurface in $\mathbb{C}^n$

Any smooth hypersurface M can be locally written as

$$\operatorname{Im} w = \sum_{j=1}^{n} \epsilon_j \left|z_j\right|^2 + E(z, \bar{z}, \operatorname{Re} w)$$

for $E \in O(3)$, and $\epsilon_j = -1, 0, 1$.
The form $\sum_{j=1}^{n} \epsilon_j \left|z_j\right|^2$ is the *Levi-form* at the origin.

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If $f \in \mathcal{O}(H_+ \setminus M) \cap C^\infty(H_+)$, then f is CR on M, that is, $\nu f = 0$ for every $\nu \in \Gamma(T^{0,1} M)$

Here $T_p^{0,1} M = \operatorname{span}_{\mathbb{C}} \left\{ \frac{\partial}{\partial \bar{z}_k}, \frac{\partial}{\partial \bar{w}} \right\} \cap \mathbb{C} \otimes T_p M$

Lewy extension

$$M: \quad \operatorname{Im} w = \sum_{j=1}^{n} \epsilon_j \left|z_j\right|^2 + E \qquad H_+: \quad \operatorname{Im} w \geq \sum_{j=1}^{n} \epsilon_j \left|z_j\right|^2 + E$$

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Theorem (Lewy extension)

Let $M, H_+ \subset \mathbb{C}^{n+1}$, $n \geq 1$, *be as above.*

Then $\exists$ *a neighborhood* U *of 0, such that given* $f \in CR(M) \cap C^\infty(M)$:

(i) *If the Levi-form at 0 has a positive eigenvalue, then* $\exists$ $F \in C^\infty(U \cap H_+) \cap \mathcal{O}(U \cap H_+ \setminus M)$ *such that* $F|_{M \cap U} = f|_{M \cap U}$

(ii) *If the Levi-form at 0 has eigenvalues of both signs, then* $\exists$ $F \in \mathcal{O}(U)$ *such that* $F|_{M \cap U} = f|_{M \cap U}$

Real-analytic CR functions

Theorem (Severi)

Suppose $M \subset \mathbb{C}^{n+1}$ is a real-analytic hypersurface and $f \in CR(M) \cap C^{\omega}(M)$

Then $\exists$ neighborhood U of M and $F \in \mathcal{O}(U)$ s.t. $F|_{M \cap U} = f|_{M \cap U}$.

Hypersurfaces in $\mathbb{C}^n \times \mathbb{R}$

Write coordinates as $(z, s) \in \mathbb{C}^n \times \mathbb{R}$.

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Call the sets $\mathbb{C}^n \times \{s\}$ the *leaves* of $\mathbb{C}^n \times \mathbb{R}$.
For $X \subset \mathbb{C}^n \times \mathbb{R}$ define $X_{(s)} = \{z \in \mathbb{C}^n \mid (z, s) \in X\}$

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Let $M \subset \mathbb{C}^n \times \mathbb{R}$ be a smooth real hypersurface.

$$T_p^{0,1} M = \operatorname{span}_{\mathbb{C}} \left\{ \frac{\partial}{\partial \bar{z}_k} \right\} \cap \mathbb{C} \otimes T_p M = T_p^{0,1} M_{(s)}$$

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M is CR at p if $\dim T_q^{0,1} M$ is constant on M near p.
Let M_{CR} be the set of CR points of M.

Otherwise M has a CR singularity at p.

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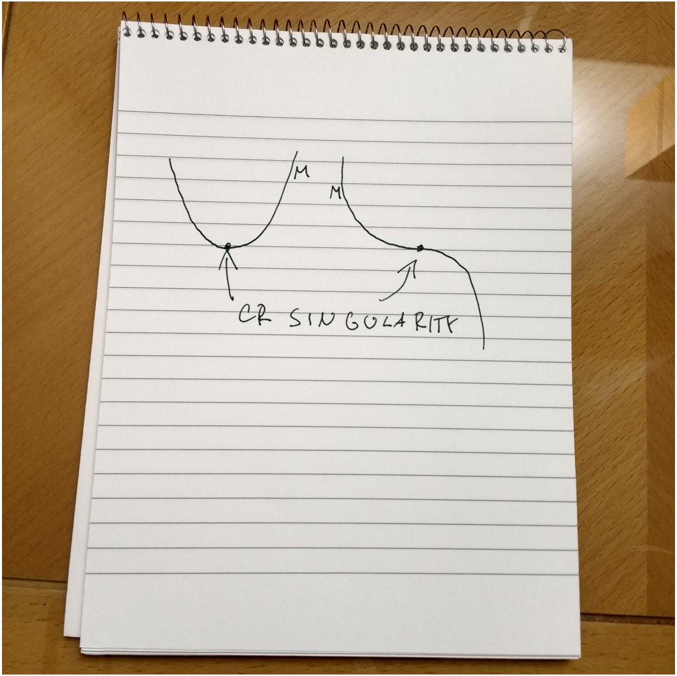
(Equivalently, $\nu f = 0$ for all $\nu \in \Gamma(\mathbb{C} \otimes TM)$ where $\nu_p \in T_p^{0,1} M$ for all p).


