Proper maps of ball complements & differences and rational sphere maps

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If U, V bounded and f extends to $\overline{U}$, then f is proper $\Leftrightarrow f(\partial U) \subset \partial V$.
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Goal: Classify all proper holomorphic maps $f\colon U \to V$.

Automorphisms of the unit disc $\mathbb{D}$:

$$z \mapsto e^{i\theta}\varphi_\alpha(z) = e^{i\theta}\frac{\alpha - z}{1 - \bar{\alpha}z} \qquad (\theta \in \mathbb{R}, \alpha \in \mathbb{D}).$$

$\varphi_\alpha(\alpha) = 0, \quad \varphi_\alpha(0) = \alpha, \quad \varphi_\alpha \circ \varphi_\alpha = \mathrm{id}.$

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Theorem (Fatou)

Every proper holomorphic map $f : \mathbb{D} \to \mathbb{D}$ is a finite Blaschke product:

$$f(z) = e^{i\theta}\prod_{k=1}^{m}\frac{a_k - z}{1 - \bar{a}_k z}$$

For $\mathbb{B}_n$ = the unit ball, $\quad \mathrm{Aut}(\mathbb{B}_n)$ = automorphisms of $\mathbb{B}_n$:

$z \mapsto U\varphi_\alpha(z), \quad U$ is unitary, $\alpha \in \mathbb{B}_n$, and

$$\varphi_\alpha(z) = \frac{\alpha - L_\alpha z}{1 - \langle z, \alpha \rangle}, \quad L_\alpha z = \left(1 - \sqrt{1 - \|\alpha\|^2}\right) \frac{\langle z, \alpha \rangle}{\|\alpha\|^2} \alpha + \sqrt{1 - \|\alpha\|^2} z, \quad L_0 = I.$$

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Theorem (Alexander, Pinchuk circa '77 (complicated history. . .))

If $f : \mathbb{B}_n \to \mathbb{B}_n$ $(n \geq 2)$ is a proper holomorphic map, then $f \in \mathrm{Aut}(\mathbb{B}_n)$.

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Theorem (Dor '90)

For every n, there exists a proper holomorphic $f : \mathbb{B}_n \to \mathbb{B}_{n+1}$ extending continuously up to the boundary and $f(\partial \mathbb{B}_n) = \partial \mathbb{B}_{n+1}$.

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Theorem (Forstnerič '89)

Suppose $2 \leq n \leq N$. If a proper holomorphic $f\colon \mathbb{B}_n \longrightarrow \mathbb{B}_N$ extends smoothly up to the boundary, then f is rational, and its degree is bounded in terms of n and N.

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In fact, the result is local:

Theorem (Forstnerič '89)

Suppose $2 \leq n \leq N$ and U is a neighborhood of $p \in \partial \mathbb{B}_n$. If $f : U \cap \overline{\mathbb{B}_n} \to \mathbb{C}^N$ is smooth, holomorphic on $U \cap \mathbb{B}_n$, and $f(U \cap S^{2n-1}) \subset S^{2N-1}$, then f is rational and extends to a proper map of $\mathbb{B}_n$ to $\mathbb{B}_N$, and its degree is bounded in terms of n and N.

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The key is to use Hartogs and Forstnerič and then study polynomial sphere maps using Cauchy–Schwarz on the reflection principle:

$$\left\langle f(z), f\left(\frac{z}{\|z\|^2}\right)\right\rangle = 1.$$

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Theorem

Suppose $n \geq 2$ *and* $B_r(c) \subset \mathbb{C}^n$, $B_R(C) \subset \mathbb{C}^N$ *are balls such that* $B_r(c) \cap \mathbb{B}_n \neq \emptyset$ *and* $B_R(C) \cap \mathbb{B}_N \neq \emptyset$*. Suppose* $f\colon \mathbb{B}_n \setminus \overline{B_r(c)} \to \mathbb{B}_N \setminus \overline{B_R(C)}$ *is proper and holomorphic.*

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Then f *is rational and extends to a rational proper map of balls* $f \colon \mathbb{B}_n \to \mathbb{B}_N$ *that takes* $(\partial B_r(c)) \cap \mathbb{B}_n$ *to* $(\partial B_R(C)) \cap \mathbb{B}_N$. *If* $c = 0$ *and* $C = 0$ *and* $f = \frac{p}{q}$ *is in lowest terms, then* $\deg q < \deg p$.

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Proposition

Suppose $n \geq 2$, $\mathbb{B}_n \cap B_r(c) \neq \emptyset$, *and* $\mathbb{B}_N \cap B_R(C) \neq \emptyset$. *There exist no proper holomorphic maps* $f \colon \mathbb{B}_n \setminus \overline{B_r(c)} \to \mathbb{C}^N \setminus \overline{\mathbb{B}_N}$ *nor* $f \colon \mathbb{C}^n \setminus \overline{\mathbb{B}_n} \to \mathbb{B}_N \setminus \overline{B_R(C)}$.

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Example: Embed $\mathbb{D}$ properly into $\mathbb{C}^N$. Take a closed ball B such that $B \cap f(\mathbb{D})$ is nontrivial but $f^{-1}(B)$ is connected, then $\mathbb{D} \setminus f^{-1}(B)$ is equivalent to an annulus.

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Definition: A rational $f : \mathbb{C}^n \dashrightarrow \mathbb{C}^N$ is an *m-fold sphere map* if there exist $2m$ numbers $0 < r_1 < r_2 < \cdots < r_m < \infty$ and $0 < R_1, R_2, \ldots, R_m < \infty$, such that the pole set of f misses $r_j S^{2n-1}$ and $f(r_j S^{2n-1}) \subset R_j S^{2N-1}$ for all $j = 1, \ldots, m$. If there are infinitely many such numbers r_j and R_j, then f is an *∞-fold sphere map*.

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Example: $f\colon \mathbb{C}^2 \to \mathbb{C}^6$ given by

$$(z_1, z_2) \mapsto \left(\frac{2}{\sqrt{5}} z_1^3, \frac{2\sqrt{2}}{\sqrt{5}} z_1^2 z_2, \frac{2}{\sqrt{5}} z_1 z_2^2, z_1 z_2, z_2^2, \frac{1}{\sqrt{5}} z_1 \right),$$

takes S^3 to S^{11} and $\frac{1}{2}S^3$ to $\frac{1}{4}S^{11}$, so it is a 2-fold map that is not a 3-fold map. Note that it is cubic.

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Definition: A rational $f\colon \mathbb{C}^n \dashrightarrow \mathbb{C}^N$ is an *m-fold sphere map* if there exist $2m$ numbers $0 < r_1 < r_2 < \cdots < r_m < \infty$ and $0 < R_1, R_2, \ldots, R_m < \infty$, such that the pole set of f misses $r_j S^{2n-1}$ and $f(r_j S^{2n-1}) \subset R_j S^{2N-1}$ for all $j = 1, \ldots, m$. If there are infinitely many such numbers r_j and R_j, then f is an ∞-*fold sphere map*.

Example: $z^{\otimes d}$ is a ∞-fold sphere map. Follows as $\|z^{\otimes d}\|^2 = \|z\|^{2d}$.

Example: If f and g are m-fold sphere maps for the same $0 < r_1 < r_2 < \cdots < r_m < \infty$, then $f \otimes g$ or $f \oplus g$ are m-fold sphere maps. Follows as $\|f \otimes g\|^2 = \|f\|^2 \|g\|^2$ and $\|f \oplus g\|^2 = \|f\|^2 + \|g\|^2$.

Example: $f\colon \mathbb{C}^2 \to \mathbb{C}^6$ given by

$$(z_1, z_2) \mapsto \left(\tfrac{2}{\sqrt{5}} z_1^3, \tfrac{2\sqrt{2}}{\sqrt{5}} z_1^2 z_2, \tfrac{2}{\sqrt{5}} z_1 z_2^2, z_1 z_2, z_2^2, \tfrac{1}{\sqrt{5}} z_1 \right),$$

takes S^3 to S^{11} and $\frac{1}{2} S^3$ to $\frac{1}{4} S^{11}$, so it is a 2-fold map that is not a 3-fold map. Note that it is cubic.

f is also a proper map of $\mathbb{B}_2 \setminus \frac{1}{2}\overline{\mathbb{B}_2} \to \mathbb{B}_6 \setminus \frac{1}{4}\overline{\mathbb{B}_6}$.

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If f *is an* ∞*-fold sphere map, then* f *is polynomial and* $\forall\, r > 0$, $\exists\, R > 0$, *s.t.* $f(rS^{2n-1}) \subset RS^{2N-1}$. *Moreover, there exists a unitary* $U \in U(\mathbb{C}^N)$ *and homogeneous sphere maps (possibly constant)* $h_j\colon \mathbb{C}^n \to \mathbb{C}^{\ell_j}$, $j = 1, \ldots, k$, *such that*

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For every m, there is a polynomial m-fold sphere map of degree $m + 1$ that is not an $(m + 1)$-fold sphere map.

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For every m, there is a rational m-fold sphere map of degree m that is not an $(m + 1)$-fold sphere map.

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For every m, there is a rational m-fold sphere map of degree m that is not an $(m+1)$-fold sphere map.
Some bound follows by a trivial argument (Bézout), but not the sharp one.

Thanks for listening!