M
M
CR SINGULARITY

(Severi strikes again)

Suppose $M \subset \mathbb{C}^n \times \mathbb{R}$ is a real-analytic CR hypersurface and $f \in CR(M) \cap C^\omega(M)$

Then $\exists$ a neighborhood $U \subset \mathbb{C}^{n+1}$ of $M \subset \mathbb{C}^n \times \mathbb{R} \subset \mathbb{C}^{n+1}$ and

$F \in \mathcal{O}(U)$ s.t. $F|_M = f$.

Smooth extension in the CR case

Theorem (Special case of Hill-Taiani '84)

Let $M \subset \mathbb{C}^n \times \mathbb{R}$, $n \geq 2$, be a real smooth CR hypersurface of CR dimension $n-1$ (not complex). Let $p = (z_0, s_0) \in M$. Let (a, b) be the number of positive and negative eigenvalues of the Levi-form of $M_{(s_0)}$ at z_0.

Then $\exists$ a neighborhood $U \subset \mathbb{C}^n \times \mathbb{R}$ of p, such that given $f \in C^\infty(M) \cap CR(M)$:

(i) *If $a \geq 1$, and H_+ is the corresponding side, then $\exists$ $F \in C^\infty(U \cap H_+) \cap CR(U \cap H_+ \setminus M)$ such that $f|_{M \cap U} = F|_{M \cap U}$.*

(ii) *If $a \geq 1$ and $b \geq 1$, then $\exists$ $F \in C^\infty(U) \cap CR(U)$ such that $f|_{M \cap U} = F|_{M \cap U}$.*

CR singular manifolds in $\mathbb{C}^2$ of real dim 2 first studied by Bishop ('65).

Later by Moser-Webster, Moser, Kenig-Webster, Gong, Huang-Krantz, Huang, Huang-Yin, etc...
Mostly interested in normal form.

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Higher dimensions far less understood.
See e.g. Huang-Yin, Burcea, Gong-L., Dolbeault-Tomassini-Zaitsev, Coffman, Slapar, (and of course L.-Noell-Ravisankar), etc...

In $\mathbb{C}^m$ for $m > 2$ generally a codimension 2 submanifold is not realizable as a submanifold of $\mathbb{C}^{m-1} \times \mathbb{R}$.

A real codimension 2 CR singular submanifold $M \subset \mathbb{C}^m$ does not in general have the extension property in the analytic case. (Harris '78, L.-Minor-Shroff-Son-Zhang).

CR singular submanifolds and $\mathbb{C}^n \times \mathbb{R}$

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Simplest example: Let M in $(z, w) \in \mathbb{C}^n \times \mathbb{C}$ be given by

$$w = z_1 \bar{z}_1$$

Then $\bar{z}_1 = \frac{w}{z_1}$ on M and so does not extend to a neighborhood.

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In early 20th century several authors considered extensions of holomorphic functions (e.g. Hartogs phenomenon) in $\mathbb{C}^n \times \mathbb{R}^k$ (e.g. Bochner, Brown, Severi, etc...)

CR singular hypersurface in $\mathbb{C}^n \times \mathbb{R}$

Let $M \subset \mathbb{C}^n \times \mathbb{R}$ be a hypersurface with a CR singularity.
Write M as

$$s = Q(z, \bar{z}) + E(z, \bar{z})$$

where Q is a real quadratic form, and $E \in O(3)$.

If Q is nondegenerate then the CR singularity is isolated.

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Write $Q(z, \bar{z}) = A(z, \bar{z}) + B(z, z) + \overline{B(z, z)}$
for Hermitian form A and bilinear B.

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Diagonalize A

$$s = \sum_{j=1}^{a} |z_j|^2 - \sum_{j=a+1}^{a+b} |z_j|^2 + B(z, z) + \overline{B(z, z)} + E(z, \bar{z}),$$

We can't generally also diagonalize B (unless $a = n$).

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Define manifold with boundary H_+ by

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Let M be defined by

$$s = \sum_{j=1}^{a} |z_j|^2 - \sum_{j=a+1}^{a+b} |z_j|^2 + B(z,z) + \overline{B(z,z)} + E(z,\bar{z})$$
$$= A + B + \overline{B} + E = Q + E$$

Theorem (L.-Noell-Ravisankar '16)

Suppose M is real-analytic (E is real-analytic), A is nondegenerate ($a+b=n$), $n \geq 2$, and $f \in C^\omega(M) \cap CR(M_{CR})$.

Then $\exists$ neighborhood U of 0 in $\mathbb{C}^n \times \mathbb{C}$ and $F \in \mathcal{O}(U)$ such that $F|_{M\cap U} = f|_{M\cap U}$.

Smooth CR singular case

$$H_+ : \ s \geq \sum_{j=1}^{a} |z_j|^2 - \sum_{j=a+1}^{a+b} |z_j|^2 + B(z,z) + \overline{B(z,z)} + E(z,\bar{z})$$
$$M : \ s = \sum_{j=1}^{a} |z_j|^2 - \sum_{j=a+1}^{a+b} |z_j|^2 + B(z,z) + \overline{B(z,z)} + E(z,\bar{z})$$

Theorem (L.-Noell-Ravisankar '17)

Suppose Q is nondegenerate, and $a \geq 2$.
Then $\exists$ a neighborhood U of 0,
such that given $f \in C^\infty(M) \cap CR(M)$:

(i) *If $a \geq 2$,*
then $\exists$ $F \in C^\infty(U \cap H_+) \cap CR(U \cap H_+ \setminus M)$
such that $F|_{M \cap U} = f|_{M \cap U}$.

(ii) *If $a \geq 2$ and $b \geq 2$,*
then $\exists$ $F \in C^\infty(U) \cap CR(U)$
such that $F|_{M \cap U} = f|_{M \cap U}$.

In either case, F has a formal power series in z and s at 0.

Levi-form on the leaves

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1) existence of the extension

2) regularity up to the boundary

For the first problem, the two eigenvalues are needed.

$$M: \ s = A(z,\bar{z}) + B(z,z) + \overline{B(z,z)} + E(z,\bar{z})$$

If A has two positive eigenvalues, then the Levi-form of $M_{(s)}$ has at least one positive eigenvalues.

Some sort of nondegeneracy is necessary

Example 1: Define M by $s = ||z||^4$ (isolated CR singularity).

The function $\sqrt{s}$ is $C^\omega(M)$ (it equals $||z||^2$ on M).

It is CR, and the unique extension to H_+ is $\sqrt{s}$,
not smooth at the origin.

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not smooth at the origin.

Example 2: Write $z = (z', z'')$. Define M by
$s = (||z'||^2 - ||z''||^2)^3$.

The function $\sqrt[3]{s}$ is $C^\omega(M)$ (equals $||z'||^2 - ||z''||^2$ on M).

It is CR, and the unique extension to H_+ is $\sqrt[3]{s}$,
not smooth at points of H^+ where $s = 0$ (including interior).

CR singularity is large. All points where $s = 0$ are CR singular.

Define $M \subset \mathbb{C}^2 \times \mathbb{R}$ and f by

$$\operatorname{Im} z_1 = s \left|z_2\right|^2, \qquad f(z,s) = \begin{cases} \frac{e^{-1/s^2}}{z_1 + is} & \text{if } s \neq 0, \\ 0 & \text{if } s = 0. \end{cases}$$

The Levi-form is zero when $s = 0$.
Extension of f to neither side is possible near 0.

CR singular case: two eigenvalues of the same sign are necessary

Define $M \subset \mathbb{C}^2 \times \mathbb{R}$ and f by

$$M: \; s = |z_1|^2 - |z_2|^2, \qquad f(z,s) = \begin{cases} \frac{1}{z_1} e^{-1/s^2} & \text{if } s > 0, \\ 0 & \text{if } s = 0, \\ \frac{1}{z_2} e^{-1/s^2} & \text{if } s < 0. \end{cases}$$

$f \in C^\infty(M) \cap CR(M)$ but no extension exists due to the poles.

Analogue of Baouendi-Treves is not true

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cannot have B-T.

There is a disc through every point attached to M, so B-T would imply extension to a neighborhood.

Question: What extra hypotheses to add to B-T to make it work.

Two eigenvalues of both signs needed for extension to a neighborhood

Define $M \subset \mathbb{C}^3 \times \mathbb{R}$ and f by

$$M: \ \ s = |z_1|^2 + |z_2|^2 - |z_3|^2, \qquad f(z,s) = \begin{cases} 0 & \text{if } s \geq 0, \\ \frac{1}{z_3} e^{-1/s^2} & \text{if } s < 0. \end{cases}$$

Again, $f \in C^\infty(M) \cap CR(M)$

And f extends above M, but not below M.

There exists an example that extends only to one side at every point.

Let $M \subset \mathbb{C}^2 \times \mathbb{R}$ be

$$s = |z_1|^2 + |z_2|^2 = \|z\|^2,$$

$g \colon S^3 \subset \mathbb{C}^2 \to \mathbb{C}$ a smooth CR function not extending to the outside of S^3 through any point (e.g. Catlin or Hakim-Sibony).

$$f(z,s) = \begin{cases} e^{-1/s^2} g\left(\frac{z}{\sqrt{s}}\right) & \text{if } s < 0, \\ 0 & \text{if } s = 0, \end{cases}$$

is smooth, CR, extends above M (to H_+), but not below through any point.

Extension fails in $n = 1$.

Let $M \subset \mathbb{C} \times \mathbb{R}$ be a nonparabolic Bishop surface

$$s = |z|^2 + \lambda z^2 + \lambda \bar{z}^2, \qquad \text{(where } 0 \leq \lambda < \infty \text{ and } \lambda \neq \tfrac{1}{2}\text{)}.$$

Define a smooth $f \colon \mathbb{C} \to \mathbb{R}$ that is zero on the first quadrant of $\mathbb{C}$ and positive elsewhere.

Parametrize M by z, then $f(z, \bar{z})$ is smooth on M.
The CR condition is vacuous.

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For every $s \neq 0$, the leaf

$$(H_+)_{(s)} = \{z \in \mathbb{C} \mid s \geq |z|^2 + \lambda z^2 + \lambda \bar{z}^2\}$$

is either empty, or has part of its boundary in the first quadrant. So f cannot extend.

Example: Topology of leaves can be a problem

Define M by

$$s = |z_1|^2 - |z_2|^2 + \lambda(z_1^2 + \bar{z}_1^2)$$

for $\lambda > \frac{1}{2}$.

For $s > 0$, the manifold with boundary $(H_+)_{(s)}$ has disconnected boundary.

Construct a function that is a different constant on each boundary component for each $(H_+)_{(s)}$.

Take $\phi(x) = \sin^2(1/x)e^{-1/x^2}$, and let M be given by

$$s = \phi(\|z\|^2)$$

$(H_+)_{(s)}$ has multiple components with disconnected boundary.

The function $\|z\|^2$ is $C^\infty(M) \cap CR(M)$ but has no extension.

The CR singularity is large (an infinite set of concentric circles).

Proof of the theorem

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3) Extend near the CR points using Hill-Taiani.

4) Construct families of analytic discs inside a single leaf attached to CR points of M shrinking to a CR point of M.

5) Apply Kontinuitätssatz to find an extension F. (technicality: proving single valuedness, $M_{(s)}$ and $(H_+)_{(s)}$ need not be connected, and $(H_+)_{(s)}$ may not be simply connected.)

Proof of the theorem, cont.

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7) Suppose M given by $s = \rho(z, \bar{z})$. Parametrize M by z and differentiate $f(z, \bar{z})$ outside the origin.

$$f_{\bar{z}_j} = (F_s|_M)\rho_{\bar{z}_j}$$

Division works formally at the origin by the formal solution. By Malgrange $F_s|_M$ is smooth. Similarly $F_{z_j}|_M$ is smooth.

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Division works formally at the origin by the formal solution. By Malgrange $F_s|_M$ is smooth. Similarly $F_{z_j}|_M$ is smooth.

8) $F_s|_M$ and $F_{z_j}|_M$ are smooth CR functions, therefore their extensions are continuous up to the boundary. Now iterate.

Notes/Questions...

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(It is not in the real-analytic/formal case).

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Question: What nondegeneracy is needed in the C^ω case?
(e.g., we can prove C^ω extension for
$s = z_1^2 + \cdots + z_n^2 + \bar{z}_1^2 + \cdots + \bar{z}_n^2$.)

Thank you